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Parallel knock-out schemes in networks *

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Abstract

We consider parallel knock-out schemes, a procedure on graphs introduced by Lampert and Slater in 1997 in which each vertex eliminates exactly one of its neighbors in each round. We are considering cases in which after a finite number of rounds, where the minimimum number is called the parallel knock-out number, no vertices of the graph are left. We derive a number of combinatorial and algorithmical results on parallel knock-out numbers. We observe that for families of sparse graphs (like planar graphs, or graphs with bounded tree-width), the parallel knock-out number grows at most logarithmically with the number n of vertices, which is basically tight for trees. Furthermore, we construct a family of bipartite graphs for which the parallel knock-out number grows proportionally to the square root of n. We characterize trees with parallel knock-out number at most 2, and show that the parallel knock-out number for trees can be computed in polynomial time via a dynamic programming approach, whereas the general problem is known to be NP-hard. Finally we show that claw-free graphs with minimum degree at least 2 have parallel knock-out number at most 2, and that the lower bound on the minimum degree is best possible.

1 Introduction

Lampert & Slater [2] introduced the following *parallel knock-out* procedure for graphs: On every vertex v of an undirected graph, there is a person standing. Every person selects one other person that stands on an adjacent vertex. Then all the selected persons are knocked out simultaneously, and the whole procedure is repeated with the surviving vertices. The game terminates, as soon as there are survivors that do not have any neighbor left to knock out. For instance, on the path $v_1 - v_2 - v_3 - v_4 - v_5$ it may happen that the persons on v_1 and v_3 both decide to knock out the person on v_2, that v_2 and v_4 both decide to knock out v_3, and that v_5 knocks out v_4. Then v_1 and v_5 are the survivors of this round, and the game terminates.

Formally, let $G = (V, E)$ be an undirected, simple, loopless graph. We denote by $N(v)$ the set of all neighbors of vertex v (not including the vertex v itself). A *KO-selection* is a function $f : V \to V$ with $f(v) \in N(v)$ for all $v \in V$. If $f(v) = u$, we will sometimes say that vertex v *knocks out* or *eliminates* vertex u, or that (in the language of chip firing games) vertex v *fires at* vertex u. For a KO-selection f, we define the corresponding *KO-successor* G^f of G as the subgraph of G that

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is induced by the vertices in $V - f(V)$; this situation will be denoted by $G \rightsquigarrow G^f$. Note that every graph G without isolated vertices has at least one KO-successor.

In this paper, we are mainly interested in the question whether for a given graph G, there exists a sequence of KO-selections and KO-successors such that

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \cdots \rightsquigarrow G_r = (\emptyset, \emptyset).$$

If no such sequence exists, then the *parallel knock-out number* of the graph G is infinite, and we write $\textsc{pko}(G) = \infty$. Otherwise, we define the parallel knock-out number $\textsc{pko}(G)$ of G as the smallest number r for which such a sequence exists. A graph G is called *KO-reducible* if and only if $\textsc{pko}(G)$ is finite.

A sequence of KO-selections that transform a KO-reducible graph into the empty graph is called a *KO-reduction scheme*. A single step in this sequence is called a *round* of the KO-reduction scheme.

It has been proved by Lampert & Slater [3] that it is an NP-complete problem to decide whether a given graph is KO-reducible. It is not difficult to check that a graph G has $\textsc{pko}(G) = 1$ if and only if G contains a spanning subgraph consisting of a number of mutually disjoint cycles and matching edges. This implies that the problem of deciding whether $\textsc{pko}(G) = 1$ is polynomially solvable; the equivalent formulation yields a folklore problem appearing in many standard books in combinatorial optimisation. The equivalence also shows that the concept of the parallel knock-out number is related to well-known concepts like perfect matching, 2-factor, and hamiltonian cycle.

2 Results of this paper

We derive a number of combinatorial and algorithmical results around parallel knock-out numbers. In Section 3 we observe that for families of sparse graphs (like planar graphs, or graphs with bounded tree-width), the parallel knock-out number grows at most logarithmically with the number n of vertices. Furthermore, we construct a family of bipartite graphs for which the parallel knock-out number grows proportionally to the square root of n. Our upper bound result on sparse graphs is basically tight for trees (up to a constant factor): Section 4 presents a corresponding lower bound construction. In Section 5 we characterize trees with parallel knock-out number at most 2. This involves a condition analogous to Hall's condition for the existence of matchings in bipartite graphs. Section 6 investigates the algorithmical behavior of the parallel knock-out number for trees: It can be computed in polynomial time via a dynamic programming approach. This seems to be one of the rare cases where a dynamic program for trees does not immediately carry over to the bounded tree-width classes: At the heart of our dynamic program for trees, there sits a certain bipartite matching problem; for higher tree-widths, this bipartite matching problem translates into something ugly. Section 7 turns to claw-free graphs: under mild conditions, i.e. $\delta(G) \geq 2$, a claw-free graph G has parallel knock-out number at most 2, and the lower bound on the minimum degree is best possible. We finish the paper with some open problems.

3 Upper and lower bounds

Lemma 3.1 *Let $\tau > 0$ be a fixed real number. Let $\mathcal{G}$ be a class of graphs that satisfies the following two properties.*

- *$\mathcal{G}$ is closed under taking vertex-induced subgraphs.*
- *Every graph $G = (V, E)$ in $\mathcal{G}$ satisfies $|E| \leq \tau\,|V|$.*

Then any KO-reducible graph $G = (V, E)$ in $\mathcal{G}$ satisfies $\text{PKO}(G) \leq \lceil 4\tau \rceil \cdot \log_2 |V|$.

Proof. Consider a KO-reduction scheme for the n-vertex graph G. We claim that after $\lceil 4\tau \rceil$ rounds, the number of surviving vertices goes down by at least a factor of two. Suppose otherwise. Then for $\lceil 4\tau \rceil$ rounds, the number of vertices is at least $n/2$. In every round, each of these $n/2$ vertices fires along some edge, and every edge is used by at most two vertices. Hence, in every round at least $n/4$ edges are removed from the graph, and so after $\lceil 4\tau \rceil$ rounds the graph would be without edges. This proves our claim. The statement in the lemma now follows by induction. ∎

Lemma 3.1 can be used to get logarithmic upper bounds e.g. for planar graphs and for graphs of bounded tree-width.

Lemma 3.2 *Let $1 \leq a \leq b$. Then the complete bipartite graph $K_{a,b}$ is KO-reducible if and only if $b \leq \frac{1}{2}a(a+1)$. If this inequality is satisfied then*

$$\text{PKO}(K_{a,b}) \;=\; \left\lceil a + \frac{1}{2} - \sqrt{(a+\frac{1}{2})^2 - 2b}\; \right\rceil \tag{1}$$

This implies $\text{PKO}(K_{a,b}) < \sqrt{2(a+b)}$.

Proof. In $K_{a,b}$ the a vertices on one side of the bipartition will be called the *left* vertices, and the remaining vertices will be called the *right* vertices. The situation after r rounds of some KO-reduction scheme is fully specified by the number a_r of surviving left vertices and by the number b_r of surviving right vertices. The initial situation is described by $a_0 = a$ and $b_0 = b$. We will denote the expression in the right hand side of (1) by $F(a, b)$. We will first show that $\text{PKO}(K_{a,b}) \leq F(a, b)$ and then that $\text{PKO}(K_{a,b}) \geq F(a, b)$.

It can be verified that $F(a, b) = 1$ implies $a = b$. Hence, these cases indeed satisfy $\text{PKO}(K_{a,b}) \leq F(a, b)$, and from now on we will assume $F(a, b) \geq 2$. Let $k = F(a, b) - 2$, and consider the following KO-reduction scheme: In the first k rounds, all right vertices fire at the same left vertex, and all left vertices fire at pairwise distinct right vertices. For $r = 1, \ldots, k$ this yields $a_r = a_{r-1} - 1$ and $b_r = b_{r-1} - a_{r-1}$, which is equivalent to $a_r = a - r$ and $b_r = b - r\,a + \frac{1}{2}r(r-1)$. It can be shown that $a_k < b_k \leq 2a_k - 1$ holds. In the $(k+1)$th round, all right vertices fire at the same left vertex, whereas the left vertices fire at $b_k - a_k + 1$ distinct right vertices. This yields $a_{k+1} = b_{k+1} = a_k - 1$. In the $(k+2)$th round, the left and right vertices fire at each other in pairs. This shows $\text{PKO}(K_{a,b}) \leq F(a, b)$.

Next, consider $K_{a,b}$ with $1 \leq a < b \leq \frac{1}{2}a(a+1)$. The first round of some KO-reduction scheme, we have $a_1 = a - x$ with $1 \leq x \leq a - 1$ and $b_1 = b - y$ with $1 \leq y \leq a$. It can be shown that for all these cases either $F(a, b) \leq F(a_1, b_1) + 1$ holds, or that otherwise the resulting graph is not KO-reducible. ∎

Lemma 3.3 *If an n-vertex tree T is KO-reducible, then* $\text{PKO}(T) \leq \lceil \log_3 n \rceil$.

Proof. The proof is done by induction on n. The statement trivially holds for $n = 2$ and $n = 3$. For the inductive argument, consider an arbitrary tree on n vertices, and let $T_1, \ldots, T_s$ be the connected components of some KO-successor of T. Clearly,

$$\text{PKO}(T) \;\leq\; 1 + \max\{\text{PKO}(T_1), \ldots, \text{PKO}(T_s)\}. \tag{2}$$

Every vertex v in some connected component T_k $(1 \leq k \leq s)$ must have eliminated some vertex $v' \notin V(T_k)$, and every such eliminated vertex v' itself must have fired at another vertex $v'' \notin V(T_k)$. It is easy to verify that distinct vertices u and v in T_k yield pairwise distinct vertices u', v', u'', and v''. Therefore T_k contains at most $n/3$ vertices. By plugging this into (2), we complete the proof. ∎

4 Trees with high parallel knock-out numbers

In this section we will construct trees with high parallel knock-out numbers. The construction is done inductively via the following two sequences $\langle Y_1, Y_2, \ldots \rangle$ and $\langle Z_1, Z_2, \ldots \rangle$ of rooted trees:

- The tree Y_1 consists of a root with one child.
- The tree Z_1 consists of a root with one child and one grandchild.
- For $\ell \geq 2$, the tree Y_ℓ consists of a root with ℓ children. The first $\ell - 2$ of these children are the roots of copies of the trees $Z_1, \ldots, Z_{\ell-2}$. The last two children are roots of copies of $Z_{\ell-1}$.
- For $\ell \geq 2$, the tree Z_ℓ consists of a root with ℓ children. These children are the roots of copies of the trees $Y_1, \ldots, Y_\ell$.

Lemma 4.1 *For every $\ell \geq 1$, there exists a sequence of ℓ KO-selections that transforms Y_ℓ into the empty graph. For every $\ell \geq 1$, there exists a sequence of ℓ KO-selections that transforms Z_ℓ into a one-vertex tree that consists of the root of Z_ℓ.*

Proof. We prove both statements simultaneously by induction on ℓ. For $\ell = 1$, we let the two vertices in Y_1 eliminate each other simultaneously, and we let the root of Z_1 fire at its child, and let the child and the grandchild fire at each other.

We use the following scheme for Y_ℓ with $\ell \geq 2$. The ℓ subtrees $Z_1, \ldots, Z_{\ell-2}, Z_{\ell-1}, Z_{\ell-1}$ that are attached at the children of the root use the KO-schemes that exist by the inductive claim. In round k with $1 \leq k \leq \ell - 1$, the root of Y_ℓ fires at the root of subtree Z_k (and makes this subtree disappear from the game). In the final round ℓ, the root of Y_ℓ and the root of the last subtree $Z_{\ell-1}$ eliminate each other simultaneously.

We use the following scheme for Z_ℓ with $\ell \geq 2$. The ℓ subtrees $Y_1, \ldots, Y_\ell$ that are attached at the children of the root use the KO-schemes that exist by the inductive claim. In round k with $1 \leq k \leq \ell$, the root of Z_ℓ fires at the root of subtree Y_k and makes this subtree disappear from the game. After round ℓ, the root of Z_ℓ is the only survivor. ■

Lemma 4.2 *Let T be a rooted tree that contains Y_ℓ (respectively Z_ℓ) as a rooted maximal subtree. Let $r(y)$ (respectively $r(z)$) denote the root of Y_ℓ (respectively, of Z_ℓ). Assume furthermore that T is KO-reducible. Then in every KO-reduction scheme for T the following statements hold:*

(a) In the first ℓ rounds, the root $r(y)$ of Y_ℓ fires at its ℓ children.

(b) In the ℓth round, the root $r(y)$ of Y_ℓ is eliminated by one of its ℓ children.

(c) The root $r(z)$ of Z_ℓ can not be eliminated by any of its children.

(d) If the root $r(z)$ of Z_ℓ is still alive in round $\ell + 1$, then it will fire at its father.

Proof. We prove all four statements (a)–(d) simultaneously by induction on ℓ. In Y_1, the leaf fires at the root in the first round. The leaf can only be eliminated, if the root fires at it in the first round. This proves (a) and (b) for $\ell = 1$. In Z_1, the child and the grandchild must fire at each other in the first round. If the root survives the first round, it can only fire at its father in the second round. This proves (c) and (d) for $\ell = 1$.

Proofs of statements (a) and (b) for $\ell \geq 2$: The children of $r(y)$ form the roots of certain subtrees Z_k. By the inductive assumption of statement (c), these children can not be eliminated by their own children. Hence, they all must be eliminated by $r(y)$, and $r(y)$ must stay alive for the first ℓ rounds. By the inductive assumption of statement (d), the roots of Z_k ($1 \leq k \leq \ell - 2$) must be eliminated in the first $\ell - 2$ rounds. In rounds $\ell - 1$ and ℓ, the roots of the two subtrees $Z_{\ell-1}$ must be eliminated; this proves (a). Moreover, in round ℓ the last surviving child of $r(y)$ eliminates $r(y)$; this proves (b).

Proofs of statements (c) and (d) for $\ell \geq 2$: The children of $r(z)$ form the roots of certain subtrees Y_k. By the inductive assumption of statements (a) and (b), these children only eliminate their own children, and eventually are eliminated by their own children. This proves (c). Since the root $r(z)$ is not eliminated by any of its children, it must be eliminated by its father. If $r(z)$ is still alive in round $\ell + 1$, it will fire at its last surviving neighbor, and this neighbor must be the father. This proves (d). ■

According to Lemma 4.1 $\text{PKO}(Y_\ell) \leq \ell$, and according to Lemma 4.2.(b) $\text{PKO}(Y_\ell) \geq \ell$. This yields $\text{PKO}(Y_\ell) = \ell$. It can be shown by induction that Y_ℓ has at most $(2 + \sqrt{2})^\ell / \sqrt{2}$ vertices, and that Z_ℓ has at most $(2 + \sqrt{2})^\ell$ vertices.

Theorem 4.3 *For arbitrarily large n, there exist KO-reducible n-vertex trees T with* $\text{PKO}(T) = \Omega(\log n)$. ■

5 Trees with low parallel knock-out numbers

In this section we will characterize trees with parallel knock-out number at most 2. This involves a condition analogous to Hall's condition for the existence of matchings in bipartite graphs.

We will start with an easy but useful observation.

Lemma 5.1 *Let T be a tree that has no matching saturating all leaves of T. Then* $\text{PKO}(T) = \infty$.

Proof. If T is KO-reducible, then in the first round of any KO-reduction scheme for T every leaf v fires at its unique neighbor u. If u does not fire at v in the first round, then after the first round v is an isolated vertex, which is not possible. Hence, if T is KO-reducible there is a matching containing one edge incident with each leaf of T. ■

Now suppose T is a KO-reducible tree, and choose a matching M of maximum cardinality subject to the condition that it saturates all leaves. It is obvious that $\text{PKO}(T) = 1$ if and only if M is a perfect matching. Assuming that M is not a perfect matching we consider the set U of unsaturated vertices. Clearly U in an independent set and by standard arguments from matching theory there are no M-alternating paths between pairs of vertices of U. For a vertex $u \in U$, a *u-triplet* is a P_3 in $T - U$ with the property that one of the end vertices of the P_3 is adjacent to u and the other end vertex has degree at least 2 in T. Thus this P_3 is an M-alternating path with two saturated end vertices in T. Let $T(u)$ denote the set of u-triplets of $u \in U$ in $T - U$. Note that a P_3 can be a u-triplet and a v-triplet for two distinct vertices u and v of U, but that in such cases u and v are adjacent to the same end vertex of the P_3; otherwise there is an M-alternating path between u and v, a contradiction. For a subset $S \subseteq U$ the set $T(S)$ of *S-triplets* is the union of all u-triplets for $u \in S$ in $T - U$. A tree T is called *Hall-perfect* if it has a matching saturating all leaves and for some maximum matching M with this property we have that either M is a perfect matching or

$$|T(S)| \geq |S| \quad \text{for all} \quad S \subseteq U, \tag{3}$$

where U is the set of unsaturated vertices of T.

Theorem 5.2 *A tree T is Hall-perfect if and only if* $\text{PKO}(T) \leq 2$.

Proof. If T is a Hall-perfect tree, we can give a KO-reduction scheme with one or two rounds. One round suffices if and only if M is a perfect matching; this is clear. Suppose M is a matching satisfying (3). By Hall's Theorem on matchings in bipartite graphs, this implies we can assign $p = |U|$ distinct U-triplets to the vertices of U, one v_i-triplet for each $v_i \in U = \{v_1, \ldots, v_p\}$. We can use the following KO-reduction scheme for each v_i and its associated v_i-triplet given by the vertices u_i, w_i, x_i, y_i of the P_3, where u_i is a neighbor of v_i. In the first round, for each i, v_i fires at one of the other neighbors (v_i is not an end vertex), u_i fires at w_i, and w_i fires at x_i, and the other matching edges of M are used to eliminate all (other) saturated vertices (except for all u_i). In the second round the edges $v_i u_i$ are used to eliminate all remaining vertices.

For the converse, suppose T can be eliminated in at most two rounds. If T needs only one round, we are done since this implies T has a perfect matching. Now suppose T needs exactly two rounds. Then in the second round the edges of a matching N between the remaining vertices after the first round are used to mutually eliminate their end vertices. Let us consider the edges $u_i v_i$ of this matching N. Each u_i has fired at a vertex x_i of T in the first round; at x_i starts a path $Q_i = x_{i_1} \ldots x_{i_{t_i}}$ of length at least one with the property that x_{i_s} fires at $x_{i_{s+1}}$ in the first round ($s = 1, \ldots, t_i - 1$), and $x_{i_{t_i}}$ fires at $x_{i_{t_i - 1}}$. Similar paths R_i exist for the vertices v_i. Clearly, Q_i and R_i do not intersect, but for each edge $u_i v_i$ of N one of the paths Q_i and R_i will have its tail in common with at least one of the paths associated with the other edges of N if $|N| \geq 2$ (since T is connected). Suppose that the two rounds in the KO-reduction scheme are chosen in such a way that $|N|$ is as small as possible. If there is only one edge in N, then $T = P_n$ with $n \geq 7$ and n odd, and it is easy to find a matching M showing that T is Hall-perfect. Suppose that $|N| \geq 2$. Consider an edge $u_i v_i \in N$ with the property that one of the associated paths Q_i and R_i, say Q_i, does not intersect with another path. Such an edge exists since T is a tree. Then R_i intersects with another path in a vertex w. We may assume that $u_i v_i$ and Q_i are chosen such that $v_i w \in E(T)$. From the choice of N we also get that Q_i has even length ≥ 2. Then we can define a matching M_i saturating all vertices (including the leaf) of $V(Q_i) \cup \{u_i\}$, and a v_i-triplet on the path induced by $V(Q_i) \cup \{u_i\}$ for the unsaturated vertex v_i. Removing the vertices of $V(Q_i) \cup \{u_i, v_i\}$, and repeating this procedure as long as there are intersecting paths, and completing the procedure by a suitable choice for the remaining path, we obtain a matching M which shows that T is Hall-perfect. ■

6 A dynamic program for trees

In this section, we describe a polynomial time algorithm for computing the parallel knock-out number of a tree T. By Lemma 3.3 $\text{PKO}(T)$ is either infinite, or it is bounded from above by $\lceil \log_3 n \rceil$, where n denotes the number of vertices in T. Without loss of generality we assume that $n \geq 3$.

We root the tree T in an arbitrary vertex called ROOT. We denote by $T(v)$ the maximal subtree of T that is rooted at vertex v. If $v \neq$ ROOT, there is some edge e_v that connects v to its father f_v. We are interested in the behavior of KO-reduction schemes inside of the subtrees $T(v)$: For $v \neq$ ROOT, the only interaction between $T(v)$ and $T - T(v)$ occurs along the edge e_v, and there is at most one round during which this edge e_v can be used. If edge e_v is used, then it is either fired upwards (the child v fires at the father f_v), or downwards (the father fires at the child), or both ways (father and child simultaneously fire at each other).

For every edge $v \neq$ ROOT and for every $r = 1, \ldots, \lceil \log_3 n \rceil$, we define three Boolean predicates UP$[v;r]$, DOWN$[v;r]$, and BOTH$[v;r]$: The predicate UP$[v;r]$ (respectively DOWN$[v;r]$, respectively BOTH$[v;r]$) is true, if there exists a KO-reduction scheme for $T(v)$, in which in round r the edge e_v is fired upwards (respectively downwards, respectively both ways). Moreover, for every vertex $v \neq$ ROOT (including the root), we introduce a Boolean predicate NONE$[v]$ which is true, if there exists a KO-reduction scheme for $T(v)$ which does not interact with vertices outside of $T(v)$; for $v \neq$ ROOT this means that the edge e_v is not used at all.

We compute the values of all these predicates by working upwards through the tree, starting in the leaves and ending in the root. For every leaf v_L, we have BOTH$[v_L;1]$ = DOWN$[v_L;1]$ = true and UP$[v_L;1]$ = NONE$[v_L]$ = false. Moreover, for all $r \geq 2$ the three predicates UP$[v_L;r]$, DOWN$[v_L;r]$, and BOTH$[v_L;r]$ are false. For non-leaf vertices v, the computation of the predicates is described in the following four lemmas.

Lemma 6.1 *For every non-leaf $v \in V(T)$ and for every $r = 1, \ldots, \lceil \log_3 n \rceil$, the value of* DOWN$[v;r]$ *can be determined in polynomial time.*

Proof. Let v be a non-leaf vertex with children $v_1, \ldots, v_d$ and father f_v. What does it mean that DOWN$[v;r]$ is true? Since the father f_v fires in round r along the edge e_v, vertex v is eliminated in round r. In the first r rounds, vertex v must have fired at r of its children. In the first $r-1$ rounds, none of the children has fired at vertex v. In round r, some of the surviving children of v may fire at v. In later rounds, none of the children can fire at v.

We model this situation via a bipartite auxiliary graph: The left vertex class in this bipartite graph has d vertices that correspond to the children $v_1, \ldots, v_d$. The right vertex class of the bipartite graph has d vertices that correspond to the possible firings along the d edges between vertex v and its children.

- For $k = 1, \ldots, r-1$ there is one edge that is used downwards during round k. We label a corresponding vertex in the bipartite graph by (DOWN, k).
- There is one edge along which v fires in round r. We label a corresponding vertex in the bipartite graph by the two labels (DOWN, r) and (BOTH, r).
- The remaining $d-r$ edges may be fired upwards in round r, or they are not being used at all. We label $d-r$ corresponding vertices in the bipartite graph by the two labels (UP, r) and (NONE).

The edges in the bipartite graph are defined as follows:

- If a vertex x in the right class has one label (DOWN, k) (respectively (UP, k), respectively (BOTH, k)), and if DOWN$[v_i;k]$ = true (respectively if UP$[v_i;k]$ = true, respectively if BOTH$[v_i;k]$ = true), then the bipartite graph has an edge between x and the vertex corresponding to v_i in the left class.
- Analogously, if a vertex x in the right class has a label (NONE) and if NONE$[v_i;k]$ = true, then the bipartite graph has an edge between x and the vertex corresponding to v_i in the left class.

There are no other edges in the auxiliary graph. It can be seen that DOWN$[v;r]$ is true if and only if the auxiliary graph contains a perfect matching. The existence of a perfect matching can be decided in polynomial time by standard methods. ■

Lemma 6.2 *For every non-leaf $v \in V(T)$ and for every $r = 1, \ldots, \lceil \log_3 n \rceil$, the value of* UP$[v;r]$ *can be determined in polynomial time.*

Proof. Let v be a non-leaf vertex with children $v_1, \dots, v_d$ and father f_v. If $\text{UP}[v;r]$ is true, then vertex v fires in round r upwards along the edge e_v. Therefore, vertex v must stay alive until it is eliminated in some round $s \geq r$. In the rounds $1, 2, \dots, r-1$ and $r+1, \dots, s$, vertex v must have fired at its children. In the first $s-1$ rounds, none of the children has fired at vertex v. In round s, some of the surviving children of v may fire at v.

Hence, if we are given the value of s, then we can model this situation as a bipartite matching problem pretty much the same way as we did in the proof of Lemma 6.1. To find the value of $\text{UP}[v;r]$, we simply test all possible values for $s = r, r+1, \dots, \lceil \log_3 n \rceil$. $\text{UP}[v;r]$ is true, if and only if at least one of these bipartite auxiliary graphs has a perfect matching. ■

Lemma 6.3 *For every non-leaf $v \in V(T)$ and for every $r = 1, \dots, \lceil \log_3 n \rceil$, the value of $\text{BOTH}[v;r]$ can be determined in polynomial time.*

Proof. Once again, let v be a non-leaf vertex with children $v_1, \dots, v_d$ and father f_v. If $\text{UP}[v;r]$ is true, then vertex v and its father f_v eliminate each other in round r. In the rounds $1, 2, \dots, r-1$ vertex v must have fired at its children, whereas none of the children has fired back at v. In round r, vertex v does not fire at its children, whereas some of the surviving children of v may fire at v. This problem can be modelled and solved as a bipartite matching problem too. ■

Lemma 6.4 *For every non-leaf $v \in V(T)$, the value of $\text{NONE}[v]$ can be determined in polynomial time.*

Proof. If $\text{NONE}[v]$ is true, then vertex v is killed by one of its children in some round s with $1 \leq s \leq \lceil \log_3 n \rceil$. In the first $s-1$ rounds, vertex v must have fired at its children, whereas none of the children has fired back at v. In round s, vertex v fires at a child, and some of the surviving children of v may fire at v. We test all possible values for s, and solve the corresponding bipartite matching problems. ■

If $\text{NONE}[\text{ROOT}]$ is true in the end, then T is KO-reducible. To find the exact value of $\text{PKO}(T)$, we remember the smallest number s in the proof of Lemma 6.4 for which a perfect matching exists. A perfect matching in a bipartite graph with α vertices and β edges can be found in $O(\beta\sqrt{\alpha})$ time. Our algorithm faces matching problems with $O(n)$ vertices and $O(n^2)$ edges, and altogether there are $O(n \log^2 n)$ matching problems to be solved. This yields the following theorem.

Theorem 6.5 *The parallel knock-out number of an n-vertex tree T can be computed in $O(n^{3.5} \log^2 n)$ time.* ■

7 Claw-free graphs

We now turn to *claw-free* graphs, i.e. graphs that contain no $K_{1,3}$ as an induced subgraph. This is a well-studied class of graphs, especially with respect to algorithmical and structural properties. We refer to [1] for an excellent survey paper on claw-free graphs.

Since claw-free graphs admit perfect matchings and 2-factors under rather mild conditions, it is natural to consider conditions that guarantee a low parallel knock-out number in a claw-free graph. Here we prove a result involving the minimum degree $\delta(G)$ of the vertices of G.

Theorem 7.1 *Let G be a claw-free graph with $\delta(G) \geq 2$. Then $\text{PKO}(G) \leq 2$.*

In fact, we prove the following slightly stronger result from which the previous result is an easy consequence. For convenience we define the notion of a *2-KO-factor* of a graph G as a spanning subgraph of G consisting of a number of mutually disjoint copies of cycles, paths on 2 vertices, paths on 4 vertices, and paths on at least 6 vertices. Note that all the components of a 2-KO-factor have parallel knock-out number at most 2.

Theorem 7.2 *Every claw-free graph G with $\delta(G) \geq 2$ has a 2-KO-factor.*

Proof. The proof is by contradiction. Let G be a claw-free graph with $\delta(G) \geq 2$ and suppose G has no 2-KO-factor. Let $S \subseteq V(G)$ denote a smallest set such that the graph $G - S$ contains a 2-KO-factor F, and assume that subject to this S and F are chosen in such a way that the number of vertices of F contained in cycles of F is maximum, and subject to this the remaining vertices are on as few paths of F as possible. Clearly S is an independent set. Let $v \in S$. We first prove four claims in order to obtain restrictions on the neighborhood of v in F.

Claim 1. v is not adjacent to a vertex on a cycle of F.

Proof of Claim 1. If v is adjacent to a vertex u on a cycle C of F, then by the choice of S and F v is not adjacent to the two neighbors x and y of u on C. Since G is claw-free this implies that $xy \in E(G)$. We obtain a contradiction to the choice of S and F by replacing xuy in F by xy and adding the path corresponding to the edge vu. This completes the proof of Claim 1.

Claim 2. v is not adjacent to a vertex on a path component with more than 6 vertices of F.

Proof of Claim 2. If v is adjacent to a vertex u on a path component P_k of F with $k \geq 7$, then by the choice of S and F u is not an end vertex of P, and v is not adjacent to the two neighbors x and y of u on P. Since G is claw-free this implies that $xy \in E(G)$. We obtain a contradiction to the choice of S and F by replacing xuy in F by xy and adding the path corresponding to the edge vu. This completes the proof of Claim 2.

Claim 3. v is not adjacent to two vertices on one path component $P \neq P_4$ of F.

Proof of Claim 3. If v is adjacent to two vertices x and y on one path component $P \neq P_4$ of F, then by the choice of S and F, $xy \notin E(P)$. Hence $P \neq P_2$. Let $P = P_k$ with $k \geq 6$. By Claim 2, $k = 6$. Clearly none of x and y is an end vertex of P and by claw-freeness there are edges in G between the two neighbors of x on P and between the two neighbors of y on P. If x or y is next to an end vertex of P, we easily find a contradiction to the choice of S and F. If both x and y are not next to an end vertex of P, we obtain a C_3 and two matching edges containing all vertices of $V(P) \cup \{v\}$, also contradicting the choice of S and F. This completes the proof of Claim 3.

Claim 4. v is not adjacent to two vertices on one path component $P = P_4$ of F.

Proof of Claim 4. If v is adjacent to two vertices x and y on one path component $P = P_4$ of F, then by the choice of S and F, x and y are neighbors on P; otherwise we obtain a C_5 or a C_4 containing at least the same number of vertices as P. None of x and y is an end vertex of P; otherwise we obtain a C_3 and a matching edge containing all vertices of $V(P) \cup \{v\}$. Next we consider an end vertex v' of P.

Clearly the choice of S and F implies that v' is neither adjacent to a second vertex of P, nor to a vertex of a cycle of F or a P_2 or P_4 of F. By the same arguments as above, v' is not adjacent to a vertex of a P_k of F with $k \geq 7$. Hence v' is adjacent to a vertex w of a P_6 of F, and by similar arguments as before, w is neither an end vertex of the P_6 nor next to an end vertex of the P_6. We can obtain a P_9 and a P_2 containing all the vertices of $V(P) \cup \{v\}$ and of the P_6. This contradiction completes the proof of Claim 4.

Since S is an independent set and $\delta(G) \geq 2$, Claims 1 to 4 imply that v is adjacent to (at least) two vertices on different path components on 2, 4 or 6 vertices of F, and to no other vertices on the same path component. In the case such a neighbor u is not an end vertex, claw-freeness implies the existence of an edge between the two neighbors of u on the path component. Below we indicate how to obtain a contradiction with the choice of S and F in all possible cases, without giving all the details. Here $v, P_i, P_j \rightarrow P_{i+j+1}$ and $v, P_i, P_j \rightarrow v', P_{i+j}$ indicate that we can obtain a longer path containing all vertices or all vertices but one, respectively.

$v, P_2, P_2 \rightarrow v', P_4$

$v, P_2, P_4 \rightarrow P_7$

$v, P_2, P_6 \rightarrow P_9$ or $v, P_2, P_6 \rightarrow v', P_8$

$v, P_4, P_4 \rightarrow P_9$

$v, P_4, P_6 \rightarrow P_{11}$ or $v, P_4, P_6 \rightarrow v', P_{10}$

$v, P_6, P_6 \rightarrow P_{13}$ or $v, P_6, P_6 \rightarrow v', P_{12}$ or we can obtain a P_9 and two matching edges containing all vertices. ■

It is easy to give examples showing that we cannot omit the degree condition in the above results. One could try to replace the minimum degree condition by the weaker condition that every vertex with degree 1 has a neighbor with a high degree, but this does not work either. Consider e.g. the claw-free graph G obtained from a complete graph K_k ($k \geq 2$) by adding $k-1$ new vertices and matching edges saturating all new vertices. One easily checks that $\text{PKO}(G) = \infty$.

8 Discussion

There remain many open problems on parallel knock-out numbers. We close the paper by posing two of them:

- Is it true that every KO-reducible n-vertex graph G satisfies $\text{PKO}(G) < 2\sqrt{n}$?
- Is there an $O(n \log n)$ algorithm for computing the parallel knock-out number of an n-vertex tree?

References

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