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Michał Walicki, Uwe Wolter, Jack Stecher

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A category for studying the standarization of reporting languages

MICHAŁ WALICKI and UWE WOLTER
Department of Informatics, University of Bergen
E-mail: {michal,Uwe.Wolter}@ii.uib.no

JACK STECHER
Norwegian School of Economics and Business Administration,
Helleveien 30, N-5045 Bergen, Norway
E-mail: jack.stecher@nhh.no

1 Introduction

The problem of information exchange among agents who see the world subjectively arises in many contexts, ranging from computer science [1] to linguistics [7] to financial accounting [2]. One aspect of this problem is the choice of whether to use a standardized terminology among all agents or to use different languages for different audiences; this has been of particular interest in international accounting contexts [3, 16].

We use topological spaces to model reporting under subjective information. The points in a topological space represent what an agent in principle could wish to communicate—i.e., the world as the agent subjectively and privately understands it. Because different agents see the world differently, an agent will not have a distinct way of conveying everything he has in mind, and some terms in any language the agent uses may have multiple meanings. Typically, an agent is only capable of giving an approximate description of what he wants to convey. We thus think of an agent as not being able to communicate specific points, but only open neighborhoods in a topological space.

There is some evidence that this idea is what standard-setting bodies have in mind in financial reporting contexts. For example, the Financial Accounting Standars Board, in [4], specifies that reported information should be "understandable" to those with a general familiarity with how businesses operate. FASB notes explicitly, "understandability of information is related to the characteristics of the decision maker as well as the characteristics of the information," and defines understandability as "the quality of information that enables users to perceive its significance." In the same document, FASB reports that reported information is subject to a "materiality" threshold. That is, reported information is approximate in nature, and should be changed only when such changes make a meaningful difference. Topologically, this says that reports differ immaterially if they are within some neighborhood of each other. The International Accounting Standards Board, which sets standards in the European Union, adopts similar definitions in [6].

Our approach follows the spirit—though not the details of the technical development—of formal topology [9, 12], which postulates separation of points and opens of a topological system into two distinct (and related) sets, i.e., structures (O, pt, D). The separation invites some degree of independence in treatment of points, D, and opens, O, with one extreme being simply removing the points completely. Keeping both sets present, one can endow O with various algebraic structures, each leading to a corresponding requirement on the relation pt between opens and points. Thus, the frames of pointfree topology are complete lattices with finite meets distributing over infinite joins. The relation to points is then required to respect these. One can think of weaker structures on O, e.g., as only meet-semilattice or just partial order, with the respective restrictions (of meetor po-compatibility) on pt. On the other extreme to that of frames, one may allow O, D to be arbitrary sets and pt an arbitrary relation. This will be our setting, which is investigated under the name "basic pair" in [10].

Compatibility of the relation pt and the structure of O ensures that the algebraic properties of O reflect, as far as possible, the topological properties of D. In case of frames, the collection of preimages $pt^{-}(o)$ for all $o \in O$ gives the full topology (all open sets) on D. In the case of meet-semilattice, such preimages yield only a basis for a possible topology. In our case of arbitrary sets and relations, a topology on D is obtained by taking the preimages of O as the subbasis.

The reason for our choice is the context of application. We intend the points in D as distinctions identifiable by an agent in his experience (or world), while we envision O as the possible reports the agent may give to describe his world. I.e., the members of O are thought of as *names* of the opens rather than the opens themselves. The spirit of formal topology allows one then to have different reports in O which are extensionally indistinguishable, i.e., which denote the same (open) sets of points. (The same spirit is discernible in the framework of "named sets" [11] and of Chu spaces [8].)

In contrast to formal topology utilizing frames, our application does not justify our putting any specific restrictions on the relation between these two sets, nor on possible structure of either. Such structure and dependencies are to be induced exclusively by the relation between the sets. For instance, one might wish to endow O with a partial ordering representing the specialization of reports (as in [13]). We find it natural to introduce such an ordering by means of the very relation between reports and points, namely, to view a report r as more specific (not more vague) than s simply when its extension is included in that of s, $pt(r) \subset pt(s)$.

The structure of the rest of this paper is as follows. The next section introduces the category of interest. As we are interested in communication between agents with different subjective views of the world and different subjective interpretations of a language, the objects in our category are structures (O, pt, D), and the morphisms are defined between such structures. After we present our definition of morphisms, we show that the result is indeed a category. In section three, we show useful properties, in particular completeness and cocompleteness. We also give some examples of the use of (co)limits to express possible relations between distinct languages and their standarization. A final section concludes.

2 The category Rep

Objects in our category are multialgebras over a signature with two sort symbols O, D, and one operation symbol $pt : O \to D$. A multialgebra A over this signature is a pair of (possibly empty) sets, O^A, D^A, with a set-valued function $pt^A : O^A \to \mathcal{P}(D^A)$.[1] Given a multilagebra A, we write $\Omega(A)$ for the topology induced on D^A by the relation pt^A, i.e., by taking as the subbasis $\mathcal{SB}(A) = \{pt^A(o) \mid o \in O^A\} \cup \{\varnothing, D^A\}$. Notice that we do not require totality or surjectivity of pt^A, e.g., there may be points $d \in D^A$ such that for all $o \in O^A : d \notin pt^A(o)$. By adding the whole set D^A to the subbasis we only ensure that a topology is always induced on the whole D^A and not only on its subset. Likewise, there may be "empty" reports $o \in O^A$ which are not related to any points, i.e., $pt^A(o) = \varnothing$. Morphisms of such structures might seem at first to present a difficulty due to all too many choices. We are able to address this issue, following the choice presented in the overview and classification of homomorphisms of multialgebras given in [14, 15]; we will justify this further in what follows.

Definition 2.1 *A homomorphism between two multialgebras, $\phi : A \to B$, is a pair of functions*

[1] Of course, set-valued functions can be viewed as relations. However, when we focus on homomorphisms, the difference between the two viewpoints becomes significant, and the more structured/algebraic character of functions turns out to be useful. All mentioned results concerning multialgebras and their categories (except for the construction of finite products in section 3.2) can be found in [15].

$\phi_O : O^A \to O^B$ *and* $\phi_D : D^A \to D^B$*, as shown on the left*

$$\begin{array}{ccc} \mathcal{P}(D^A) & \xrightarrow{\mathcal{P}(\phi_D)} & \mathcal{P}(D^B) \\ {\scriptstyle pt^A}\uparrow & & \uparrow{\scriptstyle pt^B} \\ O^A & \xrightarrow{\phi_O} & O^B \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{P}(D^A) & \xleftarrow{\mathcal{P}(\phi_D^-)} & \mathcal{P}(D^B) \\ {\scriptstyle pt^A}\uparrow & & \uparrow{\scriptstyle pt^B} \\ \mathcal{P}(O^A) & \xleftarrow{\phi_O^-} & O^B \end{array}$$

and such that: $\forall y \in O^B : pt^A(\phi_O^-(y)) = \phi_D^-(pt^B(y))$.

Function applications are extended pointwise to sets (i.e., we operate with the *weak-image*, where $x \in f(Y)$ iff $\exists y \in Y : x \in f(y)$ iff $f^-(x) \cap Y \neq \varnothing$.) Note that commutativity of the diagram goes in the direction opposite to the arrows ϕ_O, ϕ_D, as shown on the right. This opposite direction reflects the topological tradition. In the notation, we will usually confuse the two and write both simply as ϕ. The homomorphisms do compose and yield a category Rep (in [14, 15], it was called MAlg_{OT}.)

One property of this definition of homomorphism is that, for every $d \in D^B$, if d is in the image of ϕ, then so are all reports of d; i.e., the image $\phi[A]$ $(= \phi_O[O^A] \cup \phi_D[D^A])$ is closed under preimages of $pt^B : d \in \phi_O[O^A] \Rightarrow (pt^B)^-(d) \subseteq \phi_D[D^A]$. Another one is that, if o is an "empty concept" in A, i.e., $pt^A(o) = \varnothing$, then if $pt^B(\phi_O(o)) \neq \varnothing$ then $pt^B(\phi_O(o)) \cap \phi_D[D^A] = \varnothing$; i.e., the ϕ_O-image of an "empty concept" need not be "empty", but then none of its points is in the image of ϕ_D. Intuitively, the first property says that if agent A asks agent B for something, and agent B can interpret this request as including d, then anything agent B could have interpreted as including d must be acceptable to A. The second property says that, if o is in agent A's language but is meaningless to A, then any morphism to agent B's world can only translate o either into meaningless concepts for B or into terms that B uses to describe things that A does not understand. The force of this definition is that, if a standard is used to enable A and B to communicate, then they must be able to validate what they discussed: however one agent reports something must be translated to a report that the other agent can see as justified. In accounting contexts, this is called *representational faithfulness* [4].

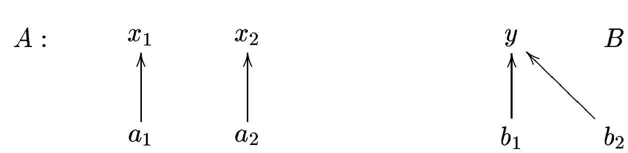
We take often advantage of the fact that morphisms are defined using functions rather than relations. For instance, generalization of the continuity condition to relations offers several choices and complications [10] while the following simple result obtains thanks to the fact that, when ϕ is a function, then $\phi^-(X \cap Y) = \phi^-(X) \cap \phi^-(Y)$.

Fact 2.2 *If* $\phi : A \to B$ *is a homomorphism, then* $\phi_D : D^A \to D^B$ *is a continuous mapping of the topologies* $\Omega(A) \to \Omega(B)$.

PROOF: We show that preimage of any element of subbasis $pt^B(y) \in \mathcal{SB}(B)$, is an open in $\Omega(A)$. (Remember the abbreviations $\mathcal{SB}(A)$ for $\mathcal{SB}(D^A)$, etc.)
For any $pt^B(y) \in \mathcal{SB}(B)$, there are two cases. If $y \notin \phi[O^A]$, then $\phi^-(pt^B(y)) = pt^A(\phi^-(y)) = \varnothing \in \Omega(A)$. Otherwise, $\phi^-(pt^B(y)) = pt^A(\phi^-(y)) = \bigcup_{x \in \phi^-(y)} pt^A(x) \in \Omega(A)$ being the union of $pt^A(x) \in \mathcal{SB}(A)$. □

The homomorphism condition is, in fact, stronger than mere continuity. This fact can be justified by the wish to provide not merely another way of doing standard topology (extended slightly by allowing multiple names for the same opens), but also a more specific framework for our intended application.

Example 2.3 *Consider two algebras:*



Viewed as topologies on the respective D sort, we have that $\Omega(A)$ is discrete and $\Omega(B) = \{\varnothing, \{y\}\}$. The mapping $\phi_D(x_i) = y$ is continuous. There is, however, no homomorphism $\phi : A \to B$. Its existence would require also a compatibility of reports (not only opens), namely, that preimage of a B's report means for A the same as the preimage of its meaning for B, i.e., for $i \in \{1,2\}$: $pt^A(\phi^-(b_i)) = \phi^-(pt^B(b_i)) = \phi^-(y) = \{x_1, x_2\}$.

Definition of continuity can vary slightly between different frameworks of formal topology, so let us only consider one simple example.

Example 2.4 *The two algebras from the previous example can be viewed as basic pairs of [10] (where pt is viewed as relation). The (continuous) morphisms proposed there are pairs of relations $r_D \subseteq D^A \times D^B$ and $r_O \subseteq O^A \times O^B$, such that $(pt^A)^-; r_O = r_D; (pt^B)^-$, where $\_;\_$ denotes the usual composition of binary relations.*

The mapping $\phi(x_i) = y$ and $\phi(a_i) = b_i$ will not satisfy this condition. However, the two structures are related by the morphism with $r_D = D^A \times D^B$ and $r_O = O^A \times O^B$.

Intuitively, we would say that a homomorphism $\phi : A \to B$ describes the space of possible "adequate communication" *from A to B*. (Notice that in the setting of the last example, any morphism $A \to B$ gives rise to the (inverse) morphism $A \leftarrow B$, i.e., using morphisms as "ways of communication", all communication becomes mutual and symmetric.) $\phi_O(a_i) = b_i$ means that A's saying a_1 is heard by B as b_1. (It may happen that also A's a_2 is heard by B as b_1, i.e., also $\phi_O(a_2) = b_1$.) The adequacy is verified at A's "intended distinctions": by a_i A intends x_i. If B hears b_1, he understands by it $pt^B(b_1)$; the preimage of these points among A's distinctions must equal what A understands by all reports which could be heard by B as b_1. In the present case, this is impossible because B has too many reports (of his "confused" distinction y) while A has too precise language. Whatever he says will be understood by B as y, but there is no imprecision in A's language, reflecting the imprecision of B's $pt^B(b_1) = y = pt^B(b_2)$. With such an interpretation, it seems appropriate to exlude any morphism from A to B. (If B did not have the report b_2, we would obtain a possible homomorphism from A; and likewise if A had more superflous words, e.g., an a_3 with $pt^A(a_3) = x_2$.) On the other hand, an "adequate communication" from B to A is possible, since B has very little to communicate: his y can be taken as A's x_1 (or x_2), in which case both reports b_1, b_2 are taken by A as a_1 (respectively, a_2). This is then a homomorphism $\psi : B \to A$, given by $\psi(y) = x_1$ and $\psi(b_i) = a_1$.

2.1 Homomorphisms, congruences and subalgebras

The following fact gives a handy and desirable characterization of epis and monos.

Fact 2.5 *A morphism is epi iff it is surjective and mono iff it is injective.*

The classical congruence condition is replaced by bireachability which is also defined generally as a relation between arbitrary two algebras.

Definition 2.6 *Given $A_1, A_2 \in \mathsf{Rep}$, a relation $\sim \subseteq A_1 \times A_2$ (i.e., a pair of relations $\sim_O \subseteq O^{A_1} \times O^{A_2}$ and $\sim_D \subseteq D^{A_1} \times D^{A_2}$) is bireachability iff:*

$$\begin{array}{rl} \forall a, b, a_1 : & a \sim_D b \wedge a \in pt^{A_1}(a_1) \Rightarrow \exists b_1 \in A_2 : b \in pt^{A_2}(b_1) \wedge a_1 \sim_O b_1 \\ \& \ \forall a, b, b_1 : & a \sim_D b \wedge b \in pt^{A_2}(b_1) \Rightarrow \exists a_1 \in A_1 : a \in pt^{A_1}(a_1) \wedge a_1 \sim_O b_1 \end{array} \tag{2.7}$$

A bireachability R between A_1 and A_2 is given a natural algebraic structure:

$$pt^R(\langle a, b \rangle) = pt^{A_1}(a) \times pt^{A_2}(b) \cap R. \tag{2.8}$$

A bireachability on A, is a bireachability between A and A.

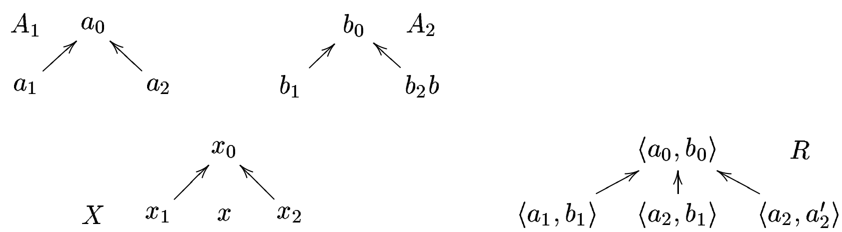
Bireachability is a "bisimilarity in the opposite direction". The name refers to the following property of such relation. If two points are bireachable, $d_1 \sim_D d_2$, and $d_1 \in pt^{A_1}(o_1)$ then there exists $o_2 \sim_O o_1$ such that $d_2 \in pt^{A_2}(o_2)$, and vice versa. Since there are no operations returning elements of sort O, arbitrary two elements of this sort can be made bireachable. One verifies easily that with the algebra structure on a bireachability R given by (2.8), the projection arrows $pr_i : R \to A_i$, $pr_i(\langle a_1, a_2\rangle) = a_i$, are morphisms in Rep.

Fact 2.9 *For every span of morphisms $\phi_i : X \to A_i$, $i \in \{1,2\}$, the relation $R = \{\langle \phi_1(x), \phi_2(x)\rangle \mid x \in X\}$ is a bireachability between A_1 and A_2.*

Bireachability between A_1 and A_2 represents some degree of "compatibility". For instance, any subset of $O^{A_1} \times O^{A_2}$ is a bireachability, which can be interpreted as saying that, as long as one does not take into account the "real distinctions" (points in D), any reports of A_1 can be related to any of A_2. However, in the moment we also want to relate some points $d_1 \in D^{A_1}$ and $d_2 \in D^{A_2}$, the "compatibility" requires that also the respective reports get related: for any report $o_1 \in (pt^{A_1})^-(d_1)$ there must be a corresponding report $o_2 \in (pt^{A_2})^-(d_2)$ and vice versa.

Although a span from X induces a bireachability R (with the associated algebra structure), it need not be the case that the morphisms factor through the induced R.

Example 2.10 *Consider two algebras:*



and two homomorphisms:

- $\phi_1 : X \to A_1$, *given by* $\phi_1(x_i) = a_i$ *and* $\phi_1(x) = a_2$, *and*
- $\phi_2 : X \to A_2$, *given by* $\phi_2(x_i) = b_i$ *and* $\phi_2(x) = b_1$.

The induced bireachability R is shown to the right. There is, however, no homomorphism $u : X \to R$ since, sending $u(x_0) = \langle a_0, b_0\rangle$, requires all the three arguments to be in the image of u, in which case the homomorphism condition for u fails for $z = u(x)$, i.e., $f^X(u^-(z)) = f^X(x) = \varnothing \neq \{x_0\} = u^-(f^R(z))$.

Obviously, the condition (2.7) is preserved under arbitrary unions. Given two algebras and a collection of bireachabilities $C_i \subseteq A_1 \times A_2$, then also thier union $\bigcup_i C_i$ satisfies trivially this condition (since the antecendent of the implication mentions only one of the bireachabilities, which can then be used to verify the consequent). Thus, collecting all bireachabilities between A and B we obtain the maximal one.

Fact 2.11 *For every $A_1, A_2 \in$ Rep there exists a (unique) maximal (wrt. set-inclusion) bireachability between A_1 and A_2.*

In particular, for every algebra A there exists a maximal bireachability on A. (It will always be total on the O-part, i.e., $O^A \times O^A$. But it need not be total on D. E.g., for $O^A = \{o\}$, $D^A = \{d_1, d_2\}$ and $pt^A(o) = \{d_1\}$, there is no bireachability making $d_1 \sim d_2$.)

A bireachability equivalence on an algebra is a bireachability which is also equivalence. Such relations play the role of congruences. The kernel of a morphism is a bireachability equivalence and any bireachability equivalence $\sim$ on A gives rise to a surjective morphism of A onto the quotient $A/_\sim$.

Fact 2.12 *The kernel of a homomorphism $\phi : A \to B$ is a bireachability equivalence on A and, given a bireachability equivalence $\sim \subseteq A \times A$, we obtain an epimorphism $e : A \to A/_\sim$, where the latter is defined as the collection of $\sim$-equivalence classes with the operation given by $pt^{A/\sim}([o]) = \{[d] \mid \exists o' \in [o]\ d' \in [d] : d' \in pt^A(o')\}$.*

Henceforth, congruence will mean bireachability equivalence. The existence of a congruence on A which is identity on O^A and non-identity on D^A implies that $\Omega(A)$ is not even T_0; i.e., there are (at least) two distinct points which belong to exactly the same opens. The above quotient by such a congruence amounts then to identifying all points which belong to the same opens.[2] One might wish to check other topological properties which are reflected in the properties of bireachabilities.

Define subalgebra relation $A \sqsubseteq B$ iff there exists a mono $A \to B$. This relation is dual to the classical one in the following sense:

Fact 2.13 *$A \sqsubseteq B$ if $A \subseteq B$ and A is closed under B-preimages of operations, i.e., $\forall d \in D^A \subseteq D^B : (pt^B)^-(d) \subseteq A$.*

This and the following two facts are special cases of the corresponding results proved for multialgebras over arbitrary signatures in [15].

Fact 2.14 *For any $A \in \mathsf{Rep}$ and every $d \in D^A$, the pair $S = \langle (pt^A)^-(d), d\rangle$ with the operation $pt^S(x) = d$ for all $x \in (pt^A)^-(d)$ is a subalgebra $S \sqsubseteq A$.*

A useful fact is

Fact 2.15 *For $B \in \mathsf{Rep}$ and $X \subseteq B$, there exists a largest $A \sqsubseteq B$ such that $A \subseteq X$.*

3 (Co)completeness of Rep

We show that the category Rep is complete and cocomplete. The latter is but a special case of the general result from [15] and we only sketch the involved constructions. Also existence of final objects and equalizers follows from this earlier work. Subsection 3.2 describes the character and construction of (binary) products, thus completing the proof of the existence of (finite) limits in Rep.

3.1 Some earlier results

Theorem 3.1 Rep *is cocomplete.*

Proof: We only sketch the used constructions. The empty algebra $(\varnothing, \varnothing, \varnothing)$ is trivially initial. The coproduct C of a collection $\{A_i \mid i \in I\}$ is given by the disjoint union of the carriers of the components with $pt^C(a_i) = pt^{A_i}(a_i)$ for all $a_i \in A_i$. To construct coequalizer for two arrows $\phi_1, \phi_2 : A \to B$, we start as usual by considering the equivalence closure $\sim$ on B of the relation $E = \{\langle \phi_1(a), \phi_2(a)\rangle \mid a \in A\}$. Denoting these classes as $[b_i]$, the operation is defined by: $[b_2] \in pt^{CE}([b_1]) \iff [b_2] \subseteq pt^B([b_1])$ or, working only with representatives: $b_2 \in pt^{CE}(b_1) \iff [b_2] \subseteq pt^B([b_1])$. □

Lemma 3.2 Rep *has final objects and equalizers.*

Proof: As a final object Z, we can take the algebra with $D^Z = \{r, u\}$ and $O^Z = \{o\}$, where $pt^Z(o) = \{r\}$. Thus, it has an "unreportable" point u, but no "empty" reports. The unique morphism from any other algebra $A \to Z$ will map all elements of O^A to o, the elements of D^A which are in the image of pt^A to r and those which are not to u.

An equalizer object E and arrow $e : E \to A$, for a pair of arrows $\phi_1, \phi_2 : A \to B$, is constructed in the more or less standard way. We let $E_0 = \{a \in A \mid \phi_1(a) = \phi_2(a)\}$ and let E be the largest subalgebra of A contained in E_0, which exists by fact 2.15. The operation is defined by $pt^E(x) = pt^A(x) \cap E$ for all $x \in E$, and the arrow $e : E \to A$ is inclusion (which is monomorphism, by fact 2.15). □

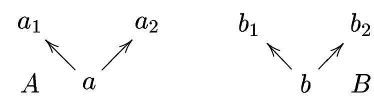
Products are a new story, and we present them in detail in the following subsection.

[2] Vickers [12], p. 62, calls this the "localification" of the space $\Omega(A)$.

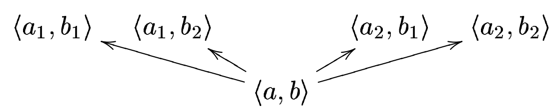
3.2 Products

We consider first the relationship between products and (maximal) bireachability between algebras. In the case of co-algebras for functors preserving mono-sources, products and maximal bisimulation coincide (theorem 8.6 in [5]). If we considered only the subcategory of multialgebras obtained as inverse from coalgebras (over a given polynomial functor), we could conclude the existence of products, namely, of maximal bireachabilities (corresponding to maximal bisimilarities between co-algebras) between the arguments. That is, if the inverse of *pt* is deterministic, the maximal bireachability becomes a product (which is the same statement as the one that maximal bisimilarity between co-algebras over a polynomial functor is their product).

Example 3.3 *Consider two algebras:*



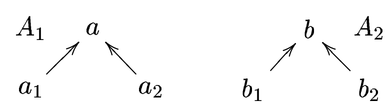
The maximal bireachability between them is (the inverse of the maximal bisimilarity between their inverses viewed as co-algebras):



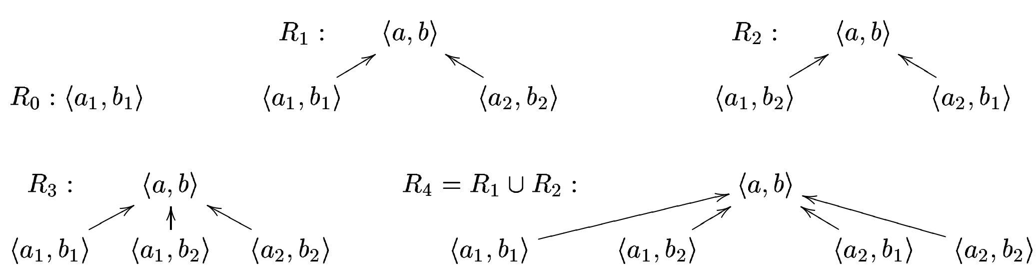
and this is the product $A \times B$.

Thus, if there are only ambiguous reports (like a or b), but no points are reported in more than one way, the product – maximal bireachability – increases "the degree of ambiguity". However, our case is more general and also more complicated. In general, maximal bireachability need not be the product. As is to be expected from the above remark, the problems and counterexamples are provided by multialgebras which are not inverse of coalgebras for polynomial functors. The problem arises when the reports are "overly precise" in the sense of several reports making no real distinctions (denoting the same points).

Example 3.4 *Consider two algebras:*

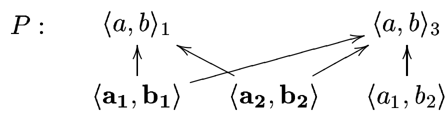


The following are examples of bireachabilities between A_1 and A_2:



R_4 is the maximal bireachability between A and B – every other bireachability is a subset of it. However, only $R_0 \sqsubseteq R_4$, while neither R_1, R_2 nor R_3 is a subalgebra of R_4: their inclusions are not homomorphisms. Consequently, R_4 can not possibly be the product $A_1 \times A_2$, as the projections from, say, R_2 would not factor through it.

Fixing the algebras A_1, A_2 and letting $\mathcal{R}_{A_1 \times A_2}$ be the collection of all bireachabilities between them – each with the canonical algebraic structure as defined in (2.8) – we consider the diagram (partial ordering) $\langle \mathcal{R}_{A_1 \times A_2}, \sqsubseteq \rangle$ and its colimit P. Consider only R_1, R_3 from the above example. They have two common subobjects, $R_0 = \langle a_1, b_1 \rangle$ and $R'_0 = \langle a_2, b_2 \rangle$, and these subobjects have to be identified. The (part of the) result will be (where the bold face indicates the identified subobjects):

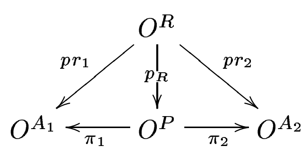


$P \not\subseteq A_1 \times A_2$, so it is no longer a bireachability between A_1 and A_2. But we do have that $R_0, R'_0, R_1, R_3 \sqsubseteq P$, where the inclusion of R_1 will map $i_1(\langle a, b \rangle) = \langle a, b \rangle_1$, while that of R_3 : $i_3(\langle a, b \rangle) = \langle a, b \rangle_3$. Thus, if we interpret a bireachability between A_1 and A_2 as a kind of potentially compatible communication, the colimit of the diagram $\langle \mathcal{R}_{A_1 \times A_2}, \sqsubseteq \rangle$ represents all such potentials, some of which need not be possible simultaneously. It collects all possible combinations of the "compatible" reports assigning to them respective "compatible" points. (But also, as in the previous example 3.3, if there are too many distinctions reportable in ambigous ways, the resulting reports acquire only more ambiguity.) The colimit P of the diagram $\langle \mathcal{R}_{A_1 \times A_2}, \sqsubseteq \rangle$ is equipped with the projections $\pi_i : P \to A_i$, $i \in \{1, 2\}$, obtained as the mediating arrows for all projections $\{p^r_A : R_r \to A_i \mid R_r \in \mathcal{R}_{A_1 \times A_2}\}$. This colimit is the product of the two algebras and we register some of its properties to be used in the proof of this claim.

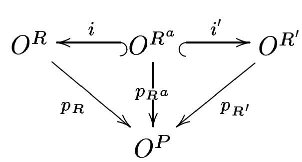
Fact 3.5 *Let P be the colimit of the diagram $\langle \mathcal{R}_{A_1 \times A_2}, \sqsubseteq \rangle$ for $A_i \in$ Rep. Then*

1. $O^P \simeq O^{A_1} \times O^{A_2}$,
2. *for every $d, d' \in D^P$, if $(pt^P)^-(d) = (pt^P)^-(d')$ and $\pi_i(d) = \pi_i(d')$ for $i \in \{1, 2\}$, then $d = d'$.*

PROOF: 1. We show that $|O^P| = |O^{A_1} \times O^{A_2}|$. All colimit arrows and projections considered below are restricted to the sort O. (i) For every bireachability $R \in \mathcal{R}_{A_1 \times A_2} : O^R \subseteq O^{A_1} \times O^{A_2}$. By the construction of colimits (coproducts and coequalizers) in Rep, the collection of all colimit arrows $\{p_R : R \to P \mid R \in \mathcal{R}_{A_1 \times A_2}\}$ is jointly epi, in particular, jointly surjective (on the sort O^P). (ii) For each bireachability $R \in \mathcal{R}_{A_1 \times A_2}$, the colimit arrow $p_R : R \to P$ is injective on the sort O^R. This follows from the fact that $O^R \subseteq O^{A_1} \times O^{A_2}$ and the commutativity of the diagram:



If for two distinct $\langle a_1, a_2 \rangle = a \neq b = \langle b_1, b_2 \rangle \in O^R$, $p_R(a) = p_R(b)$, the commutativity would be violated. Thus, in particular, if we consider the bireachability $O^{A_1} \times O^{A_2}$, we obtain an injection from it into O^P. (iii) Finally, for every $a = \langle a_1, a_2 \rangle$ and every pair of bireachabilities $R \neq R'$, if $a \in O^R \cap O^{R'}$, then also $p_R(a) = pr_{R'}(a)$. This follows from the fact that such an a is itself a bireachability ($R^a \in \mathcal{R}_{A_1 \times A_2}$, with the algebra structure given by $O^{R^a} = \{a\}, D^{R^a} = \varnothing$) and $R^a \sqsubseteq R$ and $R^a \sqsubseteq R'$. Since P with all colimit arrows is a commutative cocone over $\langle \mathcal{R}_{A_1 \times A_2}, \sqsubseteq \rangle$, we have that $p_{R^a}(a) = p_R(i(a)) = p_{R'}(i'(a))$.



This means that the joint surjectivity of all colimit arrows from (i) factors through a surjective arrow $O^{A_1} \times O^{A_2} \to O^P$. Equivalently, there is an injection in the opposite direction. Combined with the existence of an injection $O^{A_1} \times O^{A_2} \to O^P$ from (ii), this gives, by Schröder-Bernstein theorem, the required bijection.

2. Let us write $d_i = \pi_i(d) = \pi_i(d')$. We have two subalgebras $B = \langle (pt^P)^-(d), \{d\} \rangle \sqsubseteq P$ and $B' = \langle (pt^P)^-(d'), \{d'\} \rangle \sqsubseteq P$. Since the colimit arrows are jointly surjective we have two bireachabilities $R, R' \in \mathcal{R}_{A_1 \times A_2}$ with $p_R(x) = d$ (and hence $O^B \subseteq p_R[O^R]$) and $p_{R'}(x') = d'$ (and hence $O^{B'} \subseteq p_{R'}[O^{R'}]$). In particular, for any $o \in (pt^P)^-(d)$, $\langle \pi_1(o), \pi_2(o) \rangle \in R \cap R'$. But this means that we have subalgebra inclusions $i : Q \sqsubseteq R$ and $i' : Q \sqsubseteq R'$, where Q is the bireachability $\langle N, \langle d_1, d_2 \rangle \rangle$ with $N = \{\langle \pi_1(a), \pi_2(a) \rangle \mid a \in (pt^P)^-(d)\}$. Since $\langle P, \{p_r \mid R \in \mathcal{R}_{A_1 \times A_2}\}$ is a commutative cocone, we thus have that $i; p_R = p_Q = i'; p_{R'}$, in particular, $d = p_R(i(\langle d_1, d_2 \rangle)) = p_Q(\langle d_1, d_2 \rangle) = p_{R'}(i'(\langle d_1, d_2 \rangle)) = d'$.

□

Lemma 3.6 *Given $A_1, A_2 \in$ Rep, the colimit P of $\langle \mathcal{R}_{A_1 \times A_2}, \sqsubseteq \rangle$ is their product.*

PROOF: The existence of projection morphisms $\pi_i : P \to A_i$ follows from the colimit property of P. Given any X with $\phi_i : X \to A_i$, define $u : X \to P$ by:

1. for $o \in O^X : u(o) = \langle \phi_1(o), \phi_2(o) \rangle$
2. for $d \in D^X : u(d) = dd$ such that $\pi_i(dd) = \phi_i(d)$ & $(pt^P)^-(dd) = u((pt^X)^-(d))$.

We show that such a u is the unique morphism making $\phi_i = u; \pi_i$. The existence and uniqueness of u on the sort O follows from fact 3.5.1. For 2, we have to show existence and uniqueness. Then we verify the homomorphism condition, 2.1, for the whole u.

Existence: Given a $d \in D^X$, the pair $S = \langle (pt^X)^-(d), d \rangle$, with $pt^S(x) = pt^X(x) = d$ for all $x \in (pt^X)^-(d)$, is a subalgebra of X (fact 2.14). Hence ϕ_i's restricted to S are morphisms. So they induce a bireachability $R = \langle N, \{t\} \rangle \in \mathcal{R}_{A_1 \times A_2}$, where $\pi_i(t) = \phi_i(d)$ and $N = \{\langle \phi_1(x), \phi_2(x) \rangle \mid x \in (pt^X)^-(d)\}$, fact 2.9. We have the epimorphism $s : S \to R$ and, letting $r : R \to P$ be the unique morphism as in fact 3.5.2, we obtain $s; r = u' : S \to P$ (this u' is the restriction of the postulated morphism $u : X \to P$ to the subalgebra $S \sqsubseteq X$). Then, for some $dd \in D^P : u'(d) = r(t) = dd$ and $u'((pt^S)^-(t)) = u((pt^X)^-(d)) = r(N)$ which, since u' is a morphism, equals $(pt^P)^-(dd)$. This shows the existence of dd as required.

Uniqueness: Existence of two distinct $dd_1 \neq dd_2$ with $\pi_i(dd_1) = \pi_i(dd_2)$ and $(pt^P)^-(dd_1) = (pt^P)^-(dd_2)$, is excluded by fact 3.5.2.

Homomorphism condition: Let $o \in O^P$ be such that $\pi_i(o) = a_i$. Then $u^-(o) = \phi_1^-(a_1) \cap \phi_2^-(a_2)$. 1) If $u^-(o) = \varnothing$ then either 1a) $pt^P(o) = \varnothing$, in which case the condition is verified, or 1b) $pt^P(o) \neq \varnothing$. If there exists a $dd \in pt^P(o) \cap u[X]$, i.e., $dd = u(d)$ for some $d \in D^X$ then, by 2, $o \in (pt^P)^-(dd) = u((pt^X)^-(d))$, which contradicts the assumption 1. So assume 2) $u^-(o) \neq \varnothing$. We have to show two inclusions. 2a) $pt^X(u^-(o)) \subseteq u^-(pt^P(o))$: Let $x \in u^-(o)$ and $d \in pt^X(x)$. Then, since for $dd = u(d)$ we have $(pt^P)^-(dd) = u((pt^X)^-(d))$, and $o \in u((pt^X)^-(d))$, so $u(d) = dd \in pt^P(o)$ and thus $d \in u^-(pt^P(o))$. 2b) $u^-(pt^P(o)) \subseteq pt^X(u^-(o))$: Let $d \in u^-(pt^P(o))$, i.e., for some $dd \in pt^P(o) : u(d) = dd$ ($d \in u^-(dd)$). Then $o \in (pt^P)^-(dd) = u((pt^X)^-(d))$, i.e., $u^-(o) \cap (pt^X)^-(d) \neq \varnothing$. But this means that $d \in pt^X(u^-(o))$. □

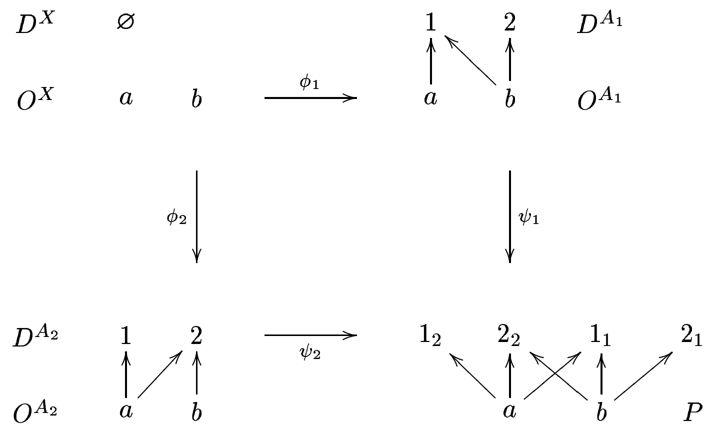
Notice that although, according to Fact 3.5, the O sort of the product P is always the cartesian product of the O sorts of the arguments, D^P can be empty even if it is not empty in any of the argument algebras. (For instance, for $A_1 = \langle \{a\}, \{1\}, pt^A \rangle$ with $pt^{A_1}(a) = \{1\}$ and $A_2 = \langle \{b\}, \{2\}, \varnothing \rangle$, the product is $\langle \{ab\}, \varnothing, \varnothing \rangle$, as the only bireachability $A_1 \sim A_2$ is $\langle a \sim_O b, \varnothing \rangle$.)

Theorem 3.7 *The category* Rep *is (finitely) complete.*

3.3 A few examples

Any span between two algebras induces a bireachability between them and we have suggested that this amounts to a kind of "compatibility" of the respective (sub)algebras. We can follow this interpretation by considering such a "compatibility" as a requirement of agreement on the two involved agents. Pushout will then yield an object which can be interpreted as the setting ensuring that both agents can communicate in (or into) it respecting the requirement.

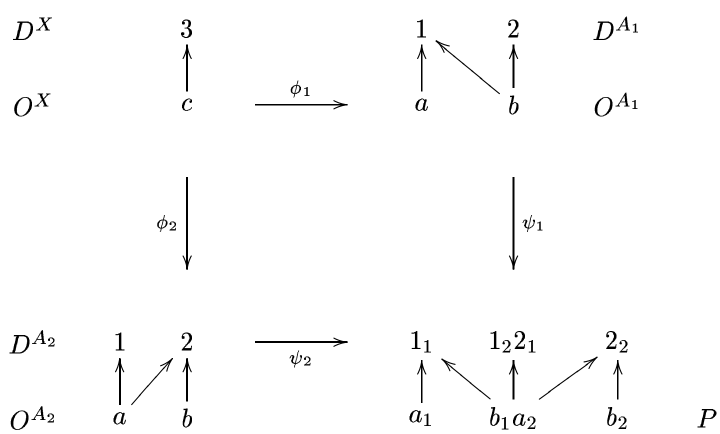
Example 3.8 *Consider a pushout of the span* $\phi_i : X \to A_i$*:*



We let $\phi_i(a) = a$ *and* $\phi_i(b) = b$*. This span can be viewed as a requirement that* A_1 *and* A_2 *must agree on using* a, b *consistently, i.e.,* a *said by* A_1 *must be heard/translated as* a *in* A_2*, and vice versa. (Notice that there is no homomorphism from* A_1 *to* A_2 *(nor other way around) mapping* a *to* a *and* b *to* b*.) There is, however, no requirement on the denoted distinctions.*

The pushout object P *contains thus* a, b *with* $\psi_i(a) = a$ *and* $\psi_i(b) = b$*. Images of* $1, 2 \in D^{A_2}$ *have subscript* $_{-2}$*, and similarly for* A_1*. In short, agents do use the same reports,* a, b*, in a consistent manner, but no actual distinctions are being communicated, since, for instance,* $a \in P$ *represents* 1_2 *and* 2_2 *for* A_2 *while* 1_1 *for* A_1*.*

Example 3.9 *Consider another requirement, where the span* $\phi_i : X \to A_i$ *is given by* $\phi_1(c) = b, \phi_1(3) = 2$ *while* $\phi_2(c) = a, \phi_2(3) = 1$*, which requires* A_2*'s* $a/1$ *to correspond to* A_1*'s* $b/2$*:*

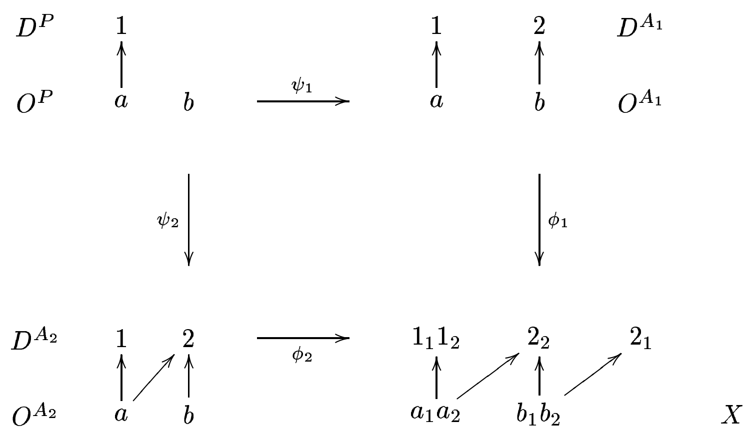


The images under ψ_i *in the pushout object are identified by the subscripts. Now, any* b *spoken by* A_1 *in the common space* P *is heard as* $b_1 a_2$ *which is denoted by* A_2 *as* a*. The distinction* $1_2 2_1$ *is the part corresponding to the original requirement. Besides that, we have "private" distinctions from both agents (not captured by any requirements), and likewise the report* a_1/b_2 *in* P *still "belongs only to"* A_1/A_2*, as* A_2/A_1 *does not have any word corresponding to it under* ψ_2/ψ_1*.*

Morphisms (quotients) from the pushout object P will allow for further agreements/coincidences or, in the extreme cases, e.g., of a final object, confusions. Pushout object can be thus taken as a maximal common communication space (standard) induced by the two agents and allowing these agents to share the part given by the initial requirement.

A different situation is obtained by interpreting pullbacks. Here we have a cospan $\phi_i : A_i \to X$ into a "common space" X (given standard), and its pullback gives us the minimal space representing the possibility of agreement, or adequate communication between A_1 and A_2, when using the standard X.

Example 3.10 *Consider a pullback of the cospan $\phi_i : A_i \to X$, where images are identified by the subscripts:*



The agreement in X concerns the common use of b_1b_2 for b both by A_1 and A_2 (albeit with reference to distinctions which do not correspond), and the fact that a_1a_2 means in either case at least $1_1 1_2$ which corresponds to the respective 1s.

4 Conclusion

We develop a formal structure for studying standardization of a language between agents who individually use different private languages. The notion of a standardized language that emerges is essentially one of a coarsening: two agents can translate their private languages into a shared one if and only if the terms in the shared language can be expressed in each agent's private language. Thus, the simplicity of standardization of information exchange comes at the cost of reducing the amount of information agents can express.

To formalize these notions, we develop a category, called Rep, of reporting environments, and model the translation of one reporting environment to another as a homomorphism in this category. We show that Rep is complete and cocomplete, and observe that homomorphisms between two agents may exist in only one direction. Thus, communicating to someone is different from understanding someone.

This category provides a promising new approach for studying standardization of communication. A fulfillment of this promise will be the point of future research, for example studying settings where agents are not aware of the same things and where agents need to consider outside parties when agreeing upon a standard. Preliminary examples of such study were given in terms of pushouts/pullbacks in Rep.

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