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Fixed-Parameter Algorithms for the (k, r)-Center in Planar Graphs and Map Graphs

*Erik D. Demaine** *Fedor V. Fomin*† *MohammadTaghi Hajiaghayi**

Dimitrios M. Thilikos‡

Abstract

The (k, r)*-center problem* asks whether an input graph G has $\leq k$ vertices (called *centers*) such that every vertex of G is within distance $\leq r$ from some center. In this paper we prove that the (k, r)-center problem, parameterized by k and r, is fixed-parameter tractable (FPT) on planar graphs, i.e., it admits an algorithm of complexity $f(k, r)n^{O(1)}$ where the function f is independent of n. In particular, we show that $f(k, r) = 2^{O(r \log r)\sqrt{k}}$, where the exponent of the exponential term grows sublinearly in the number of centers. Moreover, we prove that the same type of FPT algorithms can be designed for the more general class of *map graphs* introduced by Chen, Grigni, and Papadimitriou. Our results combine dynamic-programming algorithms for graphs of small branchwidth and a graph-theoretic result bounding this parameter in terms of k and r. Finally, a byproduct of our algorithm is the existence of a PTAS for the r-domination problem in both planar graphs and map graphs.

Our approach builds on the seminal results of Robertson and Seymour on Graph Minors, and as a result is much more powerful than the previous machinery of Alber et al. for exponential speedup on planar graphs. To demonstrate the versatility of our results, we show how our algorithms can be extended to general parameters that are "large" on grids. In addition, our use of branchwidth instead of the usual treewidth allows us to obtain much faster algorithms, and requires more complicated dynamic programming than the standard leaf/introduce/forget/join structure of nice tree decompositions. Our results are also unique in that they apply to classes of graphs that are not minor-closed, namely, constant powers of planar graphs and map graphs.

Keywords: (k, r)-center, fixed-parameter algorithms, domination, planar graph, map graph

*MIT Laboratory for Computer Science, 200 Technology Square, Cambridge, Massachusetts 02139, USA, {edemaine,hajiagha}@mit.edu

†Department of Informatics, University of Bergen, N-5020 Bergen, Norway, fomin@ii.uib.no

‡Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, Campus Nord – Mòdul C5, c/Jordi Girona Salgado 1-3, E-08034, Barcelona, Spain, sedthilk@lsi.upc.es

1 Introduction

Clustering is a key tool for solving a variety of application problems such as data mining [18], data compression [24], pattern recognition and classification [23], learning [13], and facility location [44, 29]. Among the algorithmic problem formulations of clustering are k-means, k-medians, and k-center. In all of these problems, the goal is to partition n given points into k *clusters* so that some objective function is minimized.

In this paper, we concentrate on the (unweighted) (k, r)*-center* problem [8, 21], in which the goal is to choose k *centers* from the given set of n points so that every point is within distance r from some center in the graph. In particular, the k*-center problem* [25, 28] of minimizing the maximum distance to a center is exactly (k, r)-center when the goal is to minimize r subject to finding a feasible solution. In addition, the r*-domination problem* [8, 22] of choosing the fewest vertices whose r-neighborhoods cover the whole graph is exactly (k, r)-center when the goal is to minimize k subject to finding a feasible solution.

A sample application of the (k, r)-center problem in the context of facility location is the installation of emergency service facilities such as fire stations. Here we suppose that we can afford to buy k fire stations to cover a city, and we require every building to be within r city blocks from the nearest fire station to ensure a reasonable response time. Given an algorithm for (k, r)-center, we can vary k and r to find the best bicriterion solution according to the needs of the application. In this scenario, we can afford high running time (e.g., several weeks of real time) if the resulting solution builds fewer fire stations (which are extremely expensive) or has faster response time; thus, we prefer fixed-parameter algorithms over approximation algorithms.

In this application, and many others, the graph is typically planar or nearly so. Chen, Grigni, and Papadimitriou [12] have introduced a generalized notion of planarity which allows local non-planarity. In this generalization, two countries of a map are adjacent if they share at least one point, and the resulting graph of adjacencies is called a *map graph*. (See Section 2 for a precise definition.) Planar graphs are the special case of map graphs in which at most three countries intersect at a point.

Previous results. r-domination and k-center are NP-hard even for planar graphs [21]. For r-domination, the current best approximation (for general graphs) is a $(\log n + 1)$-factor by phrasing the problem as an instance of set cover [8]. For k-center, there is a 2-approximation algorithm [25, 28] which applies more generally to the case of weighted graphs satisfying the triangle inequality. Furthermore, no $(2 - \epsilon)$-approximation algorithm exists for any $\epsilon > 0$ even for unweighted planar graphs of maximum degree 3 [36]. For geometric k-center in which the weights are given by Euclidean distance in d-dimensional space, there is a PTAS whose running time is exponential in k [1]. Approximation algorithms have also been developed for vertex-weighted k-center [37], absolute k-center [38], all-neighbor k-center [31, 34], asymmetric k-center [35], capacitated k-center [8, 32], and fault-tolerant k-center [31, 8]. Several relations between small r-domination sets for planar graphs and problems about organizing routing schemes with compact structures is discussed in [22].

The (k, r)-center problem can be considered as a generalization of the well-known dominating set problem. During the last two years in particular much attention has been paid to constructing fixed-parameter algorithms with exponential speedup for this problem. Alber et al. [2] were the first who demonstrated an algorithm checking whether a planar graph has a dominating set of size $\leq k$ in time $O(2^{70\sqrt{k}}n)$. This result was the first non-trivial result for the parameterized version of an NP-hard problem in which the exponent of the exponential term grows sublinearly in the parameter. Recently, the running time of this algorithm was further improved to $O(2^{27\sqrt{k}}n)$ [30] and $O(2^{15.13\sqrt{k}}k + n^3 + k^4)$ [19]. Fixed-parameter algorithms for solving many different problems such as vertex cover, feedback vertex set, maximal clique transversal, and edge-dominating set on planar and related graphs such as single-crossing-minor-free graphs are considered in [5, 10, 15, 33]. Most of these problems have reductions to the dominating set problem. Also, because all these problems are closed under taking minors or contractions, all classes of graphs considered so far are minor-closed.

Our results. In this paper, we focus on applying the tools of *parameterized complexity*, introduced by Downey and Fellows [16], to the (k, r)-center problem in planar and map graphs. We view both k and r as parameters to the problem. We introduce a new proof technique which allows us to extend known results on planar dominating set in two different aspects.

First, we extend the exponential speed-up for a generalization of dominating set, namely the (k, r)-center problem, on planar graphs. Specifically, the running time of our algorithm is $O((2r+1)^{6(2r+1)\sqrt{k}+12r+3/2}n + n^4)$, where n is the number of vertices. Our proof technique is based on combinatorial bounds (Section 3) derived from the Robertson, Seymour, and Thomas theorem about quickly excluding planar graphs, and on a complicated dynamic program on graphs of bounded branchwidth (Section 4). Second, we extend our fixed-parameter algorithm to map graphs which is a class of graphs that is not minor-closed. In particular, the running time of the corresponding algorithm is $O((2r+1)^{6(4r+1)\sqrt{k}+24r+3}n + n^4)$.

Notice that the exponential component of the running times of our algorithms depends only on the parameters, and is multiplicatively separated from the problem size n. Moreover, the contribution of k in the exponential part is sublinear. In particular, our algorithms have polynomial running time if $k = O(\log^2 n)$ and $r = O(1)$, or if $r = O(\log n/\log\log n)$ and $k = O(1)$. We stress the fact that we design our dynamic-programming algorithms using branchwidth instead of treewidth because this provides better running times.

Finally, in Section 6, we present several extensions of our results, including a PTAS for the r-dominating set problem and a generalization to a broad class of graph parameters.

2 Definitions and preliminary results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We let n denote the number of vertices of a graph when it is clear from context. For every nonempty $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge contractions is said to be a *contraction* of G. A graph H is a *minor* of a graph G if H is a subgraph of a contraction of G. We use the notation $H \preceq G$ (resp. $H \preceq_c G$) for H is a minor (a contraction) of G.

(k, r)-center. We define the *r-neighborhood* of a set $S \subseteq V(G)$, denoted by $N_G^r(S)$, to be the set of vertices of G at distance at most r from at least one vertex of S; if $S = \{v\}$ we simply use the notation $N_G^r(v)$. We say a graph G has a *(k, r)-center* or interchangeably has an *r-dominating set of size k* if there exists a set S of centers (vertices) of size at most k such that $N_G^r(S) = V(G)$. We denote by $\gamma_r(G)$ the smallest k for which there exists a (k, r)-center in the graph. One can easily observe that for any r the problem of checking whether an input graph has a (k, r)-center, parameterized by k is $W[2]$-hard by a reduction from dominating set. (See Downey and Fellows [16] for the definition of the W Hierarchy.)

Map graphs. Let Σ be a sphere. A *Σ-plane* graph G is a planar graph G drawn in Σ. To simplify notation, we usually do not distinguish between a vertex of the graph and the point of Σ used in the drawing to represent the vertex, or between an edge and the open line segment representing it. We denote the set of regions (faces) in the drawing of G by $R(G)$. (Every region is an open set.) An edge e or a vertex v is *incident* to a region r if $e \subseteq \bar{r}$ or $v \subseteq \bar{r}$, respectively. ($\bar{r}$ denotes the *closure* of r.)

For a Σ-plane graph G, a *map* $\mathcal{M}$ is a pair (G, ϕ), where $\phi : R(G) \to \{0, 1\}$ is a two-coloring of the regions. A region $r \in R(G)$ is called a *nation* if $\phi(r) = 1$ and a *lake* otherwise.

Let $N(\mathcal{M})$ be the set of nations of a map $\mathcal{M}$. The graph F is defined on the vertex set $N(\mathcal{M})$, in which two vertices r_1, r_2 are adjacent precisely if $\bar{r}_1 \cap \bar{r}_2$ contains at least one edge of G. Because f is the subgraph of the dual graph G^* of G, it is planar. Chen, Grigni, and Papadimitriou [12] defined the following generalization of planar graphs. A *map graph $G_{\mathcal{M}}$ of a map $\mathcal{M}$* is the graph on the vertex set $N(\mathcal{M})$ in which two vertices r_1, r_2 are adjacent in $G_{\mathcal{M}}$ precisely if $\bar{r}_1 \cap \bar{r}_2$ contains at least one vertex of G.

For a graph G, we denote by G^k the *kth power of* G, i.e., the graph on the vertex set $V(G)$ such that two vertices in G^k are adjacent precisely if the distance in G between these vertices is at most k. Let G be a bipartite graph with a bipartition $U \cup W = V(G)$. The *half square* $G^2[U]$ is the graph on the vertex set U and two vertices are adjacent in $G^2[U]$ precisely if the distance between these vertices in G is 2.

Theorem 2.1 ([12]). *A graph $G_{\mathcal{M}}$ is a map graph if and only if it is the half-square of some planar bipartite graph H.*

Here the graph H is called a *witness for $G_{\mathcal{M}}$*. Thus the question of finding a (k, r)-center in a map graph $G_{\mathcal{M}}$ is equivalent to finding in a witness H of $G_{\mathcal{M}}$ a set $S \subseteq V(G_{\mathcal{M}})$ of size k such that every vertex in $V(G_{\mathcal{M}}) - S$ has distance $\leq 2r$ in H from some vertex of S.

The proof of Theorem 2.1 is constructive, i.e., given a map graph $G_{\mathcal{M}}$ together with its map $\mathcal{M} = (G, \phi)$, one can construct a witness H for $G_{\mathcal{M}}$ in time $O(|V(G_{\mathcal{M}})| + |E(G_{\mathcal{M}})|)$. One color class $V(G_{\mathcal{M}})$ of the bipartite graph H corresponds to the set of nations of the map $\mathcal{M}$. Each vertex v of the second color class $V(H) - V(G_{\mathcal{M}})$ corresponds to an intersection point of boundaries of some nations, and v is adjacent (in H) to the vertices corresponding to the nations it belongs. What is important for our proofs are the facts that

1. in such a witness, every vertex of $V(H) - V(G_{\mathcal{M}})$ is adjacent to a vertex of $V(G_{\mathcal{M}})$, and
2. $|V(H)| = O(|V(G_{\mathcal{M}})| + |E(G_{\mathcal{M}})|)$.

Thorup [43] provided a polynomial-time algorithm for constructing a map of a given map graph in polynomial time. However, in Thorup's algorithm, the exponent in the polynomial time bound is about 120 [11]. So from practical point of view there is a big difference whether we are given a map in addition to the corresponding map graph. Below we suppose that we are always given the map.

Branchwidth. Branchwidth was introduced by Robertson and Seymour in their Graph Minors series of papers. A *branch decomposition* of a graph G is a pair (T, τ), where T is a tree with vertices of degree 1 or 3 and τ is a bijection from $E(G)$ to the set of leaves of T. The *order function* $\omega : E(T) \to 2^{V(G)}$ of a branch decomposition maps every edge e of T to a subset of vertices $\omega(e) \subseteq V(G)$ as follows. The set $\omega(e)$ consists of all vertices of $V(G)$ such that, for every vertex $v \in \omega(e)$, there exist edges $f_1, f_2 \in E(G)$ such that $v \in f_1 \cap f_2$ and the leaves $\tau(f_1)$, $\tau(f_2)$ are in different components of $T - \{e\}$. The *width* of (T, τ) is equal to $\max_{e \in E(T)} |\omega(e)|$ and the *branchwidth* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G.

It is well-known that, if $H \preceq G$ or $H \preceq_c G$, then $\mathbf{bw}(H) \leq \mathbf{bw}(G)$.

The following deep result of Robertson, Seymour, and Thomas (Theorems (4.3) in [39] and (6.3) in [40]) plays an important role in our proofs.

Theorem 2.2 ([40]). *Let $k \geq 1$ be an integer. Every planar graph with no $(k \times k)$-grid as a minor has branchwidth $\leq 4k - 3$.*

Branchwidth is the main tool in this paper. All our proofs can be rewritten in terms of the related and better-known parameter *treewidth*, and indeed treewidth would be easier to handle in our dynamic program. However, branchwidth provides better combinatorial bounds resulting in exponential speed-up of our algorithms.

3 Combinatorial bounds

Lemma 3.1. *Let $\rho, k, r \geq 1$ be integers and G be a planar graph having a (k,r)-center and with a $(\rho \times \rho)$-grid as a minor. Then $k \geq (\frac{\rho-2r}{2r+1})^2$.*

Proof. We set $V = \{1,\dots,\rho\} \times \{1,\dots,\rho\}$. Let

$$F = (V, \{((x,y),(x',y')) \mid |x-x'| + |y-y'| = 1\})$$

be a plane $(\rho \times \rho)$-grid that is a minor of some plane embedding of G. W.l.o.g. we assume that the external (infinite) face of this embedding of F is the one that is incident to the vertices of the set $V_{\text{ext}} = \{(x,y) \mid x = 1 \text{ or } x = \rho \text{ or } y = 1 \text{ or } y = \rho\}$, i.e., the vertices of F with degree < 4. We call the rest of the faces of F *internal faces*. We set $V_{\text{int}} = \{(x,y) \mid r+1 \leq x \leq \rho - r,\ r+1 \leq y \leq \rho - r\}$, i.e., V_{int} is the set of all vertices of F within distance $\geq r$ from all vertices in V_{ext}. Notice that $F[V_{\text{int}}]$ is a sub-grid of F and $|V_{\text{int}}| = (\rho - 2r)^2$. Given any pair of vertices $(x,y),(x',y') \in V$ we define $\delta((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$.

We also define $d_F((x,y),(x',y'))$ to be the distance between any pair of vertices (x,y) and (x',y') in F. Finally we define J to be the graph occurring from F by adding in it the edges of the following sets:

$$\{((x,y),(x+1,y+1) \mid 1 \leq x \leq \rho - 1,\ 1 \leq y \leq \rho - 1)\}$$

$$\{((x,y+1),(x+1,y) \mid 1 \leq x \leq \rho - 1,\ 1 \leq y \leq \rho - 1)\}$$

(In other word we add all edges connecting pairs of non-adjacent vertices incident to its internal faces). It is easy to verify that $\forall (x,y),(x',y') \in V \quad \delta((x,y),(x',y')) = d_J((x,y),(x',y'))$. This implies the following.

$$\text{If } R \text{ is a subgraph of } J, \text{ then } \forall (x,y),(x',y') \in V\ \delta((x,y),(x',y')) \leq d_R((x,y),(x',y')) \tag{1}$$

For any $(x,y) \in V$ we define $B_r((x,y)) = \{(a,b) \in V \mid \delta((x,y),(a,b)) \leq r\}$ and we observe the following:

$$\forall_{(x,y) \in V}\ |V(B_r((x,y)))| \leq (2r+1)^2. \tag{2}$$

Consider now the sequence of edge contractions/removals that transform G to F. If we apply on G only the contractions of this sequence we end up with a planar graph H that can obtained by the $(\rho \times \rho)$-grid F after adding edges to non-consecutive vertices of its faces. This makes it possible to partition the additional edges of H into two sets: a set denoted by E_1 whose edges connect non-adjacent vertices of some square face of F and another set E_2 whose edges connect pairs of vertices in V_{ext}. We denote by R the graph obtained by F if we add the edges of E_1 in F. As R is a subgraph of J, (1) implies that

$$\forall_{(x,y) \in V}\ N_R^r((x,y)) \subseteq B_r((x,y)) \tag{3}$$

We also claim that

$$\forall_{(x,y) \in V}\ N_H^r((x,y)) \subseteq B_r((x,y)) \cup (V - V_{\text{int}}) \tag{4}$$

To prove (4) we notice first that if we replace H by R in it then the resulting relation follows from (3). It remains to prove that the consecutive addition of edges of E_2 in R does not introduce in $N_R^r((x,y))$ any vertex of V_{int}. Indeed, this is correct because any vertex in V_{ext} is in distance $\geq r$ from any vertex in V_{int}. Notice now that (4) implies that $\forall_{(x,y)\in V}\ N_H^r((x,y)) \cap V_{\text{int}} \subseteq B_r((x,y)) \cap V_{\text{int}}$ and using (2) we conclude that

$$\forall_{(x,y)\in V}\ |N_H^r((x,y)) \cap V_{\text{int}}| \leq (2r+1)^2 \tag{5}$$

Let S be a (k',r)-center in the graph H. Applying (5) on S we have that the r-neighborhood of any vertex in S contains at most $(2r+1)^2$ vertices from V_{int}. Moreover, any vertex in V_{int} should belong to the r-neighborhood of some vertex in S. Thus $k' \geq |V_{\text{int}}|/(2\rho+1)^2 = (\rho-2r)^2/(2\rho+1)^2$ and therefore $k' \geq (\frac{\rho-2r}{2r+1})^2$.

Clearly, the conditions that G has an r-dominating set of size k and $H \preceq_c G$ imply that H has an r-dominating set of size $k' \leq k$. (But this is not true for $H \preceq G$.) As H is a contraction of G and G has a (k,r)-center, we have that $k \geq k' \geq (\frac{\rho-2r}{2r+1})^2$ and lemma follows. □

We are ready to prove the main combinatorial result of this paper:

Theorem 3.2. *For any planar graph G having a (k,r)-center,* $\mathbf{bw}(G) \leq 4(2r+1)\sqrt{k} + 8r + 1$.

Proof. Suppose that $\mathbf{bw}(G) > p = 4(2r+1)\sqrt{k}+8r+\epsilon-3$ for some ϵ, $0 < \epsilon \leq 4$, for which $p+3 \equiv 0 \pmod 4$. By Theorem 2.2, G contains a $(\rho\times\rho)$-grid as a minor where $\rho = (2r+1)\sqrt{k}+2r+\frac{\epsilon}{4}$. By Lemma 3.1, $k \geq (\frac{\rho-2r}{2r+1})^2 = (\frac{(2r+1)\sqrt{k}+\frac{\epsilon}{4}}{2r+1})^2$ which implies that $\sqrt{k} \geq \sqrt{k}+\frac{\epsilon}{8r+4}$, a contradiction. □

Notice that the branchwidth of a map graph is unbounded in terms of k and r. For example, a clique of size n is a map graph and has a $(1,1)$-center and branchwidth $\geq 2/3n$.

Theorem 3.3. *For any map graph $G_{\mathcal{M}}$ having a (k,r)-center and its witness H,* $\mathbf{bw}(H) \leq 4(4r+3)\sqrt{k} + 16r + 9$.

Proof. The question of finding a (k,r)-center in a map graph $G_{\mathcal{M}}$ is equivalent to finding in a witness H of $G_{\mathcal{M}}$ a set $S \subseteq V(G_{\mathcal{M}})$ of size k such that every vertex $V(G_{\mathcal{M}}) - S$ is at distance $\leq 2r$ in H from some vertex of S. By the construction of the witness graph, every vertex of $V(H) - V(G_{\mathcal{M}})$ is adjacent to some vertex of $V(G_{\mathcal{M}})$. Thus H has a $(k, 2r+1)$-center and by Theorem 3.2 the proof follows. □

4 (k,r)-Centers in Graphs of Bounded Branchwidth

In this section, we present a dynamic-programming approach to solve the (k,r)-center problem on graphs of bounded branchwidth. It is easy to prove that, for a *fixed* r, the problem is in MSOL (monadic second-order logic) and thus can be solved in linear time on graphs of bounded treewidth (branchwidth). (See the survey of Bodlaender [9] for more information on bounded treewidth and MSOL.) However, for r part of the input, the situation is more difficult. Additionally, we are interested in not just a linear-time algorithm but in an algorithm with running time $f(k,r)n$.

It is worth mentioning that our algorithm requires more than a simple extension of Alber et al.'s algorithm for dominating set in graphs of bounded treewidth [2], which corresponds to the case $r = 1$. In fact, finding a (k,r)-center is similar to finding homomorphic subgraphs, which has been solved only for special classes of graphs and even then only via complicated dynamic programs [26]. The main difficulty is that the path $v = v_0, v_1, v_2, \ldots, v_{\leq r} = c$ from a vertex v to its assigned center c may wander up and down the branch decomposition repeatedly, so that c and v may be in radically different 'cuts' induced by the branch decomposition. All we can guarantee is that the next vertex v_1 along the path from v to c is somewhere in a common 'cut' with v, and that vertex v_1 and v_2 are in a common 'cut', etc. In this way, we must propagate information through the v_i's about the remote location of c.

Let (T', τ) be a branch decomposition of a graph G with m edges and let $\omega' : E(T') \to 2^{V(G)}$ be the order function of (T', τ). We choose an edge $\{x, y\}$ in T', put a new vertex v of degree 2 on this edge, and make v adjacent to a new vertex r. By choosing r as a root in the new tree $T = T' \cup \{v, r\}$, we turn T into a rooted tree. For every edge of $f \in E(T) \cap E(T')$, we put $\omega(f) = \omega'(f)$. Also we put $\omega(\{x, v\}) = \omega(\{v, y\}) = \omega'(\{x, y\})$ and $\omega(\{r, v\}) = \emptyset$.

For an edge f of T we define E_f (V_f) as the set of edges (vertices) that are "below" f, i.e., the set of all edges (vertices) g such that every path containing g and $\{v, r\}$ in T contains f. With such a notation, $E(T) = E_{\{v,r\}}$ and $V(T) = V_{\{v,r\}}$. Every edge f of T that is not incident to a leaf has two children that are the edges of E_f incident to f. We denote by G_f the subgraph of G induced by the vertices incident to edges from the following set

$$\{\tau^{-1}(x) \mid x \in V_f \wedge (x \text{ is a leaf of } T')\}.$$

The subproblems in our dynamic program are defined by a coloring of the vertices in $\omega(f)$ for every edge f of T. Each vertex will be assigned one of $2r + 1$ colors

$$\{0, \uparrow 1, \uparrow 2, \ldots, \uparrow r, \downarrow 1, \downarrow 2, \ldots, \downarrow r\}.$$

The meaning of the color of a vertex v is as follows:

- 0 means that the vertex v is a chosen center.
- $\downarrow i$ means that vertex v has distance exactly i to the closest center c. Moreover, there is a neighbor $u \in V(G_f)$ of v that is at distance exactly $i - 1$ to the center c. We say that neighbor u *resolves* vertex v.
- $\uparrow i$ means that vertex v has distance exactly i to the closest center c. However, there is *no* neighbor of v in $V(G_f)$ resolving v. Thus we are guessing that any vertex resolving v is somewhere in $V(G) - V(G_f)$.

Intuitively, the vertices colored by $\downarrow i$ have already been resolved (though the vertex that resolves it may not itself be resolved), whereas the vertices colored by $\uparrow i$ still need to be assigned vertices that are closer to the center.

We use the notation $\updownarrow i$ to denote a color of either $\uparrow i$ or $\downarrow i$. Also we use $\updownarrow 0 = 0$.

For an edge f of T, a coloring of the vertices in $\omega(f)$ is called *locally valid* if the following property holds: for any two adjacent vertices v and w in $\omega(f)$, if v is colored $\updownarrow i$ and w is colored $\updownarrow j$, then $|i - j| \le 1$. (If the distance from some vertex v to the closest center is i, then for every neighbor u of v the distance from u to the closest center can not be less than $i - 1$ or more than $i + 1$.)

For every edge f of T we define the mapping

$$A_f : \{0, \uparrow 1, \uparrow 2, \ldots, \uparrow r, \downarrow 1, \downarrow 2, \ldots, \downarrow r\}^{|\omega(f)|} \to \mathbb{N} \cup \{+\infty\}.$$

For a locally valid coloring $c \in \{0, \uparrow 1, \uparrow 2, \ldots, \uparrow r, \downarrow 1, \downarrow 2, \ldots, \downarrow r\}^{|\omega(f)|}$, the value $A_f(c)$ stores the size of the "minimum (k, r)-center restricted to G_f and coloring c". More precisely, $A_f(c)$ is the minimum cardinality of a set $D_f(c) \subseteq V(G_f)$ such that

- For every vertex $v \in \omega(f)$,
  - $c(v) = 0$ if and only if $v \in D_f(c)$, and

– if $c(v) = \downarrow i$, $i \geq 1$, then $v \notin D_f(c)$ and either there is a vertex $u \in \omega(f)$ colored by $\updownarrow j$, $j < i$, at distance $i - j$ from v in G_f, or there is a path P of length i in G_f connecting v with some vertex of $D_f(c)$ such that no inner vertex of P is in $\omega(f)$.

- Every vertex $v \in V(G_f) - \omega(f)$ whose closest center is at distance $i \leq r$, either is at distance i in G_f from some center in $D_f(c)$, or is at distance j, $j < i$, in G_f from a vertex $u \in \omega(f)$ colored $\updownarrow(i - j)$.

We put $A_f(c) = +\infty$ if there is no such a set $D_f(c)$, or if c is not a locally valid coloring. Because $\omega(\{r, v\}) = \emptyset$ and $G_{\{r,v\}} = G$, we have that $A_{\{r,v\}}(c)$ is the smallest size of an r-dominating set in G.

We start computations of the functions A_f from leaves of T. Let x be a leaf of T and let f be the edge of T incident with x. Then G_f is the edge of G corresponding to x. We consider all locally valid colorings of $V(G_f)$ such that if a vertex $v \in V(G_f)$ is colored by $\downarrow i$ for $i > 0$ then there is an adjacent vertex w in $V(G_f)$ colored $\updownarrow i - 1$. For each such coloring c we define $A_f(c)$ to be the number of vertices colored 0 in $V(G_f)$. Otherwise, $A_f(c)$ is $+\infty$, meaning that this coloring c is infeasible. The brute-force algorithm takes $O(rm)$ time for this step.

Let f be a non-leaf edge of T and let f_1, f_2 be the children of f. Define $X_1 = \omega(f) - \omega(f_2)$, $X_2 = \omega(f) - \omega(f_1)$, $X_3 = \omega(f) \cap (\omega(f_1) \cap \omega(f_2))$, and $X_4 = (\omega(f_1) \cup \omega(f_2)) - \omega(f)$.

Notice that

$$\omega(f) = X_1 \cup X_2 \cup X_3. \tag{6}$$

By the definition of ω, it is impossible that a vertex belongs to exactly one of $\omega(f), \omega(f_1), \omega(f_2)$. Therefore, condition $u \in X_4$ implies that $u \in \omega(f_1) \cap \omega(f_2)$ and we conclude that

$$\omega(f_1) = X_1 \cup X_3 \cup X_4, \tag{7}$$

and

$$\omega(f_2) = X_2 \cup X_3 \cup X_4. \tag{8}$$

We say that a coloring $c \in \{0, \uparrow 1, \uparrow 2, \ldots, \uparrow r, \downarrow 1, \downarrow 2, \ldots, \downarrow r\}^{|\omega(f)|}$ of $\omega(f)$ is *formed* from a coloring c_1 of $\omega(f_1)$ and a coloring c_2 of $\omega(f_2)$ if

1. For every $u \in X_1$, $c(u) = c_1(u)$;
2. For every $u \in X_2$, $c(u) = c_2(u)$;
3. For every $u \in X_3$,
   (a) If $c(u) = \uparrow i$, $1 \leq i \leq r$, then $c(u) = c_1(u) = c_2(u)$. Intuitively, because vertex u is unresolved in $\omega(f)$, this vertex is also unresolved in $\omega(f_1)$ and in $\omega(f_2)$.
   (b) If $c(u) = 0$ then $c_1(u) = c_2(u) = 0$.
   (c) If $c(u) = \downarrow i$, $1 \leq i \leq r$, then $c_1(u), c_2(u) \in \{\downarrow i, \uparrow i\}$ and $c_1(u) \neq c_2(u)$. We avoid the case when both c_1 and c_2 are colored by $\downarrow i$ because it is sufficient to have the vertex u resolved in at least one coloring. This observation helps to decrease the number of colorings forming a coloring c. (Similar arguments using a so-called "monotonicity property" are made by Alber et al. [2] for computing the minimum dominating set on graphs of bounded treewidth.)
4. For every $u \in X_4$,

(a) either $c_1(u) = c_2(u) = 0$ (in this case we say that u is *formed by* 0 *colors*),

(b) or $c_1(u), c_2(u) \in \{\downarrow i, \uparrow i\}$ and $c_1(u) \neq c_2(u)$, $1 \leq i \leq r$ (in this case we say that u is *formed by* $\{\downarrow i, \uparrow i\}$ *colors*).

This property says that every vertex u of $\omega(f_1)$ and $\omega(f_2)$ that does not appear in $\omega(f)$ (and hence does not appear further) should finally either be a center (if both colors of u in c_1 and c_2 were 0), or should be resolved by some vertex in $V(G_f)$ (if one of the colors $c_1(u), c_2(u)$ is $\downarrow i$ and one $\uparrow i$). Again, we avoid the case of $\downarrow i$ in both c_1 and c_2.

Notice that every coloring of $\omega(f)$ is formed from some colorings of $\omega(f_1)$ and $\omega(f_2)$. Moreover, if $D_f(c)$ is the restriction to G_f of some (k,r)-center and such a restriction corresponds to a coloring c of $\omega(f)$ then $D_f(c)$ is the union of the restrictions $D_{f_1}(c_1)$, $D_{f_2}(c_2)$ to G_{f_1}, G_{f_2} of two (k,r)-centers where these restrictions correspond to some colorings c_1, c_2 of $\omega(f_1)$ and $\omega(f_2)$ that form the coloring c.

We compute the values of the corresponding functions in a bottom-up fashion. The main observation here is that if f_1 and f_2 are the children of f, then the vertex sets $\omega(f_1)$ $\omega(f_2)$ "separate" subgraphs G_{f_1} and G_{f_2}, so the value $A_f(c)$ can be obtained from the information on colorings of $\omega(f_1)$ and $\omega(f_2)$.

More precisely, let c be a coloring of $\omega(f)$ formed by colorings c_1 and c_2 of f_1 and f_2. Let $\#_0(X_3, c)$ be the number of vertices in X_3 colored by color 0 in coloring c and and let $\#_0(X_4, c)$ be the number of vertices in X_4 formed by 0 colors. For a coloring c we assign

$$A_f(c) = \min\{A_{f_1}(c_1) + A_{f_2}(c_2) - \#_0(X_3, c_1) - \#_0(X_4, c_1) \mid c_1, c_2 \text{ form } c\}. \tag{9}$$

(Every 0 from X_3 and X_4 is counted in $A_{f_1}(c_1) + A_{f_2}(c_2)$ twice and $X_3 \cap X_4 = \emptyset$.) The time to compute the minimum in (9) is given by

$$O\Big(\sum_c \big|\{\{c_1, c_2\} \mid c_1, c_2 \text{ form } c\}\big|\Big).$$

Let $x_i = |X_i|$, $1 \leq i \leq 4$. For a coloring c let z_3 be the number of vertices colored by $\downarrow$ colors in X_3. Also we denote by z_4 the number of vertices in X_4 formed by $\{\downarrow i, \uparrow i\}$ colors, $1 \leq i \leq r$. Thus the number of pairs forming c is $2^{z_3+z_4}$. The number of colorings of $\omega(f)$ such that exactly z_3 vertices of X_3 are colored by $\downarrow$ colors and such that exactly z_4 vertices of X_4 are formed by $\{\downarrow, \uparrow\}$ colors is

$$(2r+1)^{x_1}(2r+1)^{x_2}(r+1)^{x_3-z_3}\binom{x_3}{z_3}r^{z_3}\binom{x_4}{z_4}r^{z_4}.$$

Thus the number of operations needed to estimate (9) for all possible colorings of $\omega(f)$ is

$$\sum_{p=0}^{x_3}\sum_{q=0}^{x_4} 2^{p+q}(2r+1)^{x_1+x_2}(r+1)^{x_3-p}\binom{x_3}{p}r^p\binom{x_4}{q}r^q = (2r+1)^{x_1+x_2+x_4}(3r+1)^{x_3}.$$

Let ℓ be the branchwidth of G. By (6), (7) and (8),

$$\begin{aligned} x_1 + x_2 + x_3 &\leq \ell \\ x_1 + x_3 + x_4 &\leq \ell \\ x_2 + x_3 + x_4 &\leq \ell. \end{aligned} \tag{10}$$

The maximum value of the linear function $\log_{3r+1}(2r+1) \cdot (x_1 + x_2 + x_4) + x_3$ subject to the constraints in (10) is $\frac{3\log_{3r+1}(2r+1)}{2}\ell$. (This is because the value of the corresponding LP achieves maximum at $x_1 = x_2 = x_4 = 0.5\ell$, $x_3 = 0$.) Thus one can evaluate (9) in time

$$(2r+1)^{x_1+x_2+x_4}(3r+1)^{x_3} \leq (3r+1)^{\frac{3\log_{3r+1}(2r+1)}{2}\ell} = (2r+1)^{\frac{3}{2}\cdot\ell}.$$

It is easy to check that the number of edges in T is $O(m)$ and the time needed to evaluate $A_{\{r,v\}}(c)$ is $O((2r+1)^{\frac{3}{2}\cdot\ell}m)$. Moreover, it is easy to modify the algorithm to obtain an optimal choice of centers by bookkeeping the colorings assigned to each set $\omega(f)$.

Summarizing, we obtain the following theorem:

Theorem 4.1. *For a graph G on m edges and with a given branch decomposition of width $\leq \ell$, and integers k, r, the existence of a (k,r)-center in G can be checked in $O((2r+1)^{\frac{3}{2}\cdot\ell}m)$ time and, in case of a positive answer, constructs a (k,r)-center of G in the same time.*

Similar result can be obtained for map graphs.

Theorem 4.2. *Let H be a witness of a map graph $G_{\mathcal{M}}$ on n vertices and let k, r be integers. If a branch-decomposition of width $\leq \ell$ of H is given, the existence of a (k,r)-center in $G_{\mathcal{M}}$ can be checked in $O((2r+1)^{\frac{3}{2}\cdot\ell}n)$ time and, in case of a positive answer, constructs a (k,r)-center of G in the same time.*

Proof. We give a sketch of the proof here. H is bipartite graph with a bipartition $(V(G_{\mathcal{M}}), V(H) - V(G_{\mathcal{M}}))$. There is a (k,r)-center in $G_{\mathcal{M}}$ if and only if H has a set $S \subseteq V(G_{\mathcal{M}})$ of size k such that every vertex $V(G_{\mathcal{M}}) - S$ is at distance $\leq 2r$ in H from some vertex of S. We check whether such a set S exists in H by applying arguments similar the proof of Theorem 4.1. The main differences in the proof are the following. Now we color vertices of the graph H by $\updownarrow i$, $0 \leq i \leq 2r$, where i is even. Thus we are using $2r+1$ numbers. Because we are not interested whether the vertices of $V(H) - V(G_{\mathcal{M}})$ are dominated or not, for vertices of $V(H) - V(G_{\mathcal{M}})$ we keep the same number as for a vertex of $V(G_{\mathcal{M}})$ resolving this vertex. For a vertex in $V(G_{\mathcal{M}})$ we assign a number $\downarrow i$ if there is a resolving vertex from $V(H) - V(G_{\mathcal{M}})$ colored $\updownarrow(i-2)$. Also we change the definition of locally valid colorings: for any two adjacent vertices v and w in $\omega(f)$, if v is colored $\updownarrow i$ and w is colored $\updownarrow j$, then $|i-j| \leq 2$.

Finally, H is planar, so $|E(H)| = O(|V(H)|) = O(n)$. □

5 Algorithms for the (k,r)-center problem

For a planar graph G and integers k, r, we solve (k,r)-center problem on planar graphs in three steps.

Step 1: We check whether the branchwidth of G is at most $4(2r+1)\sqrt{k}+8r+1$. This step requires $O((|V(G)|+|E(G)|)^2)$ time according to the algorithm due to Seymour & Thomas (algorithm (7.3) of Section 7 of [41] — for an implementation, see the results of Hicks [27]). If the answer is negative then report that G has no any (k,r)-center and stop. Otherwise go to the next step.

Step 2: Compute an optimal branch-decomposition of a graph G. This can be done by the algorithm (9.1) in the Section 9 of [41] which requires $O((|V(G)|+|E(G)|)^4)$ steps.

Step 3: Compute, if it exists, a (k,r)-center of G using the dynamic-programming algorithm of Section 4.

It is crucial that, for practical applications, there are no *large hidden constants* in the running time of the algorithms in Steps 1 and 2 above. Because for planar graphs $|E(G)| = O(|V(G)|)$, we conclude with the following theorem:

Theorem 5.1. *There exists an algorithm finding, if it exists, a (k,r)-center of a planar graph in $O((2r+1)^{6(2r+1)\sqrt{k}+12r+3/2}n + n^4)$ time.*

Similar arguments can be applied to solve the (k,r)-center problem on map graphs. Let $G_{\mathcal{M}}$ be a map graph. To check whether $G_{\mathcal{M}}$ has a (k,r)-center, we compute optimal branchwidth of its witness H. By Theorem 3.3, if $\mathbf{bw}(H) > 4(4r+3)\sqrt{k}+16r+9$, then $G_{\mathcal{M}}$ has no (k,r)-center. If $\mathbf{bw}(H) \le 4(4r+3)\sqrt{k}+16r+9$, then by Theorem 4.2 we obtain the following result:

Theorem 5.2. *There exists an algorithm finding, if it exists, a (k,r)-center of a map graph in $O((2r+1)^{6(4r+1)\sqrt{k}+24r+13.5}n+n^4)$ time.*

By a straightforward modification to the dynamic program, we obtain the same results for the *vertex-weighted (k,r)-center* problem, in which the vertices have real weights and the goal is to find a (k,r)-center of minimum total weight.

6 Concluding remarks

In this paper, we presented fixed-parameter algorithms with exponential speed-up for the (k,r)-center problem on planar graphs and map graphs. Our methods for (k,r)-center can also be applied to algorithms on more general graph classes like constant powers of planar graphs, which are not minor-closed family of graphs. Extending these results to other non-minor-closed families of graphs would be instructive. Faster algorithms for (k,r)-center for planar graphs and map graphs can be obtained by adopting the proof techniques for planar dominating set from [19]. The disadvantage of this approach is that proofs (but not the algorithm itself) become much more difficult.

In addition, there are several interesting variations on the (k,r)-center problem. In *multiplicity-m (k,r)-center*, the k centers must satisfy the additional constraint that every vertex is within distance r of at least m centers. In *f-fault-tolerant (k,r)-center* [8], every non-center vertex must have f vertex-disjoint paths of length at most r to centers. (For this problem with $r=\infty$, [8] gives a polynomial-time $O(f\log|V|)$-approximation algorithm for k.) In *L-capacitated (k,r)-center* [8], each of the k centers can satisfy only L "customers", essentially forcing the assignment of vertices to centers to be load-balanced. (For this problem, [8] gives a polynomial-time $O(\log|V|)$-approximation algorithm for r.) In *connected (k,r)-center* [42], the k chosen centers must form a connected subgraph. In all these problems, the main challenge is to design the dynamic program on graphs of bounded treewidth/branchwidth. We believe that our approach can be used as the main guideline in this direction.

More generally, it seems that our approach should extend other graph algorithms (not just dominating-set-type problems) to apply to the rth power and/or half-square of a graph (and hence in particular map graphs). It would be interesting to explore to which other problems our approach can be applied. Also, obtaining "fast" algorithms for problems like feedback vertex set or vertex cover on constant powers of graphs of bounded branchwidth (treewidth), as we did for dominating set, would be interesting.

Map graphs can be seen as contact graphs of disc homeomorphs. A question is whether our results can be extended for another geometric classes of graphs. An interesting candidate is the class of unit-disk graphs. The current best algorithms for finding a vertex cover or a dominating set of size k on these graphs have $n^{O(\sqrt{k})}$ running time [4].

To demonstrate the versatility of our approach, notice that a direct consequence of our approach is the following theorem.

Theorem 6.1. *Let $\mathbf{p}$ be a function mapping graphs to non-negative integers such that the following conditions are satisfied:*

(1) *There exists an algorithm checking whether $\mathbf{p}(G) \le w$ in $f(\mathbf{bw}(G))n^{O(1)}$ steps.*

(2) *For any $k \ge 0$, the class of graphs where $\mathbf{p}(G) \le k$ is closed under taking of contractions.*

(3) *If R is any partially triangulated $(j \times j)$-grid*[1] *then* $\mathbf{p}(R) = \Omega(j^2)$.

Then there exists an algorithm checking whether $p(G) \leq k$ on planar graphs in $O(f(\sqrt{k}))n^{O(1)}$ steps.

For a wide source of parameters satisfying condition (1) we refer to the theory of Courcelle [14] (see also [6]). For parameters where $f(\mathbf{bw}(G)) = 2^{O(\mathbf{bw}(G))}$, this result is a strong generalization of Alber et al.'s approach which requires that the problem of checking whether $\mathbf{p}(G) \leq k$ should satisfy the "layerwise separation property" [3]. Moreover, the algorithms involved are expected to have better constants in their exponential part comparatively to the ones appearing in [3]. Similar results can also be obtained for constant powers of planar graphs and for map graphs.

Finally, let us note that combining Theorems 4.1 and 4.2 with ideas of [7] (see also [17] and F [20]) adapted to branch decompositions instead of tree decompositions, we are able to obtain a PTAS for r-dominating set on planar and map graphs. We summarize these results in the following theorems:

Theorem 6.2. *For any integer $p \geq 1$, the r-dominating set problem on planar graphs has a $(1 + 2r/p)$-approximation algorithm with running time $O(p(2r + 1)^{3(p+2r)}m))$.*

Theorem 6.3. *For any integer $p \geq 1$, the r-dominating set problem on map graphs has a $(1+4r/p)$-approximation algorithm with running time $O(p(4r + 3)^{3(p+4r)}m))$.*

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