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Modalities as Interactions between Classical and Constructive Logics

Michał Walicki

29th May 2006

Abstract

The "possible world" semantics of modal logics has proven so powerful and successful, that one has been willing to disregard some of its technical disadvantages and, not less importantly, the problems with philosophical interpretation of the "possible worlds". The main such problem which is addressed here concerns the disappearance of the actual world, which in any Kripke structure can be chosen arbitrarily. Even then, it is not the common world which is, in one way or another, shared by all the agents, but only one of the possibilities which may even be inaccessible to some of the involved parties. This may, certainly, offer various advantages when modelling specific problems and we do not contest this matter. But we would consider it a drawback when seen in a more philosophical perspective.

We offer a view of modalities which are anchored in one common world governed by the classical (propositional) logic. We review the relations between the classical, constructive and modal propositional logics and show that S4 not only admits embedding of the other two logics, but can be seen as their natural and minimal union. We reformulate topological algebras interpreting S4 as boolean algebras equipped with constructive negation. The constructive substructure of such an algebra can be then seen as an "epistemic subuniverse", and modalities arise from the interaction between the constructive and boolean negations or, we might perhaps say, between the epistemic and the ontological aspects. As an example of the generality of the obtained formalization, we apply it also to S5.

As our semantics, underlying the above view of modalities, is based on the boolean algebras with operators, it should not be considered in an opposition to the "possible worlds" semantics: the latter can be obtained by the well known transformation of the former. Thus, apparent incompatibility of the interpretations notwithstanding, we would view them as complementary rather than as contrary.

1 Introduction

We begin by recalling the algebraic semantics of the three propositional logics and some of the standard results on their interconnections. In addition to the obvious functor from the category of topological algebras to that of Heyting algebras, $I : \mathcal{TA} \to \mathcal{HA}$, we also give a functor C in the opposite direction and show that the two are adjoint (with the unit of adjunction being identity). This allows us than, in section 2, to state the embedding of constructive logic into S4 in a compact way, which shows also a stronger character of this embedding than the mere preservation and reflection of the semantic consequence. Various classical theorems follow as corollaries and so sections 1-2 present only a compact view of the known results (except for the functor C which we have not encountered in the literature).

The involved constructions give rise to a new formulation of topological algebras (providing the semantics for S4), as $\mathcal{IC}$-algebras, which are presented in section 3. The main novelty consists in making explicit and transparent the relationships between the constructive and boolean logics which, together, yield S4. An advantage of this formulation exemplified in section 3 is that proofs of embeddings and dependencies between these three logics, performed traditionally at the metalevel and typically by analysis of the respective proof systems, become internalized in the common language of $\mathcal{IC}$-algebras. We give a couple of simple examples. Section 4 shows how the new formulation adapts to extensions of S4 exemplified by S5.

Section 5 gives a proposal for an informal reading of the operations of $\mathcal{IC}$-algebras, gives some examples and discusses the emerging view of modalities as combinations of boolean and constructive negation.

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To make the paper entirely self-contained and to introduce the notation to be used, we state a few usual definitions.

A lattice is a triple $\langle L; \cap, \cup \rangle$ where L is a set and the binary operations satisfy the following equalities:

$$\begin{array}{lrcllrcl} \textit{comm.} & a \cap b & = & b \cap a & & a \cup b & = & b \cup a \\ \textit{assoc.} & (a \cap b) \cap c & = & a \cap (b \cap c) & & (a \cup b) \cup c & = & a \cup (b \cup c) \\ \textit{absorp.} & a \cap (a \cup b) & = & a & & a \cup (a \cap b) & = & a \end{array}$$

Idempotency follows. Given a lattice, we write $\subseteq$ for the partial order: $x \subseteq y \Leftrightarrow x \cup y = y$.

Lattice is distributive iff also

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \tag{1.1}$$

(the dual is equivalent). It is bounded iff it has least and greatest elements, $\mathbf{0}, \mathbf{1}$, i.e.,

$$x \cup \mathbf{1} = \mathbf{1} \quad \textit{and} \quad x \cup \mathbf{0} = x. \tag{1.2}$$

Definition 1.3 *Heyting algebra is a tuple $\mathbf{H} = \langle H; \sqcap, \sqcup, \hookrightarrow, \mathbf{0} \rangle$ such that:*

1. *$\langle H; \sqcap, \sqcup \rangle$ is a lattice*
2. *H is (relatively) pseudo-complemented: $\forall x, y\ \exists x \hookrightarrow y = max\{z \mid x \sqcap z \sqsubseteq y\}$*
   (one has $x \sqsubseteq a \hookrightarrow b \ \Leftrightarrow\ a \sqcap x \sqsubseteq b$)
3. *$\mathbf{0}$ is the least element.*

The greatest element $\mathbf{1} = x \hookrightarrow x$ exists in any pseudo-complemented lattice. Pseudo-complement is defined as $\div x = x \hookrightarrow \mathbf{0}$. Heyting algebra is also a distributive lattice and meet distributes over infinite joins (when these exist): $x \sqcap \bigsqcup_{y \in Y} y = \bigsqcup_{y \in Y} (x \sqcap y)$.

The class of Heyting algebras with homomorphisms gives the category $\mathcal{HA}$.

To easily distingush whether an operator of a Heyting or a boolean lattice is meant, we will use the symbols from the above definition for the operations in Heyting algebras, and the general lattice symbols introduced before it for boolean algebras.

Definition 1.4 *Boolean algebra is a tuple $\mathbf{B} = \langle B; \cap, \cup, - \rangle$ where*

1. *$\langle B; \cap, \cup, - \rangle$ is a distributive lattice*
2. *which is complemented, i.e., $x \cap -x = \mathbf{0}$ and $x \cup -x = -\mathbf{0} = \mathbf{1}$.*

A toplogical algebra $\mathbf{T} = \langle T; \cap, \cup, -, \mathbf{c} \rangle$ is a boolean algebra with a closure operator $\mathbf{c} : T \to T$ (or, dually, interior operator $\mathbf{i}(x) = -\mathbf{c}(-x)$), satisfying the equations:

$$\begin{array}{llll} \mathbf{c1}. & x \subseteq \mathbf{c}(x) & \mathbf{i1}. & x \supseteq \mathbf{i}(x) \\ \mathbf{c2}. & \mathbf{c}(\mathbf{c}(x)) = \mathbf{c}(x) & \mathbf{i2}. & \mathbf{i}(\mathbf{i}(x)) = \mathbf{i}(x) \\ \mathbf{c3}. & \mathbf{c}(x \cup y) = \mathbf{c}(x) \cup \mathbf{c}(y) & \mathbf{i3}. & \mathbf{i}(x \cap y) = \mathbf{i}(x) \cap \mathbf{i}(y) \\ \mathbf{c4}. & \mathbf{c}(\mathbf{0}) = \mathbf{0} & \mathbf{i4}. & \mathbf{i}(\mathbf{1}) = \mathbf{1} \end{array}$$

An element $x \in T$ is open/closed iff $x = \mathbf{i}(x)/x = \mathbf{c}(x)$.

(Unit $\mathbf{1}$ exists in every pseudo-complemented lattice; for boolean algebras it is $-x \cup x$; and so we also obtain the zero element $\mathbf{0} = -\mathbf{1}$)

The class of topological algebras, i.e., boolean algebras with closure operator, with the respective homomorphisms, gives the category $\mathcal{TA}$.

Theorem 1.5 *$I : \mathcal{TA} \to \mathcal{HA}$, restricting the objects and morphisms to the open elements, is a functor.*

Proof. We map $\mathbf{T} = \langle T; \cap, \cup, -, \mathbf{c} \rangle$ onto $\mathbf{H} = \langle H; \sqcap, \sqcup, \hookrightarrow, \mathbf{0} \rangle$ where

- $H = \{x \in T \mid x = \mathbf{i}(x)\}$
- by $\mathbf{i1}$, $\mathbf{i}(\mathbf{0}) \subseteq \mathbf{0}$, i.e., $\mathbf{i}(\mathbf{0}) = \mathbf{0}$ and hence $\mathbf{0} \in H$
- $x \sqcap y = x \cap y$ and $x \sqcup y = x \cup y$
- $x \hookrightarrow y = \mathbf{i}(-x \cup y)$.

The fact that $\mathbf{H}$ is Heyting algebra is well known, e.g., theorem 1.14 from [9], or IV.1.4 from [11]. Homomorphism condition for the reduced mapping follows for the operations inherited from the source $\mathbf{T}$, and is easily verified for $\hookrightarrow$ (IV.2.1 in [11]). □

The standard, though involved, part of the proof of the following theorem concerns the embedding of a Heyting algebra into a boolean one. Functoriality is more involved and we show it explicitly.

Theorem 1.6 *There is a functor $C : \mathcal{HA} \to \mathcal{TA}$, such that $C; I = ID_{\mathcal{HA}}$.*

Proof. The fact that every Heyting algebra can be obtained as algebra of open elements of a toplogical algebra is the theorem 1.15 from [9] (for the dual formulation in terms of closed elements), or IV.3.1 from [11]. To establish the result, we have to find "the right" topological algebra for every Heyting algebra. We repeat here the construction given in [9] which will be needed to verify functoriality.

Given $\mathbf{H} = \langle H; \sqcap, \sqcup, \hookrightarrow, \mathbf{0}\rangle$, we consider it first as a bounded distributive lattice $\langle H, \sqcap, \sqcup, \mathbf{0}, \mathbf{1}\rangle$. By [7] or [11] 3.1, $\mathbf{H}$ can be extended uniquely to a boolean algebra $\mathbf{T} = \langle T; \cap, \cup, -\rangle$ where

1. $\mathbf{H}$ is a sublattice of $\mathbf{T}$ (i.e., $\forall x, y \in H : x \cup y = x \sqcup y$ and $x \cap y = x \sqcap y$)
2. every element $b \in T$ is of the form $\bigcap_1^n -a_i \cup b_i$ for some finite set of $a_i, b_i \in H$.[1]

Using the fact that

$$a \hookrightarrow b \ \subseteq \ -a \cup b, \tag{1.7}$$

one shows that the choice of representatives in 2 is inessential for the definition of the interior/closure operator, namely, $\forall a_i, b_i, a_j, b_j \in H$:

$$\bigcap_1^n -a_i \cup b_i = \bigcap_1^m -a_j \cup b_j \ \Rightarrow \ \bigcap_1^n a_i \hookrightarrow b_i = \bigcap_1^m a_j \hookrightarrow b_j \tag{1.8}$$

Interior is defined for every b (of the form 2):

$$\mathbf{i}(\bigcap_1^n -a_i \cup b_i) = \bigcap_1^n a_i \hookrightarrow b_i. \tag{1.9}$$

Letting $C(\mathbf{H}) = \mathbf{T}$ gives $I(C(\mathbf{H})) = \mathbf{H}$.

Given a homomorphism $h : \mathbf{H1} \to \mathbf{H2}$, let ∇_h denote the filter in $\mathbf{H1}$ given by $\nabla_h = \{x \in H1 \mid h(x) = \mathbf{1}^{\mathbf{H2}}\}$, which determines the congruence on $\mathbf{H1}$, the kernel of h, given by $x \sim_h y \Leftrightarrow (x \hookrightarrow^{\mathbf{H1}} y) \in \nabla_h \wedge (y \hookrightarrow^{\mathbf{H1}} x) \in \nabla_h$. Given a filter ∇ in a (pseudo-complemtend) algebra $\mathbf{H}$, we write $\mathbf{H}/_{\nabla}$ for the quotient algebra by the congruence $\sim_{\nabla}$. It is the standard fact that the mapping $q : \mathbf{H} \to \mathbf{H}/_{\nabla}$ sending each $x \in H$ onto its equivalence class $[x]^{\nabla}$ is a homomorphism satisfying also the condition:

$$[x]^{\nabla} \sqsubseteq [y]^{\nabla} \Leftrightarrow x \hookrightarrow y \in \nabla. \tag{1.10}$$

Let $\mathbf{Ti} = C(\mathbf{Hi})$ be as described above. Let $\nabla_{C(h)}$ be the filter in $\mathbf{T1}$ generated by ∇_h, and $\sim_{C(h)}$ the respective congruence on $\mathbf{T1}$ (relative pseudo-complement $x \hookrightarrow y$ becoming relative complement, i.e., $x \sim_{C(h)} y \Leftrightarrow (-x \cup y) \in \nabla_{C(h)} \wedge (-y \cup x) \in \nabla_{C(h)}$). We let $q : \mathbf{T1} \to \mathbf{T1}/_{\nabla_{C(h)}} = \mathbf{T1}'$ be the quotient mapping.

1. Assume first that h is onto. For the moment, consider only the boolean part of all involved algebras $\mathbf{T1}, C(\mathbf{H1}/_{\nabla_h})$, etc.

Let $\nabla_{C(h)}$ be the filter in $\mathbf{T1}$ generated by ∇_h, i.e., one conisting of all elements b for which there exist some $y_1, ..., y_n \in \nabla_h$ with $y_1 \cap ... \cap y_n \subseteq b$. We have that $\forall z \in T1$:

$$\mathbf{i}(z) \in \nabla_h \ \Leftrightarrow \ z \in \nabla_{C(h)}. \tag{1.11}$$

$\Rightarrow$ is obvious since $\nabla_h \subseteq \nabla_{C(h)}$ and $\mathbf{i}(z) \subseteq z$, while $\Leftarrow$ follows since $z \in \nabla_{C(h)}$ implies that $\bigcap_i^n y_i \subseteq z$ for some $y_1, ..., y_n \in \nabla_h$, but then also $\bigcap_i^n y_i \subseteq \mathbf{i}(z)$, and hence $\mathbf{i}(z) \in \nabla_h$,

In particular, this gives $\forall a, b \in H1 : \mathbf{i}(-a \cup b) = a \hookrightarrow b \in \nabla_h \Leftrightarrow (-a \cup b) \in \nabla_{C(h)}$, so $\forall a, b \in H1$:

$$a \sim_h b \ \Leftrightarrow \ a \sim_{C(h)} b. \tag{1.12}$$

[1] There are more specific conditions for the canonical elements to which we will return shortly.

This will be used to show the commutativity of the following diagram, where we view Cs simply as inclusions ($q : \mathbf{T1} \to \mathbf{T1}/_{\nabla_{C(h)}}$ is the quotient morphism):

$$\begin{tikzcd}
\mathbf{H1} \arrow[r, hook, "C"] \arrow[dd, "h"'] & \mathbf{T1} \arrow[d, "q"] \\
& \mathbf{T1'} = \mathbf{T1}/_{\nabla_{C(h)}} \arrow[d, phantom, "\simeq"] \\
\mathbf{H1'} = \mathbf{H1}/_{\nabla_h} \arrow[r, hook, "C"] & C(\mathbf{H1'})
\end{tikzcd} \tag{1.13}$$

To show the claimed isomorphism, we also need the more specific conditions on the canonical elements in the boolean extension $\mathbf{T}$ of Heyting algebra $\mathbf{H}$, given in [7] and mentioned in footnote 1, and some of their consequences:

r1. every $b \in T$ has the form $\bigcap_1^n -a_i \cup b_i$ where for all $1 \leq i < n : b_{i+1} \sqsubseteq a_{i+1} \sqsubseteq b_i \sqsubseteq a_i$

r2. equivalently, each $b \in T$ can be written as $\bigcup_1^n a_i \cap -b_i$ with the restriction as above;

r3. given b as in **r2**, its complement $-b = \bigcup_1^{n+1} b_{i-1} \cap -a_i$ where $b_0 = \mathbf{1}$ and $a_{n+1} = \mathbf{0}$;

r4. $\bigcup_1^n a_i \cap -b_i \subseteq \bigcup_1^m c_j \cap -d_j$ iff $a_i \sqcap d_{j-1} \sqsubseteq b_i \sqcup c_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m+1$ (where we complete the representation of the rhs with the element $\mathbf{1} \cap -\mathbf{1}$ for index 0 and $\mathbf{0} \cap -\mathbf{0}$ for index $m+1$, i.e., so that $d_0 = \mathbf{1}$ and $c_{m+1} = \mathbf{0}$)

Let $x = \bigcup_1^n [a_i]^{\nabla_h} \cap -[b_i]^{\nabla_h}$ and $y = \bigcup_1^m [c_j]^{\nabla_h} \cap -[d_j]^{\nabla_h}$ be arbitrary elements in $C(\mathbf{H1'})$, with $a_i, b_i, c_j, d_j \in H1$.

$$\begin{array}{lll}
x \subseteq^{C(\mathbf{H1'})} y & \overset{\mathbf{r4}}{\iff} ([a_i]^{\nabla_h} \sqcap [d_{j-1}]^{\nabla_h} \sqsubseteq [b_i]^{\nabla_h} \sqcup [c_j]^{\nabla_h})^{\mathbf{H1'}} & \text{for respective } i, j \\
& \overset{(1.10)}{\iff} (a_i \sqcap d_{j-1} \hookrightarrow b_i \sqcup c_j)^{\mathbf{H1}} \in \nabla_h & \text{for the same } i, j \\
& \overset{(1.11)}{\iff} (-(a_i \cap d_{j-1}) \cup (b_i \cup c_j))^{\mathbf{T1}} \in \nabla_{C(h)} & \text{for the same } i, j \\
& \overset{(1.10)}{\iff} ([a_i \cap d_{j-1}]^{\nabla_{C(h)}} \subseteq [b_i \cup c_j]^{\nabla_{C(h)}})^{\mathbf{T1'}} & \text{for the same } i, j \\
& \overset{q\ homo}{\iff} ([a_i]^{\nabla_{C(h)}} \cap [d_{j-1}]^{\nabla_{C(h)}} \subseteq [b_i]^{\nabla_{C(h)}} \cup [c_j]^{\nabla_{C(h)}})^{\mathbf{T1'}} & \text{for the same } i, j \\
\iff \bigcup_1^n [a_i]^{\nabla_{C(h)}} \cap -[b_i]^{\nabla_{C(h)}} \subseteq^{\mathbf{T1'}} \bigcup_1^m [c_j]^{\nabla_{C(h)}} \cap -[d_j]^{\nabla_{C(h)}} & &
\end{array}$$

The last equivalence follows, assuming the restriction from **r2**, for any boolean algebra.

Since all elements of $C(\mathbf{H1'})$ have the form as just considered, while all elements of $\mathbf{T1'}$ the form as either side of $\subseteq^{\mathbf{T1'}}$ in the last line, this shows that $\forall x \in H1 : [x]^{\nabla_h} = [x]^{\nabla_{C(h)}}$ and so the correspondence

$$C(\mathbf{H1'}) \ni \bigcup_1^n [a_i]^{\nabla_h} \cap -[b_i]^{\nabla_h} \quad \leftrightarrow \quad \bigcup_1^n [a_i]^{\nabla_{C(h)}} \cap -[b_i]^{\nabla_{C(h)}} \in \mathbf{T1'} \tag{1.14}$$

is an isomorphism between the two boolean algebras. (Preservation of complements follows by **r3**, while of unions by the very representation of elements in **r2** as unions; these two conditions are sufficient.) By this isomorphism, we can identify the boolean parts of both algebras, $C(\mathbf{H1'}) = \mathbf{T1'}$, which yields also that $\forall x \in H1 : h(x) = q(x) = C(h)(x)$.

So far, we have not addressed the interior operator in $\mathbf{T1'}$. By (1.11), $\nabla_{C(h)}$ is an **i**-filter ($\forall z : z \in \nabla_{C(h)} \Rightarrow \mathbf{i}(z) \in \nabla_{C(h)}$), and hence q is a toplogical algebraic homomorphism, i.e., also $q(\mathbf{i}(x))^{\mathbf{T1}} = \mathbf{i}(q(x))^{\mathbf{T1'}}$ (e.g., theorem III.12.1 in [11]).

By the above identification, showing that $\mathbf{i}([x])^{\mathbf{T1'}} = \mathbf{i}([x])^{C(\mathbf{H1'})}$, will complete the proof of the isomorphism. But this follows easily, since the above correspondence can be written equivalently, using representation from **r1**, as

$$C(\mathbf{H1'}) \ni \bigcap_1^n -[a_i]^{\nabla_h} \cup [b_i]^{\nabla_h} \quad \leftrightarrow \quad \bigcap_1^n -[a_i]^{\nabla_{C(h)}} \cup [b_i]^{\nabla_{C(h)}} \in \mathbf{T1'}.$$

$$\begin{array}{rcll}
\mathbf{i}(\bigcap_1^n -[a_i]^{\nabla_h} \cup [b_i]^{\nabla_h})^{C(\mathbf{H1'})} & = & \mathbf{i}(\bigcap_1^n -[a_i]^{\nabla_{C(h)}} \cup [b_i]^{\nabla_{C(h)}})^{\mathbf{T1'}} & [x]^{\nabla_h} = [x]^{\nabla_{C(h)}} \\
& = & \mathbf{i}(q(\bigcap_1^n -a_i \cup b_i))^{\mathbf{T1'}} &
\end{array}$$

2. Consider now a diagram like (1.13) with injective homomorphism $i_h : \mathbf{H1'} \to \mathbf{H2}$.

The respective congruences are now identities, and extending i_h to i by $i(\bigcap_1^n -a_j \cup b_j) = \bigcap_1^n -i_h(a_j) \cup i_h(b_j)$ yields an injective boolean homomorphism. The image $i_h(\mathbf{H1}')$ is a subalgebra of $\mathbf{H2}$; hence also $C(i_h(\mathbf{H1}'))$ is a subalgebra of $C(\mathbf{H2})$. Applying the previous argument to $\mathbf{H1}'$ and $i_h(\mathbf{H1}')$, gives an injective (since $\nabla_h = \{\mathbf{1}\}$ so $\nabla_{C(h)} = \mathbf{1}$) toplogical algebra homomorphism $\mathbf{T1}' \to C(i_h(\mathbf{H1}'))$, and inclusion into $C(\mathbf{H2})$ the desired $C(i_h) : C(\mathbf{H1}') \to C(\mathbf{H2})$. Again, $\forall x \in H1' : i_h(x) = C(i_h)(x)$.

3. Given an arbitrary homomorphism $h : \mathbf{H1} \to \mathbf{H2}$, we have a factorization $h = q_h; i_h$

$$\mathbf{H1} \xrightarrow{q_h} \mathbf{H1}/_{\nabla_h} \xhookrightarrow{i_h} \mathbf{H2}$$

Then $C(h) = C(q_h); C(i_h) : C(\mathbf{H1}) \to C(\mathbf{H2})$ is a toplogical algebra homomorphism which coincides with h for all (opens) $x \in H1$.

4. C preserves compositions of morphisms. For, given $h : \mathbf{H} \to \mathbf{H1}$ and $g : \mathbf{H1} \to \mathbf{H2}$, with the respective filters ∇_h, ∇_g in $\mathbf{H}, \mathbf{H1}$, the composite $h; g$ is obtained from the filter in $\mathbf{H} : \nabla_{h;g} = \bigcup_{[x]^{\nabla_h} \in \nabla_g} [x]^{\nabla_h} \supseteq \nabla_h$. So

$$\begin{aligned} C(h); C(g)(\textstyle\bigcap_1^n -a_i \cup b_i)^{\mathbf{T}} &= C(g)(\textstyle\bigcap_1^n -[a_i]^{\nabla_h} \cup [b_i]^{\nabla_h})^{\mathbf{T1}} \\ &= (\textstyle\bigcap_1^n -[[a_i]^{\nabla_h}]^{\nabla_g} \cup [[b_i]^{\nabla_h}]^{\nabla_g})^{\mathbf{T2}} \\ &= (\textstyle\bigcap_1^n -[a_i]^{\nabla_{h;g}} \cup [b_i]^{\nabla_{h;g}})^{\mathbf{T2}} \\ &= C(h; g)(\textstyle\bigcap_1^n -a_i \cup b_i)^{\mathbf{T}}. \end{aligned}$$

One verifies easily that $C(id_{\mathbf{H}}) = id_{C(\mathbf{H})}$ and so $C : \mathcal{HA} \to \mathcal{TA}$ is a functor.

Inspecting the construction of $C(h)$ for a given $h : \mathbf{H} \to \mathbf{H1}$, one observes that restricting $C(h)$ to $\mathbf{H}$ gives h (i.e., for every (open) $x \in H : C(h)(x) = h(x)$). That is, also for homomorphisms we have that $I(C(h)) = h$. □

Remark 1.15 *Observe that the proof entails a stronger claim than stated in the theorem. Namely,*

obs1. *Not only is C strongly persistent (i.e., $C; I = ID_{\mathcal{HA}(\Gamma)}$), but also the morphism $C(h) : C(\mathbf{H}) \to C(\mathbf{H1})$ is unique such that it coincides with h on the sublattice $\mathbf{H}$ of $C(\mathbf{H})$, i.e., unique such that $I(C(h)) = h$.*

obs2. *$C : \mathcal{HA} \to \mathcal{TA}$ is full and faithful.*

obs3. *When h is surjective/injective then $C(h)$ is surjective/injective.*

obs4. *$I : \mathcal{TA} \to \mathcal{HA}$ is surjective on objects (and full).*

This tight correspondence is strengthened even further by the following result.

Theorem 1.16 *The functors $C : \mathcal{HA} \to \mathcal{TA}$ and $I : \mathcal{TA} \to \mathcal{HA}$ are adjoint, $C \dashv I$, with unit being identity.*

Proof. Given an $\mathbf{H} \in \mathcal{HA}$ and $\mathbf{T1} \in \mathcal{TA}$, and a morphism $h : \mathbf{H} \to I(\mathbf{T1})$, we have to show existence of a unique morphism $g : C(\mathbf{H}) \to \mathbf{T1}$ such that $I(g) = h$ (this simplification obtains since we have $C; I = ID_{\mathcal{HA}(\Gamma)}$, i.e., the unit of adjunction will be identity.)

Denote $\mathbf{H1} = I(\mathbf{T1})$ and $\mathbf{T} = C(\mathbf{H})$. Since $\mathbf{H1}$ is a sublattice of $\mathbf{T1}$ so there exists an element $\bigcup_1^n a_i \cap -b_i \in T1$ for every (finite) combination of $a_i, b_i \in H1$. Define the mapping $g : \mathbf{T} \to \mathbf{T1}$ by

$$g(\bigcup_1^n a_i \cap -b_i)^{\mathbf{T}} = (\bigcup_1^n h(a_i) \cap -h(b_i))^{\mathbf{T1}}. \tag{1.17}$$

In particular, $\forall x \in H : g(x) = h(x)$. To verify that it is indeed a homomorphism, we show first that it preserves the ordering, $\forall a_i, b_i, a_j, b_j \in H$:

$$\begin{array}{l} (\bigcup_1^n a_i \cap -b_i)^{\mathbf{T}} = x \ \subseteq \ y = (\bigcup_1^m c_j \cap -d_j)^{\mathbf{T}} \quad \Rightarrow \\ \qquad (\bigcup_1^n h(a_i) \cap -h(b_i))^{\mathbf{T1}} = g(x) \ \subseteq \ g(y) = (\bigcup_1^m h(c_j) \cap -h(d_j))^{\mathbf{T1}} \end{array} \tag{1.18}$$

The proof of this fact uses the restricted representation **r2** of $\mathbf{T}$-elements and is based on the characterisation of $\subseteq^{\mathbf{T}}$ in terms of $\sqsubseteq^{\mathbf{H}}$ given in **r4** on page 4. (It is essentially the same as the one following that characterisation.)

$$
\begin{array}{llll}
x \subseteq^{\mathbf{T}} y & \overset{\mathbf{r4}}{\iff} & a_i \sqcap d_{j-1} \sqsubseteq^{\mathbf{H}} b_i \sqcup c_j & \text{for respective } i,j \\
& \overset{h\ homo}{\Rightarrow} & h(a_i) \sqcap h(d_{j-1}) \sqsubseteq^{\mathbf{H1}} h(b_i) \sqcup h(c_j) & \text{for the same } i,j \\
& \overset{\mathbf{H1}\ sublattice\ \mathbf{T1}}{\iff} & h(a_i) \cap h(d_{j-1}) \subseteq^{\mathbf{T1}} h(b_i) \cup h(c_j) & \text{for the same } i,j \\
& \overset{\mathcal{BA}}{\iff} & \bigcup_1^n h(a_i) \cap -h(b_i) \subseteq^{\mathbf{T1}} \bigcup_1^m h(c_j) \cap -h(d_j) & \text{for the same } i,j \\
& \overset{(1.17)}{\iff} & g(x) \subseteq^{\mathbf{T1}} g(y) &
\end{array}
$$

The equivalence marked $\mathcal{BA}$ holds for all boolean algebras, assuming the restriction **r2**.

Homomorphism condition follows now easily. For preservation of complements, we use again the restricted representation **r2** of **T**-elements and the characterisation **r3** of complement from page 4:

$$
\begin{array}{llll}
g(-\bigcup_1^n a_i \cap -b_i)^{\mathbf{T}} & \overset{\mathbf{r3.}}{=} & g(\bigcup_1^{n+1} b_{i-1} \cap -a_i)^{\mathbf{T}} & b_0 = \mathbf{1},\ a_{n+1} = \mathbf{0} \\
& \overset{(1.17)}{=} & (\bigcup_1^{n+1} h(b_{i-1}) \cap -h(a_i))^{\mathbf{T1}} & \\
& \overset{\mathcal{BA}}{=} & (-\bigcup_1^n h(a_i) \cap -h(b_i))^{\mathbf{T1}} & \\
& = & (-\bigcup_1^n g(a_i) \cap -g(b_i))^{\mathbf{T1}} & \forall x \in H : g(x) = h(x)
\end{array}
$$

where the equation marked $\mathcal{BA}$ holds for all boolean algebras, when the respective inclusions, required by **r2**, hold. But as they hold for $a_i, b_i \in \mathbf{T}$, they also hold for $g(a_i/b_i) = h(a_i/b_i)$ in **T1** by (1.18). Preservation of unions follows by the representation of elements in **T** (as unions).

To complete the proof that g is a homomorphism of toplogical algebras, we show that it also preserves the interior operator. Now, (1.9) holds for any toplogical algebra **T1** and the lattice of its open sets $\mathbf{H1} = I(\mathbf{T1})$, i.e., $\forall a, b \in H1 : \mathbf{i}(-a \cup b) = a \hookrightarrow b$ (e.g., theorem IV.1.4 in [11]). We therefore obtain, $\forall a_i, b_i \in H$:

$$
\begin{array}{lll}
g(\mathbf{i}(\bigcap_1^n -a_i \cup b_i)^{\mathbf{T}}) & \overset{(1.9)}{=} & g((\bigcap_1^n a_i \hookrightarrow b_i)^{\mathbf{H}}) \\
& = & h((\bigcap_1^n a_i \hookrightarrow b_i)^{\mathbf{H}}) \\
& \overset{h\ homo}{=} & (\bigcap_1^n h(a_i) \hookrightarrow h(b_i))^{\mathbf{H1}} \\
& = & (\bigcap_1^n h(a_i) \hookrightarrow h(b_i))^{\mathbf{T1}} \\
& \overset{(1.9)}{=} & \mathbf{i}(\bigcap_1^n -h(a_i) \cup h(b_i))^{\mathbf{T1}}) \\
& \overset{(1.17)}{=} & \mathbf{i}(g((\bigcap_1^n -a_i \cup b_i)^{\mathbf{T}}))
\end{array}
$$

Restriction of (1.17) to elements of **H** makes it obvious that $I(g) = h$. To complete the proof of adjointness, we show uniqueness of g. But this follows trivially, since g is induced by h. Any other homomorphism g' which coincides with g on all elements of **H** will also coincide there with h, and hence will have to satisfy (1.17). □

To appreciate the meaning of this adjunction, and in particular of its unit being the identity, compare this to the possibility of a similar relation between (classical) boolean algebras $\mathcal{BA}$ and $\mathcal{TA}$. There is the obvious forgetful functor $U : \mathcal{TA} \to \mathcal{BA}$, which simply forgets the existence of **i**. There is, however, a multiplicity of possible definitions of a functor $T : \mathcal{BA} \to \mathcal{TA}$, since there are many ways of adding the interior operator **i** to a boolean algebra (e.g., $\mathbf{i}(x) = x$, $\mathbf{i}(x) = \mathbf{1}$ for $x \neq \mathbf{0}$ and $\mathbf{i}(\mathbf{0}) = \mathbf{0}$, etc.). None of such definitions will yield an adjunction $T \dashv U$ if we, at the same time, want to obtain the identity on objects $U(T(\mathbf{B})) = \mathbf{B}$. To obtain an adjunction, we have to let $T(\mathbf{B})$ be a free extension of $\mathbf{B}$, i.e., an algebra with freely added elements $\mathbf{i}(x)$ for all $x \in \mathbf{B}$, then freely closed under all operations of $\mathcal{TA}$ and, finally, quotiented by the congruence induced by the equations axiomatizing toplogical algebras. This will not be a (strongly) persistent extension, but should still yield a conservative extension, in the sense that we should obtain, as the unit of adjunction, an inclusion $\iota_{\mathbf{B}} : \mathbf{B} \to U(T(\mathbf{B}))$ for every $\mathbf{B} \in \mathcal{BA}$.

2 Embedding CL into $\mathsf{S4}$

We consider the syntax of constructive propositional logic CL and modal logic $\mathsf{S4}$ over a fixed alphabet X of propositional variables. McKinsey-Tarski embedding of the sytax is given by:

$$
\begin{array}{rcl}
\mathsf{CL} & \mapsto & \mathsf{S4} \\
\hline
a \in X : tr(a) & = & \Box a \\
tr(\phi_1 \wedge \phi_2) & = & tr(\phi_1) \wedge tr(\phi_2) \\
tr(\phi_1 \vee \phi_2) & = & tr(\phi_1) \vee tr(\phi_2) \\
tr(\phi_1 \to \phi_2) & = & \Box(tr(\phi_1) \to tr(\phi_2))
\end{array}
\tag{2.1}
$$

Models of CL are Heyting algebras and of $\mathsf{S4}$ toplogical algebras. Satisfaction relation is given by the obvious extension of an assignment $v : X \to M$ to formulae and by the standard condition $\mathbf{M} \models_v \phi \iff v(\phi) = \mathbf{1}$. Explicitly:

$$
\begin{array}{rcl|rcl}
\multicolumn{3}{l|}{\mathcal{HA} \ni \mathbf{H} \models_v \phi \text{ iff :}} & \multicolumn{3}{l}{\mathcal{TA} \ni \mathbf{T} \models_w \phi \text{ iff :}} \\
\hline
a & \Leftrightarrow & v(a) = \mathbf{1} & a & \Leftrightarrow & w(a) = \mathbf{1} \\
\phi_1 \wedge \phi_2 & \Leftrightarrow & v(\phi_1) \sqcap v(\phi_2) = \mathbf{1} & \phi_1 \wedge \phi_2 & \Leftrightarrow & w(\phi_1) \cap w(\phi_2) = \mathbf{1} \\
\phi_1 \vee \phi_2 & \Leftrightarrow & v(\phi_1) \sqcup v(\phi_2) = \mathbf{1} & \phi_1 \vee \phi_2 & \Leftrightarrow & w(\phi_1) \cup w(\phi_2) = \mathbf{1} \\
\phi_1 \to \phi_2 & \Leftrightarrow & v(\phi_1) \hookrightarrow v(\phi_2) = \mathbf{1} & \phi_1 \to \phi_2 & \Leftrightarrow & -w(\phi_1) \cup w(\phi_2) = \mathbf{1} \\
 & & & \Box\phi_1 & \Leftrightarrow & \mathbf{i}(w(\phi_1)) = \mathbf{1}
\end{array}
$$

The classical result concerning this embedding states that $\Gamma \models_{\mathsf{CL}} \phi \iff tr(\Gamma) \models_{\mathcal{TA}} tr(\phi)$. However, a stronger relation obtains from which also this result follows.

Following the suggestions from [1], one introduced the concept of institution, [3], as a general semantic concept of a logical system. We could expand our definitions to view both CL and $\mathsf{S4}$ as institutions, but this would only require a lot of bureaucracy. We therfore restrict our attention to the essential aspect which can be presented as if the two logics (for a given alphabet X) belonged to the same institution. The essential aspect of the definition is the so called satisfaction condition (translation condition in [1]) which amounts to the requirement that satisfaction relation remains invariant under translation of the syntax. In our particular situation, the condition reads as follows:

$$
\forall \phi \in \mathsf{CL}\ \forall \mathbf{T} \in \mathcal{TA} : I(\mathbf{T}) \models \phi \iff \mathbf{T} \models tr(\phi)
\tag{2.2}
$$

Intuitively, an embedding of CL into $\mathsf{S4}$ consists of a pair $E = (tr, I)$, where tr is the translation (inclusion) of CL-formulae into $\mathsf{S4}$-formulae, while I is a functor which from any $\mathcal{TA}$-algebra recovers the part corresponding to the CL-syntax, namely, a Heyting algebra. We have the following diagram, and the satisfaction condition connects a formula $\phi \in \mathsf{CL}$ and a toplogical algebra $\mathbf{T}$:

$$
\begin{tikzcd}
\mathsf{CL} \arrow[rr, "E"] & & \mathsf{S4} \\
\phi \arrow[rr, "tr"] \arrow[d, dotted, dash, "\models_{\mathcal{HA}}"'] & & tr(\phi) \arrow[d, dotted, dash, "\models_{\mathcal{TA}}"] \\
I(\mathbf{T}) & & \mathbf{T} \arrow[ll, "I"']
\end{tikzcd}
$$

$$
I(\mathbf{T}) \models \phi \qquad \iff \qquad \mathbf{T} \models tr(\phi) \tag{2.2}
$$

Verification of the condition presents no serious difficulties.

PROOF. We show a stronger statement. Given an assignment $w : X \to \mathbf{T}$, we define $\overline{w} : X \to I(\mathbf{T})$ as $\overline{w}(x) = \mathbf{i}(w(x))$ for all $x \in X$. We show that $\forall \phi \in \mathsf{CL}\ \forall \mathbf{T} \in \mathcal{TA}\ \forall w : w(tr(\phi)) = \overline{w}(\phi)$, by induction on ϕ. (This is (5) in the proof of theorem XI.8.6 from [11], which states equivalence of validity of CL-formualae in $\mathcal{HA}$ and their translations in $\mathcal{TA}$. (2.2) is a more general condition with wider consequences.)

Let $w : X \to \mathbf{T}$ be arbitrary. For atomic $\phi = a : w(tr(a)) = \mathbf{i}(w(a)) = \overline{w}(a)$. Induction passes trivially through $\wedge, \vee$, and for $\to$ we get $w(tr(\phi_1 \to \phi_2)) = w(\Box(tr(\phi_1) \to tr(\phi_2))) = \mathbf{i}(-w(tr(\phi_1)) \cup w(tr(\phi_2))) \overset{IH}{=} \mathbf{i}(-\overline{w}(\phi_1) \cup \overline{w}(\phi_2)) \overset{(1.9)}{=} \overline{w}(\phi_1) \hookrightarrow \overline{w}(\phi_2) = \overline{w}(\phi_1 \to \phi_2)$.

The main claim follows. Assume $\mathbf{T} \models tr(\phi)$ and let $v : X \to I(\mathbf{T})$ be arbitrary. But since $I(T) \subseteq T$, so v can be obtained as a $\overline{w}$ for some $w : X \to \mathbf{T}$, so the claim follows by assumption and $w(tr(\phi)) = \overline{w}(\phi)$. For the converse, assume $I(\mathbf{T}) \models \phi$ and let $w : X \to \mathbf{T}$ be arbitrary. The claim follows now by assumption since $\overline{w}(\phi) = w(tr(\phi))$ for any w. □

Putting a requirement on each particular model, this establishes a tighter relation between the logics than the classical embedding which merely ensures preservation and reflection of validity, [10].

For any (class of) CL-formulae Γ and the class $\mathcal{HA}(\Gamma)$ of Heyting algebras which are its models, on the one hand and, on the other hand, the translation $tr(\Gamma)$ and the class $\mathcal{TA}(tr(\Gamma))$ of its toplogical algebraic models, we obtain the restriction of our functors I, C with the following specialization of the adjunction from theorem 1.16 to the respective model classes (which is standard property of an institution, following from (2.2) whenever the functor corresponding to our C is persistent.)

Theorem 2.3 *For every (class of) CL-formulae Γ, the functors $C : \mathcal{HA}(\Gamma) \to \mathcal{TA}(tr(\Gamma))$ and $I : \mathcal{TA}(tr(\Gamma)) \to \mathcal{HA}(\Gamma)$ are adjoint, $C \dashv I$, with unit being identity.*

PROOF. By (2.2), if $\mathbf{H} \models \phi$ then $C(\mathbf{H}) \models tr(\phi)$ (since C is persistent, i.e., $I(C(\mathbf{H})) = \mathbf{H}$). On the other hand, again by (2.2), when $\mathbf{T} \models tr(\phi)$ then $I(\mathbf{T}) \models \phi$. That is, the functors can be considered as mapping the respective model classes into each other. The rest is proven as for 1.16. □

One sees easily that also the observations from remark 1.15 still hold, when the formulations are restricted to the model classes $\mathcal{HA}(\Gamma), \mathcal{TA}(tr(\Gamma))$. In particular, the functor $I : \mathcal{TA}(tr(\Gamma)) \to \mathcal{HA}(\Gamma)$ is surjective on objects, **obs4**.

As a simple corollary we obtain then the classical result about the translation $tr(\_)$ preserving and reflecting validity of CL-formulae, e.g., [10], theorem 5.1 (with provability replaced by validity). But we also obtain a stronger consequence, namely, preservation and reflection of semantic consequence. The latter is defined by:

$$\Gamma \models_{\mathcal{K}} \phi \iff \mathcal{K}(\Gamma) \models_{\mathcal{K}} \phi \tag{2.4}$$

where we instantiate $\mathcal{K}$ either to $\mathcal{HA}$ or $\mathcal{TA}$, and $\mathcal{K}(\Gamma) = \{M \in \mathcal{K} \mid M \models_{\mathcal{K}} \Gamma\}$.

Corollary 2.5 *The following equivalences hold:*

1. $\models_{\mathcal{HA}} \phi \Leftrightarrow \models_{\mathcal{TA}} tr(\phi)$ *(5.1, [10])*
2. $\Gamma \models_{\mathcal{HA}} \phi \Leftrightarrow tr(\Gamma) \models_{\mathcal{TA}} tr(\phi)$ *(XI.8.6, [11])*

PROOF. As 1 is a special case of 2, we verify the latter. Assume $\mathcal{HA}(\Gamma) \models \phi$ and let $T \in \mathcal{TA}(tr(\Gamma))$ be arbitrary. Then $I(T) \in \mathcal{HA}(\Gamma)$ by 2.3 and so $T \models tr(\phi)$ by the assumption $I(T) \models \phi$ and (2.2).

Conversely, assume $\mathcal{TA}(tr(\Gamma)) \models tr(\phi)$ and let $H \in \mathcal{HA}(\Gamma)$ be arbitrary. By **obs4**, there is a $T \in \mathcal{TA}(tr(\Gamma))$ such that $H = I(T)$. But then the assumption $T \models tr(\phi)$ and (2.2) imply that $H \models \phi$. □

In other words, the extension of an CL-theory Γ to its S4-image $tr(\Gamma)$ is conservative. Notice, however, that this relation holds pointwise for every model, i.e., we can strengthen the above to the following:

Corollary 2.6 *For every $\mathbf{H} \in \mathcal{HA}$ and $\phi \in$ CL $: \mathbf{H} \models_{\mathcal{HA}} \phi \iff C(\mathbf{H}) \models_{\mathcal{TA}} tr(\phi)$.*

PROOF. $C(\mathbf{H}) \models_{\mathcal{TA}} tr(\phi) \Leftrightarrow I(C(\mathbf{H})) \models_{\mathcal{HA}} \phi$, but by persistency of C, $I(C(\mathbf{H})) = \mathbf{H}$. □

As another corollary, of theorem 1.16 and **obs3**, we obtain for instance theorems 1.17 and 1.18 from [9].

Corollary 2.7 *(1.17) For any $\mathbf{H} \in \mathcal{HA}$, $\mathbf{T1} \in \mathcal{TA}$, if $\mathbf{H}$ is a subalgebra of $I(\mathbf{T1})$ then $C(\mathbf{H})$ is a subalgebra of $\mathbf{T1}$.*

(1.18) If $\mathbf{H} \simeq \mathbf{H}'$, then $C(\mathbf{H}) \simeq C(\mathbf{H}')$ (and the latter isomorphism is uniquely determined by the former).

The adjunction from theorem 2.3 establishes a kind of canonicity, or freeness, of the extension from CL to S4. The functor C from theorem 1.6 yields not only some boolean algebra related vaguely to the source Heyting algebra, but a free such algebra. (MacNeille observed that all relations of the extension are "required" by the extended algebra, while Tarski/McKinsey showed that the extension is "minimal". Both these observations are captured by the adjunction result.) This adjunction finds its real use in the semantic considerations. Thus, for instance, as left adjoints preserve colimits and the right ones limits, we can obtain coproducts in $\mathcal{TA}$ as C-images of coproducts in $\mathcal{HA}$ and, on the other hand, products in $\mathcal{HA}$ as I-images of products in $\mathcal{TA}$. As such issues are not in our current focus, we leave them aside.

3 $\mathcal{IC}$-algebras as models of S4

For every Heyting algebra $\mathbf{H}$ and $x \in H$, we obtain in the extension $\mathbf{T} = C(\mathbf{H})$:

$$\mathbf{c}(x) = -\mathbf{i}(-x) = -\mathbf{i}(-x \cup \mathbf{0}) = -(x \hookrightarrow \mathbf{0}) = - \div x, \tag{3.1}$$

i.e., the introduced operator of closure/interior is just a combination of the constructive and boolean negation. Thus the constructive negation survives the embedding, albeit in a disguised and hidden form.

Seen from the opposite direction: the closure/interior operator of any topological algebra $\mathbf{T}$ contains an aspect of constructive negation which, in fact, is what makes the straightforward reduction of such algebras to Heyting algebras possible when defininig the functor I in theorem 1.5. (3.1) gives also the dual fact:

$$\mathbf{i}(x) = \div - x \tag{3.2}$$

The apparent "problem" is that $\div$ is defined only over $\mathbf{H}$, while we want to use this, or (3.1), for arbitrary elements of our algebra. This is only apparent, since in any $\mathcal{TA}$-algebra, we can define the $\div$ operation by:

$$\div x = \mathbf{i}(-x) \tag{3.3}$$

Also, in any $\mathcal{TA}$-algebra we can define Heyting arrow:

$$x \hookrightarrow y = \mathbf{i}(-x \cup y) = \div - (-x \cup y). \tag{3.4}$$

This suggests the possibility of combining in one structure the boolean and constructive elements according to the following definition.

Definition 3.5 *Consider $\mathcal{IC}$-algebras ("intuitionistic-classical") $\langle C; \cup, \cap, -, \div\rangle$ where $\langle C; \cup, \cap, -\rangle$ is a boolean algebra, and a unary operation $\div$ (constructive negation) satisfies the following axioms:*

s1. $\div x \subseteq -x$
s2. $\div x = \div - \div x$
s3. $\div(x \cup y) = \div x \cap \div y$
s4. $\div \mathbf{0} = \mathbf{1}$

Trivially, we can convert every $\mathcal{TA}$-algebra into such an $\mathcal{IC}$-algebra using (3.3), while an $\mathcal{IC}$-algebra can be converted into a $\mathcal{TA}$-algebra using (3.2). This last claim follows by the choice of the axioms **s**, since the formulations of **s1**, **s2** are equivalent to those in **i1** and **i2** (see **l1**, **l2** below). **s4** is trivially equivalent to **i4**, and **s3** to **i3**.

3.1 Some tautologies

l1. **s1** $\Leftrightarrow$ **i1**, i.e., $\div x \subseteq -x \iff \div - x \subseteq x$
l2. **s2** $\Leftrightarrow$ **i2**, i.e., $\div x = \div - \div x \iff \div - x = \div - \div - x$
l3. $\div \mathbf{1} = \mathbf{0}$
l4. $x \subseteq y \Rightarrow \div x \supseteq \div y$
l5. $\div - (\div x \cup \div y) = \div x \cup \div y$
l6. $\div x = x \hookrightarrow \mathbf{0}$ using (3.4) as definition of $\hookrightarrow$:
l7. $x \cap \div x = \mathbf{0}$
l8. $x \subseteq \div - x \stackrel{\mathbf{l1}}{\Longrightarrow} x = \div - x$

l9. $(x_1 \hookrightarrow y) \cap (x_2 \hookrightarrow y) = (x_1 \cup x_2) \hookrightarrow y$
l10. $a \cap x \subseteq b \Leftarrow a \subseteq x \hookrightarrow b = \div - (-x \cup b) = \div(x \cap -b)$
l11. $a \cap x \subseteq b \Rightarrow a \subseteq x \hookrightarrow b$ when (*) $a = \div a'$ (in particular, when $a = \div - a$)
l12. $x \subseteq \div \div x$ when (*) $x = \div x'$ (for all $x : x \subseteq - \div x$)
l13. $\div - x \subseteq \div \div x$
l14. $\div x \cup \div y \subseteq \div(x \cap y)$
l15. $\div \div x = \div \div \div \div x$
l16. $- \div \div - x \subseteq - \div x$.
l17. $\div x = \mathbf{1} \Rightarrow x = \mathbf{0}$
l18. $\div x = \mathbf{0} \not\Rightarrow x = \mathbf{1}$.

We prove these statements below:

pl1. $\div(-x) \overset{\mathbf{s1}}{\subseteq} -(-x) = x$ – and conversely: $\div - x \subseteq x \Rightarrow \div - (-x) \subseteq -x$, i.e., $\div x \subseteq -x$
pl2. **s2** $\Leftrightarrow$ **i2** (where **i2** is $\div - x = \div - \div - x$):

s2 $\Rightarrow$ **i2** is obvious; for the opposite simplify $\div - (-x) \overset{\mathbf{i2}}{=} \div - \div - (-x)$ to **s2**.

pl3. $\div \mathbf{1} = \mathbf{0}$ since: $\div \mathbf{1} \overset{\mathbf{s1}}{\subseteq} -\mathbf{1} = \mathbf{0} \Rightarrow \div \mathbf{1} = \mathbf{0}$
pl4. $x \subseteq y \Rightarrow \div x \supseteq \div y$

$x \subseteq y \iff x \cup y = y \Longrightarrow \div(x \cup y) = \div y \overset{\mathbf{s3}}{\iff} \div x \cap \div y = \div y$, i.e., $\div x \supseteq \div y$

pl5. $\subseteq$ follows from **l1**, and the opposite inclusion is shown as follows:

$$\begin{aligned}
&\div - (\div x \cup \div y) \cap (\div x \cup \div y) \quad = \\
& \qquad \div - (\div x \cup \div y) \cap \div x \quad \cup \quad \div - (\div x \cup \div y) \cap \div y \\
\overset{\mathbf{s2}}{=} & \qquad \div - (\div x \cup \div y) \cap \div - \div x \quad \cup \quad \div - (\div x \cup \div y) \cap \div - \div y \\
\overset{\mathbf{s3}}{=} & \qquad \div - ((\div x \cup \div y) \cap \div x) \quad \cup \quad \div - ((\div x \cup \div y) \cap \div y) \\
= & \qquad \div - ((\div x \cap \div x) \cup (\div y \cap \div x)) \quad \cup \quad \div - ((\div x \cap \div y) \cup (\div y \cap \div y)) \\
= & \qquad \div - (\div x \cup (\div y \cap \div x)) \quad \cup \quad \div - ((\div x \cap \div y) \cup \cap \div y) \\
= & \qquad \div - (\div x) \quad \cup \quad \div - (\div y) \\
\overset{\mathbf{s2}}{=} & \qquad \div x \quad \cup \quad \div y
\end{aligned}$$

pl6. $x \hookrightarrow \mathbf{0} \overset{(3.4)}{=} \div - (-x \cup \mathbf{0}) = \div - (-x) = \div x$
pl7. $x \cap -x = \mathbf{0} \overset{\mathbf{s1}}{\Longrightarrow} x \cap \div x = \mathbf{0}$
pl8. $x \subseteq \div - x \overset{\mathbf{l1}}{\Longrightarrow} x = \div - x$
pl9.
$$\begin{aligned}
lhs &\overset{(3.4)}{=} \div - (-x_1 \cup y) \cap \div - (-x_2 \cup y) \\
&\overset{\mathbf{s3}}{=} \div - ((-x_1 \cup y) \cap (-x_2 \cup y)) \\
&= \div - ((-x_1 \cap -x_2) \cup y) \quad = \quad \div - (-(x_1 \cup x_2) \cup y) \quad = \quad rhs
\end{aligned}$$

pl10.
$$\begin{aligned}
a \subseteq \div - (-x \cup b) &\iff a \cap (\div(x \cap -b)) = a \\
&\overset{\mathbf{s1}}{\Longrightarrow} a \cap (-(x \cap -b)) = a \\
&\iff a \cap (x \cap -b) = \mathbf{0} \quad \iff \quad a \cap x \subseteq b
\end{aligned}$$

pl11.
$$\begin{aligned}
a \cap x \subseteq b &\iff a \subseteq -(x \cap -b) \qquad\qquad - \text{ is } \cap\text{-complement} \\
&\iff -a \supseteq - - (x \cap -b) \\
&\iff -(\div a') \supseteq (x \cap -b) \\
&\overset{\mathbf{l4}}{\Longrightarrow} \div - \div a' \subseteq \div(x \cap -b) \\
&\overset{\mathbf{s2}}{\iff} \div a' \subseteq \div(x \cap -b) \quad \overset{(*)}{\iff} \quad a \subseteq \div(x \cap -b)
\end{aligned}$$

pl12. $-x \overset{\mathbf{s1}}{\supseteq} \div x \overset{\mathbf{l4}}{\Longrightarrow} \div - x \subseteq \div \div x \overset{(*)}{\iff} \div - \div x' \subseteq \div \div \div x' \overset{\mathbf{s2}}{\iff} \div x' \subseteq \div \div \div x'$

The inclusion typically fails when x is not open. E.g., for the usual topology on the real line, $[0,1] \not\subseteq (0,1) = \div((-\infty,0) \cup (1,+\infty)) = \div \div [0,1]$

pl13. $-x \overset{\mathbf{s1}}{\supseteq} \div x \overset{\mathbf{l4}}{\Longrightarrow} \div - x \subseteq \div \div x$.
pl14.
$$\begin{aligned}
(\div x \cup \div y) \cap (x \cap y) \subseteq \mathbf{0} \quad & \text{(by distributivity and } \mathbf{l7}) \\
\overset{\mathbf{l11+l5}}{\Longrightarrow} \quad & (\div x \cup \div y) \subseteq (x \cap y) \hookrightarrow \mathbf{0} \\
\overset{\mathbf{l6}}{\iff} \quad & (\div x \cup \div y) \subseteq \div(x \cap y)
\end{aligned}$$

The inclusion may be strict, e.g., let $\mathbf{1} = \mathsf{R}$ with the usual topology, $x = \mathsf{Q}, y = \mathsf{R} - \mathsf{Q}$; then $\div(x \cap y) = \mathbf{1} \neq \mathbf{0} = \div x \cup \div y$.

pl15. $\div\div x \overset{\mathbf{l12}}{\subseteq} \div\div\div\div x$, while $\div x \overset{\mathbf{l12}}{\subseteq} \div\div\div x \overset{\mathbf{l4}}{\Longrightarrow} \div\div x \supseteq \div\div\div\div x$

pl16. $\div - x \subseteq x \Longrightarrow - \div\div - x \subseteq - \div x$. I.e., $\mathbf{ci} \subseteq \mathbf{c}$, and the inclusion may be strict: $x = \mathsf{Q}$, then $\mathbf{i}(\mathsf{Q}) = \emptyset$, and so $\mathbf{c}(\mathbf{i}(\mathsf{Q})) = \emptyset \neq \mathsf{Q} = \mathbf{c}(\mathsf{Q})$.

pl17. $\div x = \mathbf{1} \Longrightarrow - \div x = \mathbf{0} \Longrightarrow x = \mathbf{0}$, since $x \subseteq - \div x$.

pl18. the dual of the above fails, of course, since dense elements, $\div x = \mathbf{0}$, need not be $\mathbf{1}$...; we have some standard things about them, e.g., each such element has the form $a \cup \div a$ (only if follows by taking $a = x$, while if using **s3**)...

As one would expect from topological spaces, every open element is a complement of a closed one and vice versa: open means to be of the form $\div x$, then $- \div x$ is closed, while $\div x$ is exactly complement of the latter. On the other hand, to be closed means to be of the form $- \div x$, which is $-(\div x)$, i.e., complement of an open $\div x$.

3.2 Relating tautologies

Lemmata **l** give only a few direct proofs of a vast variety of tautologies. We can easily conclude that the following hold in $\mathcal{IC}$-algebras (the respective restriction of the language is given to the right):

β) any boolean tautology $\qquad \beta ::= x \mid \beta \cap \beta \mid \beta \cup \beta \mid -\beta \mid \mathbf{0}$

θ) any topological/S4 tautology $\qquad \theta ::= x \mid \theta \cap \theta \mid \theta \cup \theta \mid -\theta \mid \div - \theta \mid \mathbf{0}$

γ) any constructive tautology $\qquad \gamma ::= \div - x \mid \gamma \cap \gamma \mid \gamma \cup \gamma \mid \div - (-\gamma \cup \gamma) \mid \mathbf{0}$

(In β and θ, we added $\mathbf{0}$ merely to ease comparison.) The restriction on the variables in γ simply makes sure that all constructive formulae address only the constructive/open elements of the algebras. (Equivalently, we might only require $\div x$.) For instance, **l15** would constructively be formulated with one $\div$ less. In our case, it acquires this additional $\div$, because the constructive tautology $\div x = \div\div\div x$, holds in our case for the open, but not necessarily for other elements. Similarly, the constructive tautology **l12** may fail when x is not open. On the other hand, some constructive tautologies survive in the unchanged form and can be applied to all elements, not only the open ones, e.g., **l10**, **l14**.

Validity of all (instances of) boolean tautologies follows since $\mathcal{IC}$-algebras are boolean algebras, and validity of all topological tautologies since they also are topological algebras, with interior written as $\div-$. Finally, validity of all constructive tautologies follows since, by the restriction on the variables which must be preceded by $\div-$, they address only open elements of an $\mathcal{IC}$-algebra, that is, only and all elements of its substructure which is Heyting algebra. (In the proofs, we may mark some transitions by γ, θ if we merely refer to a tautology from the respective logic.) McKinsey-Tarski embedding (2.1) turns out to be simply an inclusion of the sublanguage $L(\gamma) \subset L(\theta)$.

In addition, we also have tautologies which do not belong to any of the above sublanguages, for instance, **s1** or **s2**. We can view these as tautologies "connecting" the different sublanguages. In particular, they will allow us to recover some of the classical results relating the different logics involved and express them in the internal language of $\mathcal{IC}$.

3.2.1

One such result was given in corollary 2.5 based on the translation (2.1) $tr(\_) : L(\gamma) \to L(\theta)$. In the present case, there is actually no translation and $tr(\phi) = \phi$: CL is simply a syntactically identifiable fragment of S4. (Of course, so is BL, and this is the simplest possible proof of soundness of the rule including among S4-provable formulae all instances of propositional tautologies. This rule, present in all modal logics, reflects the fact that such logics relate to boolean algebras – with appropriate operators.)

3.2.2

Constructive logic emerges now as a syntactic subset of $\mathcal{IC}$-logic, inluding the constructive negation. The specificity of this logic appears thus to be the consequence of restricting the domain of interpretation (assignments to variables), which is reflected in the basic case of its grammar. It follows trivially (by **s2**, **s3** and **l5**) that all elements interpreting expressions of γ in an $\mathcal{IC}$-algebra are open.

As an example illustrating that we obtain "constructive" connectives by restricting attention to the "constructive" elements, we show the disjunction property:

Lemma 3.6 *The following implications hold (x may be a sequence of variables):*

1. *If $\mathcal{IC} \models -\div\phi_1(x) \cap -\div\phi_2(x) = \mathbf{0}$ then either $\mathcal{IC} \models \phi_1(x) = \mathbf{0}$ or $\mathcal{IC} \models \phi_2(x) = \mathbf{0}$.*
2. *If $\mathcal{IC} \models \div\phi_1(x) \cup \div\phi_2(x) = \mathbf{1}$ then either $\mathcal{IC} \models \div\phi_1(x) = \mathbf{1}$ or $\mathcal{IC} \models \div\phi_2(x) = \mathbf{1}$.*

PROOF. 1. This is theorem 4.12 from [8] ($-\div x = \mathbf{c}(x)$). It holds here because each $\mathcal{IC}$-algebra can be seen as a toplogical algebra (used in that theorem) and vice versa. In particular, any derived $\mathcal{IC}$-operator ϕ_i is expressible as a derived $\mathcal{TA}$-operator and vice versa.

The disjunction property 2 follows from 1. The assumption is equivalent to $\mathcal{IC} \models (-\div\phi_1(x)) \cap (-\div\phi_2(x)) = \mathbf{0}$. Then either $\mathcal{IC} \models \phi_1(x) = \mathbf{0}$ or $\mathcal{IC} \models \phi_2(x) = \mathbf{0}$, by 1. In either case, $\div\phi_i(x) = \mathbf{1}$ by **s4**. □

3.2.3

Since $\div$ is in our case the "switch" which brings an element over into the "constructive subuniverse", some classical results, in any case those involving double constructive negation, should obtain a natural – and internal – expression. For instance, we have:

$$-a = \mathbf{1} \iff \div a = \mathbf{1} \tag{3.7}$$

i.e., $-a = \mathbf{1} \Rightarrow a = \mathbf{0} \Rightarrow \div a = \mathbf{1}$, and the opposite holds since $\div a \subseteq -a$. This gives immediately the corollary:

$$\mathcal{IC} \models -\phi(x) = \mathbf{1} \iff \mathcal{IC} \models \div\phi(x) = \mathbf{1}. \tag{3.8}$$

But it is not exactly the classical theorem saying that, $\mathcal{IC} \models -\phi(x) = \mathbf{1}$ (where ϕ is a boolean derived operator) iff $\mathcal{IC} \models \div\phi'(x) = \mathbf{1}$ (where ϕ' is an appropriate translation into constructive connectives).

Likewise

$$a = \mathbf{1} \overset{\mathbf{l3}}{\Longrightarrow} \div a = \mathbf{0} \overset{\mathbf{s4}}{\Longrightarrow} \div\div a = \mathbf{1}, \tag{3.9}$$

gives the general statement:

$$\mathcal{IC} \models \phi(x) = \mathbf{1} \Longrightarrow \mathcal{IC} \models \div\div\phi(x) = \mathbf{1}. \tag{3.10}$$

In a sense, this is stronger than the classical result, since ϕ can now contain richer combinations of boolean *and* constructive connectives. But it is weaker when the relation between the respective logics is concerned, because $\div\div\phi(x)$ need not be a constructive formula.

The opposite of (3.9) does not hold! (Take $\mathbf{1} = \mathsf{R}^2$ and $x = \mathsf{R}^2 - (0,0)$; then $\div x = \emptyset = \mathbf{0}$, and so $\div\div x = \mathbf{1}$. But $x \neq \mathbf{1}$.) This counter-example works actually for open elements like $x = \div - y = \div - (0,0)$. The impossibility is related to the failure of **l18**.

3.2.4 Glivenko's theorem

Just like $\div-$ "switches" an element into its constructive version, so $\div\div$ switches a tautology into a quasi-constructive one (3.10). It is only quasi-constructive because ϕ remains unchanged and may involve non-constructive expressions. The translation is obtained by restricting somewhat the form of the considered expressions. In the view of definition (3.4) and lemma 3.6, one direction of Glivenko's theorem will have the following form.[2] Let $\phi(x)$ be a boolean derived operator (i.e., with no $\div$, cf. grammar β on p. 11), then:

$$\mathcal{IC} \models \phi(x) = \mathbf{1} \Longrightarrow \mathcal{IC} \models \div\div\phi'(\div - x) = \mathbf{1} \tag{3.11}$$

where ϕ is assumed to be in conjunctive normal form (to simplify the presentation) and ϕ' is ϕ with all $-$ replaced by $\div$. Notice that this yields formulae which still contain boolean complement but only at the very variables and preceded by $\div$. Hence they conform to the grammar γ (the "ungrammatical" occurences of $\div\_$ can be repalced by $\_ \hookrightarrow \mathbf{0}$). This is

[2]We do not show the opposite implication which is a trivial consequence of the completeness results for CL and BL, and the observation that CL-provability is contained in BL-provability.

the only difference from the classical formulation which is due to the fact that we are now interpreting both BL- and CL-formulae in the same algebras, and constructive satisfaction concerns only open elements.

So let ϕ be in conjunctive normal form, i.e., $\bigcap_i(\bigcup_j \overline{x}_{ij})$ where each $\overline{x}$ is x or $-x$. $\mathcal{IC} \models \phi = \mathbf{1}$ means that every $\mathcal{IC}$-algebra $\mathbf{T} \models \phi = \mathbf{1}$, and we conduct the proof for an arbitrary such algebra:

$$
\begin{array}{cll}
\bigcap_i(\bigcup_j \overline{x}_{ij}) = \mathbf{1} \iff & \bigcup_j \overline{x}_{ij} = \mathbf{1} & \text{for all } i \\
\iff & \div - \bigcup_j \overline{x}_{ij} = \mathbf{1} & \mathbf{i}(x) \subseteq x \ \& \ \mathbf{i}(\mathbf{1}) = \mathbf{1} \\
= & \div - (\bigcup_{ijm} -x_{ijm} \cup \bigcup_{ijp} x_{ijp}) = \mathbf{1} & \\
\iff & \div(\bigcap_{ijm} x_{ijm} \cap \bigcap_{ijp} -x_{ijp}) = \mathbf{1} & \\
\stackrel{(subst)}{\Longrightarrow} & \div(\bigcap_{ijm} \div\div x_{ijm} \cap \bigcap_{ijp} -\div\div x_{ijp}) = \mathbf{1} & x \mapsto \div\div x \\
\stackrel{\mathbf{l15}}{\iff} & \div(\bigcap_{ijm} \div\div\div\div x_{ijm} \cap \bigcap_{ijp} -\div\div x_{ijp}) = \mathbf{1} & \\
\stackrel{\mathbf{s1,l4}}{\Longrightarrow} & \div(\bigcap_{ijm} \div\div\div\div x_{ijm} \cap \bigcap_{ijp} \div x_{ijp}) = \mathbf{1} & x \subseteq -\div x \\
\stackrel{\mathbf{s3}}{\iff} & \div\div(\bigcup_{ijm} \div\div\div x_{ijm} \cup \bigcup_{ijp} x_{ijp}) = \mathbf{1} & \\
\stackrel{(subst)}{\Longrightarrow} & \div\div(\bigcup_{ijm} \div\div\div\div -x_{ijm} \cup \bigcup_{ijp} \div - x_{ijp}) = \mathbf{1} & x \mapsto \div - x \\
\stackrel{\mathbf{l15}}{\iff} & \div\div(\bigcup_{ijm} \div\div -x_{ijm} \cup \bigcup_{ijp} \div - x_{ijp}) = \mathbf{1} & \\
= & \div\div(\bigcup_{ijm} \div x'_{ijm} \cup \bigcup_{ijp} x'_{ijp}) = \mathbf{1} & \\
= & \div\div\bigcup_{ij} \overline{x}'_{ij} = \mathbf{1} & \text{for all } i \\
\iff & \bigcap_i \div\div\bigcup_j \overline{x}'_{ij} = \mathbf{1} & \\
\stackrel{\mathbf{s3}}{\iff} & \div\bigcup_i \div\bigcup_j \overline{x}'_{ij} = \mathbf{1} & \\
\stackrel{\mathbf{l14},\gamma}{\iff} & \div\div\bigcap_i \bigcup_j \overline{x}'_{ij} = \mathbf{1} &
\end{array}
$$

The resulting $\overline{x}'_{ij}$ have the form $\div - x_{ij}$ (second substitution) and those which were preceded by $-$ are now preceded by $\div$ instead (line -5/-4). Note that this is an internal proof (not a metaproof) as all transitions rely exclusively on the formulae valid in every $\mathcal{IC}$-algebra.

To complete the proof for arbitrary tautologies, not only in CNF, we only observe that any CL-formula $\div\div\phi(x)$ is (constructively) equivalent to $\div\div\phi'(x)$ where ϕ' is obtained from ϕ by boolean transformations (e.g., $\div\div(\div\psi_1(x) \cup \div\psi_2(x)) = \div\div\div(\psi_1(x) \cap \psi_2(x))$, $\div\div(\div\psi_1(x) \cup \psi_2(x)) = \div\div(\psi_1(x) \hookrightarrow \psi_2(x))$, etc. Also, $x \hookrightarrow y \subseteq -x \cup y$. Hence, if our boolean tautology is not, initially, in CNF, we transform it into CNF, apply the above result, and then transform the final constructive formula into the corresponding constructive form using these equivalences. This transformation is valid by the observations at the beginning of 3.2.

3.2.5 BL $\rightarrow$ CL

Gödel's embedding is the following:

$$
\begin{array}{rcl}
\text{BL} & \mapsto & \text{CL} \\
\hline
a \in X : tr(a) & = & \div\div a \\
tr(\phi_1 \wedge \phi_2) & = & tr(\phi_1) \wedge tr(\phi_2) \\
tr(\phi_1 \vee \phi_2) & = & \div(\div tr(\phi_1) \wedge \div tr(\phi_2)) \\
tr(\phi_1 \rightarrow \phi_2) & = & tr(\phi_1) \hookrightarrow tr(\phi_2) \\
tr(-\phi) & = & \div tr(\phi)
\end{array}
\tag{3.12}
$$

To show that $\mathcal{BA} \models \phi \iff \mathcal{HA} \models tr(\phi)$, we first utilize the observation that for any formula $tr(\phi) \in L(\gamma)$: CL $\models tr(\phi) \iff \mathcal{TA} \models tr(\phi)$. Again, we can assume that boolean tautology is in CNF. Translation of $\vee$ is motivated by the analysis of provability. We have $\div(\div x \cap \div y) = \div\div(x \cup y)$, so *we* can use this latter formulation. (As in 3.2.4, the implication $tr(\phi) = \mathbf{1} \Rightarrow \phi = \mathbf{1}$ follows trivially by inspecting the respective proof systems, so we address only the opposite implication.)

$$
\begin{array}{lcll}
\bigcap_i \bigcup_j \overline{x}_{ij} = \mathbf{1} & \iff & \bigcup_j \overline{x}_{ij} = \mathbf{1} & \text{for all } i \\
& = & \bigcup_{jn} -x_{ijn} \cup \bigcup_{jp} x_{ijp} = \mathbf{1} & \\
& \Longrightarrow & \div\div(\bigcup_{jn} -x_{ijn} \cup \bigcup_{jp} x_{ijp}) = \mathbf{1} & \\
& \Longrightarrow & \div\div(\bigcup_{jn} - \div\div x_{ijn} \cup \bigcup_{jp} \div\div x_{ijp}) = \mathbf{1} & x \mapsto \div\div x \\
& \iff & \div(\bigcap_{jn} \div - \div\div x_{ijn} \cap \bigcap_{jp} \div\div\div x_{ijp}) = \mathbf{1} & \\
& \iff & \div(\bigcap_{jn} \div\div x_{ijn} \cap \bigcap_{jp} \div\div\div x_{ijp}) = \mathbf{1} & \\
& \iff & \div(\bigcap_{jn} \div\div\div\div x_{ijn} \cap \bigcap_{jp} \div\div\div x_{ijp}) = \mathbf{1} & \\
& \iff & \div\div(\bigcup_{jn} \div(\div\div x_{ijn}) \cup \bigcup_{jp} \div\div x_{ijp}) = \mathbf{1} & \text{for all } i \\
& = & \div\div(\bigcup_{jn} \div(tr(x_{ijn})) \cup \bigcup_{jp} tr(x_{ijp})) = \mathbf{1} & \text{for all } i \\
& = & \div\div \bigcup_j tr(\overline{x}_{ij}) = \mathbf{1} & \text{for all } i \\
& = & tr(\bigcup_j \overline{x}_{ij}) = \mathbf{1} & \text{for all } i \\
& \iff & \bigcap_i tr(\bigcup_j \overline{x}_{ijn}) = \mathbf{1} & \\
& \iff & tr(\bigcap_i \bigcup_j \overline{x}_{ijn}) = \mathbf{1} &
\end{array}
$$

As before, translation of implication is compatible with the above proof schema. We would obtain

$$
\begin{array}{rcl}
tr(\phi \to \psi) & = & tr(-\phi \cup \psi) \\
& = & \div\div(tr(-\phi) \cup tr(\psi)) \\
& = & \div\div(\div tr(\phi) \cup tr(\psi)) \\
& \stackrel{\gamma}{=} & \div\div(tr(\phi) \hookrightarrow tr(\psi)) \\
& \stackrel{\gamma}{=} & \div\div tr(\phi) \hookrightarrow \div\div tr(\psi)
\end{array}
$$

which is equivalent with the formula obtained by pushing the double $\div\div$ inside and, eventually, reducing $\div\div\div\div x = \div\div x$, i.e., with $tr(\phi) \hookrightarrow tr(\psi)$ which results from (3.12)

4 $\mathcal{IC}$-models for S5

The development in section 3 is not limited to S4. What is specific about S4 is only that it contains the constructive logic in an unmodified form. Further extensions will, typically, affect this aspect and we illustrate it by an extension to S5.

To the $\mathcal{IC}$-axioms **s1-s4**, we add the S5-axiom:

s5. $-\div x \subseteq \div\div x$

which is just $-\div x \subseteq \div - - \div x$, i.e., $\Diamond x \to \Box\Diamond x$. Combined with axiom **s1**, this entails $-\div x = \div\div x$. That is, in $\mathcal{IC}$-algebras for S5, the negation of open elements equals the interior of their negation, i.e., the complement of an open is open.

An equivalent definition of S5-algebras, e.g., [4], requires that complement of every closed element is closed, i.e.,

$$(*) \quad \forall x \exists y : -(-\div x) = -\div y.$$

(**s5** $\Rightarrow$ (*) follows since $-\div x = \div\div x \Rightarrow -(-\div x) = -\div(\div x)$, so we can take $y = \div x$. For the opposite implication, let x be arbitrary, then $-\div x$ is closed, and so $-(-\div x) \stackrel{(*)}{=} -\div y$, i.e., $\div x = -\div y$. But then $-\div(\div x) = -\div - \div y = -\div y = \div x \Rightarrow \div\div x = -\div x$.)

Hence also complement of every open is open. (I.e., $-(\div x) = -\div x$ and by the above its complement is closed, i.e., $- - \div x = -\div y \Rightarrow -\div x = \div y$, which is open.) Since in every $\mathcal{IC}$-algebra, opens are complements of the closed elements (and vice versa), this means that in S5-algebras all opens are closed and vice versa, i.e., we have only elements wich are either clopen or neither closed nor open. (Or else, just see the verification of **s5** $\Rightarrow$ (*) above, which shows that $\div x = -\div(\div x)$, i.e., every open is closed.) I.e., as is well known, a topological algebra is an S5-algebra iff the topology is almost discrete (open = closed).

Yet another equivalent formualtion of the **s5** axiom is given in [4]:

$$(**) \quad \forall x \forall y : y = -\div y \Rightarrow -\div(x \cap y) = -\div x \cap y.$$

Some properties of such algebras:

s5-l1. $-\div\div - x = \div - x$ (as: $-\div\div - x = - - \div - x = \div - x$)

s5-l2. $\div\div\div x = \div x$ (as: $\div(\div\div x) = \div(-\div x) = \div x$)

The essential difference between $\mathcal{IC}$-algebras for S4 and for S5 is that the former contain genuine Heyting substructures. In the latter, where complement of an open is open, the subalgebra of opens is actually a boolean algebra. This can be now seen as the crucial collapse enforced by S5: its modalities, still present, express no longer a relation between constructive and boolean worlds, but between one boolean world and its substructure which is itself boolean. The remaining section is devoted to a discussion of a possible interpretation of the relations between the studied logics and modalities from the perspective of $\mathcal{IC}$-algebras.

5 A note on a *possible* reading

Triviality of the equivalence (between the $\mathcal{TA}$ and the $\mathcal{IC}$ algebras) notwithstanding, the reformulation seems nevertheless to throw a new light on the relations between boolean, constructive and modal logics. We are not attempting here to draw any deep philosophical conclusions (which should never be drawn from any formal or scientific results). But we think it is legitimate to attempt a more "intuitive" reading which might enhance the understanding of the pure formalism.

First, $\div$ is not a modality! Well, this depends on a definition, but given the algebraic tradition, it is more or less standard to consider a modality to be an operator on a boolean algebra. An operator is, [5, 6], a strict additive function, i.e., (for a unary one) satisfying $op(\mathbf{0}) = \mathbf{0}$ and $op(x \cup y) = op(x) \cup op(y)$. Our $\div$ does not satisfy either of these two (**s4** and **l14**).[3]

5.1 One world – many epistemolgies

Each $\mathcal{IC}$-algebra $\mathbf{T}$ contains a subset of elements constituting a Heyting algebra $\mathbf{H}$, as given in theorem 1.5. We can think of the latter as a universe of "constructive elements" contained in the universe of all ("boolean") elements of $\mathbf{T}$. Given an arbitrary element $x \in \mathbf{T}$, we "switch" it to a constructive element by $\div - x$. Since every $y \in \mathbf{T}$ can be written as $-x$, we see that $\div$ does the "actual switching", i.e., every element of $\mathbf{H}$ can be written as $\div x$ for some $x \in \mathbf{T}$. (Precisely: every $\div x$ is open by **s2**, while every open is defined by $y = \div - y$, i.e, has the form $\div x$.)

5.1.1. The first question concerns the reading of these two universes $\mathbf{H} \subseteq \mathbf{T}$. Following Heyting's interpretation of classical logic as the logic of ontology and the constructive logic as the logic of epistemology, we would propose to read the universe of all boolean elements as the universe of all truths. The elements of the constructive subuniverse might then be read as:

1. actually provable truths
2. potentially provable truths
3. actually accessible/epistemic truths
4. potentially accessible/epistemic truths
5. ...

We certainly do not want to enter all too detailed discussions of the differences between such interpretations. But observing the tension between the actually and potentially accessible, we should at least settle for one of them. Now, admitting the possibility of an omniscient agent who knows all ontological truths, one might risk the accusations of some form of idealism, if not theologism. On the other extreme, one can meet dedicated scientism which preaches the ultimate rationality and comprehensibility of the whole world. The opposition between these two extremes notwithstanding, they seem to defend the same point as far as the potentiality of an epistemic access to the whole ontology (whether by us or by others) is concerned. Both will presumably grant also that, at present, we are not in such a position, and this will be granted by all who fall inbetween these two extremes. We therefore prefer to view the constructive subuniverse as the totality of actually accepted truths, whether of the kind 1 or 3. The former can be plausibly seen as a subset of the latter (not everything we know is provable), so let us choose the reading 3: the constructive subset of the boolean universe represents finitary combinations of pieces of knowledge (the open elements). The number of

[3] **l14** gives a weaker statement, but consult the proof to see that it can not be strengthened.

such pieces and their combinations will be probably extended in the futre (but this temporal aspect falls entirely outside the scope of the present discussion). And so CL is the logic of (constructible) epistemology, while BL of the whole ontology. The combination of these two universes in a one framework illustrates also why S4 is a more adequate logic of knowledge than CL: the latter is only the logic of solipsistic knowledge unrelated to any world outside its finite constructions, while the former (and its refinement presented here) allows one to consider both dimensions.

Thus, 'opens' can be viewed as the objects knowledge is actually using, while all other elements as objects which knowledge can be about. (Of course, knowledge can also concern the epistemic elements.) We will therefore refer to all the boolean elements as "ontological", while to the constructive subset as "epistemic" (or "constructive").

All the involved vagueness notwithstanding, it has already some implications for the reading of the more specific elements which we will address shortly. But first let us return to the point marked in the abstract.

5.1.2. Working with the possible world semantics (of modal logic), one has repeatedly emphasized that the "possible worlds" are not to be interpreted as some strange other-worldly entities but simply as possible variations of the states of affairs obtainning in the world in which we are actually living. Unfortunately, this intuitive point does not find any natural expression in the semantic model where, indeed, different possible worlds can have nothing in common (except for the boolean logic). If one points at one world claiming that this is the actual one, there is still nothing in the framework ensuring that all agents actually share in this particular world; there may even be agents to whom this world remains inaccessible. (A residual trace of the "common world" can be found, for instance, in the concept of rigid designators whose role (apart from the attempt to give an interpretation of proper names) is exactly to establish a common ontology shared by all possible worlds.)

The present setting can be viewed as addressing exactly this problem by introducing a distinction between the boolean world of ontology, on the one hand, and its "epistemic substructure" of epistemic elements which approximate all elements, on the other hand. The "approximation" can be best thought of in terms of the epistemic elements (of a Heyting algebra) as corresponding to the open sets of a topological space which, indeed, approximate (better or worse) all elements. A variety of possibilities is then simply a potential multiplicity of such "epistemic worlds" (Heyting algebras) which all are substructures of the same (boolean) world. Formally, one would simply consider a multiplicity of $\div_i$, one for each agents i.[4]

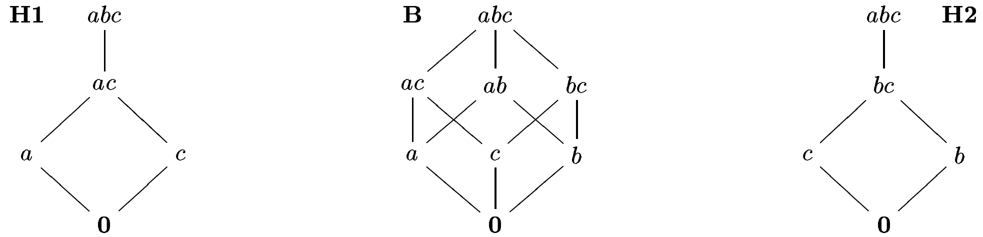
5.2 Two examples

To give an impression of the relations and interactions between the epistemic (constructive), the ontological (boolean) and the modal elements implied by the presented formalization, we give two simple examples for, respectively, S4 and S5, algebras. (We should emphasize the simplicity of these examples which, being finite and small, are not fully representative. They should, nevertheless, give the impression of the involved relations.)

5.2.1. S4. Consider a simple boolean world $\mathbf{B} = \mathcal{P}(\{a, b, c\})$ and two possible epistemic substructures, Heyting algebras **H1**, **H2**. (We denote joins by concatenation, e.g., $a \cup b$ is

[4]To the dedicated adherents of Kripke semantics, we could say that our proposal can be taken as a complementary, and not as a contrary, view of modalities. Simply, a boolean algebra with operators can be represented according to the theorem 3.10 from [5], as an algebra of complexes (multialgebra), namely a boolean algebra with the operator/modality being a set-valued function from which one can recover the reachability relation (see also chap. 5 of [2]).

written ab.)



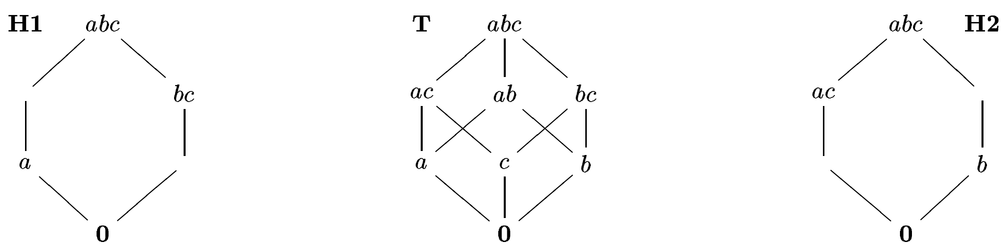
We have, for instance:

		H1	H2		H1	H2		H1	H2
1.	$-c =$	ab	ab	$-b =$	ac	ac	$-bc =$	a	a
2.	$\div c =$	a	b	$\div b =$	ac	c	$\div bc =$	a	$\mathbf{0}$
3.	$\div - c =$	c	c	$\div - b =$	$\mathbf{0}$	b	$\div - bc =$	c	bc
4.	$- \div c =$	bc	ac	$- \div b =$	b	ab	$- \div bc =$	bc	abc
5.	$\div \div c =$	c	c	$\div \div b =$	$\mathbf{0}$	b	$\div \div bc =$	c	abc

The first table concerns the element c present in both **H1** and **H2**. The differences in rows 2. and 4. reflect the differences between the respective epistemologies. Reading $\div c$ as "recognized impossibility of c", for **H1** it can be only a while for **H2** only b. This is then reflected in what appears as c's possibility in row 4. In either case it can be c itself, but for **H1**, possibly also b – as it does not belong to its epistemic world, the possibilities it harbours are not recognizable by **H1**.

The second and third table concern elements which are in the epistemic world **H2** but not **H1**. Thus, either a or c of **H1** amount to impossibility of b, while for **H2**, it is only c. Dually, the necessity of b, row 3., does not obtain in **H1**, while it is present in **H2** as the element b itself. (Note in the third table that, although $bc \notin$ **H1**, its necessity still obtains as the element b – the greatest open included in this epistemically absent element.) The possibility of b, row 4., is not however absent for **H1**, although it does not meet any elements of his world – it is an "external" possibility, which obtains only due to the ontological structure of the whole world. For **H2**, this possibility is further extended by the element a which is not part of his epistemic world (and hence might, potentially for **H2**, harbour the possibility of b, even if it actually does not).

5.2.2. S5. Consider the same boolean world as in the previous example and two possible epistemic substructures **H1**, **H2**. Since **s5** axiom makes complements of opens open, the Heyting substructure will here be a boolean algebra in which all opens are also closed.



We have, for instance:

		H1	H2		H1	H2		H1	H2
1.	$-c =$	ab	ab	$-b =$	ac	ac	$-bc =$	a	a
2.	$\div c =$	a	b	$\div b =$	a	ac	$\div bc =$	a	$\mathbf{0}$
3.	$\div - c =$	$\mathbf{0}$	$\mathbf{0}$	$\div - b =$	$\mathbf{0}$	b	$\div - bc =$	bc	b
4.	$- \div c =$	bc	ac	$- \div b =$	b	ab	$- \div bc =$	bc	abc
5.	$\div \div c =$	bc	ac	$\div \div b =$	b	ab	$\div \div bc =$	bc	abc

Note that although the epistemic substructures are now boolean algebras, the epistemic negation $\div$ does not coincide with the ontological one $-$. The difference concerns the epistemically absent elements. Thus, for instance, in the first table, c is epistemically absent from **H1**, but its impossibility, $\div c$, amounts only to the epistemically available contraries, namely, a, and not to its ontological negation ab. Likewise, in the third table, $bc \notin \mathbf{H2}$, but its epistemic impossibility amounts to contradictiction $\mathbf{0}$, although ontologically it can be also obtained as a.

5.3 Three logics

Before addressing some more specific aspects of the proposed interpretation, let us first comment on the obtained relations between CL and BL with S4 as their "union".

5.3.1. On the one hand, CL is a real extension of expressivity of BL, requiring additional operator $\div$. Yet, this amounts really to CL restricting BL, namely, by addressing only some specific kinds of elements: the "constructive" elements of the form $\div - x$ and, in fact, already of the form $\div x$. That is, conceptually, CL extends the boolean logic by the concept of $\div$, but its specificity (and the specificity of its associates, like $\hookrightarrow$) arises only from the consideration of a subset of all (boolean) objects. Namely, each $\mathcal{IC}$-algebra contains a substructure of the epistemic elements, a Heyting algebra, and every such Heyting algebra is contained in some (in fact, in may different) $\mathcal{IC}$-alegbras.

This seems entirely plausible given the basic tenets of intuitionism: it throws away a lot of mathematics, restricting its attention to constructible objects. Perhaps, we could thus give $\div x$ the intended interpretation: not as a negation of x but as its absence: while $-x$ means not-x, so $\div x$ means a counter-proof of x or, perhaps, the impossibility of x.

5.3.2. Although $\hookrightarrow, \sqcup$ are the fundamental operators of Heyting algebras, when viewed in the context of $\mathcal{IC}$-algebras, constructive negation seems to acquire much more central status: other constructive operators are definable: $\hookrightarrow$ by (3.4) and disjunction by $x \sqcup y = \div - x \cup \div - y$ (lemma 3.6). Strictly speaking, they are not any new connectives (just abbreviations), but their specificity comes from the fact that they work on "constructive" elements only. And clearly, they are interdefinable, so one can still use $\hookrightarrow$, instead of $\div$, as the primitive (though this would require axiomatization of $\hookrightarrow$.)

5.3.3. Modal logic S4 arises now as a natural combination or union of the boolean and constructive logics and not merely as some logic which only happens to admit embedding of the other two. S4-modalities arise as combinations of the boolean and constructive negation, $\Box x = \div - x$, respectively, $\Diamond x = - \div x$ (with their "duality" simply as the associativity of operations: $-\Diamond x = -(\div - x) = (-\div) - x = \Box -x$).

5.4 The epistemic impossibility

The epistemic aspect of constructivism is well expressed in the common interpretation of negation as possession of a counter-proof. In our context, a series of readings of $\div x$ – of various strength and flavour – might be acceptable, e.g.:

1. possession of a counter-proof of x
2. existence of a counter-proof of x
3. lack of proof of x
4. non-existence of a proof of x
5. possession of a counter-example to x
6. ontological impossibility of x
7. epistemic impossibility (inadmissibility, unimaginability, counter-proof) of x
8. ...

Our decision expressed at the end of 5.1.1 leads to an epistemic turn in the interpretation of impossibility and other modalities, which speaks against the readings 2, 4 and 6. Being an epistemic element, $\div x$ is not metaphysical (or ontological) impossibility of x. But we should be clear here. Given the epistemic reading of opens, we do not have any ontological impossibility in our framework: we only have ontological absence, non-actuality: $-x$ does not say that x is ontologically impossible; it only says that it simply does not obtain.

Reading $\div x$ as epistemic impossibly of x (whatever it might be), 7, axiom **s1** becomes the statement that (such) impossibility of x entails the actual negation of x. One should emphasize here the finitary/constructive flavor of epistemic impossibility. For, certainly, we can have a directly available negative knowledge! Seeing that Per is not in the room, we know that he is not there. But do we really? Seeing that he is not here, is only not seeing him to be here (not seeing him wherever we are actually looking). Although the latter, which corresponds to $\div x$, is all we have, we actually act as if he was not here, as if $-x$. In epistemic terms, this seems to be the only way of constructing the knowledge of not-x. We sense how the finitarism of constructive logic approaches pure phenomenalism, where even the most immediate negation must be constructed from the epistemic evidence to the impossibility.

To avoid torturous arguments, while aiming at most possible generality compatible with the formalism, we adopt the reading 7.

5.5 Knowledge and necessity

Since $\Box$ has been treated as 'knowledge' or 'necessity' operator, let us consider briefly such interpretations.

$\Box x$ arises as 'the (epistemic) impossibility of the negation of x', 5.3.3. Thus, $\Box$ read as knowledge, becomes an epistemic impossibility (unimaginability) of the contrary. As impossibility entails the actual negation, knowledge, i.e., epistemic necessity of x, entails the (ontological) actuality of $x : \div - x \subseteq x$.

5.5.1. This reading squares very well with reading of $\Box$ as necessity, provided that we grant its epistemic interpretation. According to Ockham's arguments (commonly associated with Hume), necessity is an epistemic modality, we might say, the impossibility of alternatives.[5] Now, alternatives are, in the 'possible worlds' parlance, other possibilities, which to a large extent are very mental entities. "To a large extent" because there may always be possibilities which are not taken into account. This ontological aspect of possibility – something more can be the case than what we are able to imagine – comes equally nicely forth in our formulation: $\Diamond x = - \div x \supseteq \div \div x$. I.e., x is possible not only when its impossibility is (appears) impossible, $\div \div x$, but also when it actually – ontologically – does not obtain, $- \div x$. So, perhaps a bit unexpectedly, possibility has a stronger ontological flavour than has necessity.

This is also what common-sense would make out of necessity. It is simply impossibility of accepting other alternatives, as when we say: "This is unavoidable!" Of course, few things are ever unavoidable/necessary in the strict sense of logical impossibility. In the more mundane situations, logical impossibility is replaced by milder, that is, more epistemic predicates: irrelevancy, implausibility or incapacity, and x appears unavoidable just when its contrary falls under some such predicate.

5.5.2. The appealing feature of this formulation, $\Box x = \div - x$, is that it does not commit us to any such specific choices of what we want to consider as epistemic and what as ontological. It only acknowledges the distinction between the two, and obtains necessity out of their combination. Necessity and knowledge arise thus as ... synonyms. Of course, knowledge understood not merely as an acceptance of a fact, but as inadmissibility of a contrary, we might say, as a justified belief.

The strength of $\mathcal{IC}$-algebras is the combination of these two aspects. The fact that from the knowledge of x (or, from the unimaginability of the contrary) we can conclude x, $\div - x \subseteq x$, is the connection between our limited ('open') knowledge of x and x itself. Allowing such weaker readings of $\div$ (as implausible, unimaginable, etc.) marks a strong underlying current of 'default' thinking which, so it appears, need no new logics or rules, since it is inherent in this very basic view of knowledge as essentially an event of double negation: the epistemic applied to the ontological one.

5.5.3. The universe-subuniverse view finds also a natural application to the interpretation of S5. We have seen that the **s5** axiom amounts to equating the epistemic and the ontological negation when applied to epistemic elements. The epistemic subuniverse can still be distinct from the ontological one, but it is itself a boolean universe. This is the counterpart of the specific property of S5, namely, that any chain of modalities is equivalent ("collapses") to the rightmost one. In our formulation, having once entered the epistemic subuniverse (by means of $\div$), the $\div$ becomes $-$, and so no more properly modal operations are available. This seems to express well the traditional reading of S5 as the logic of metaphysical necessity: it

[5] Metaphysical necessity is, perhaps, something accessible to God but not to humans. Ockham's empirical reductionism with the associated denial of necessity in the world as *we* know it, is just an elaboration of the theme of God's omnipotence.

comprises a subset of the actual universe (all necessary truths) which, itself, is governed by the same (boolean) laws.

5.6 Knowledge of distinctions

Let us observe an entirely different aspect of epistemic modelling. Knowledge concerns the ability to draw and relate various distinctions. The topological interpretation faciliates precisely such a view without any need of adding it on the top of the view of modalities we have developed.

Consider as an example a boolean world $\mathcal{P}(\{a,b,c,d\})$ and a Heyting substructure containing (the opens) $\{\emptyset, a, b, ab, abcd\}$ with $\emptyset = \mathbf{0}$ and $abcd = \mathbf{1}$. Following the topological view, this amounts to (the agent being capable of) distinguishing a from b (having distinc opens covering them). However, c and d fall outisde the epistemic world and, consequently, they and (their join) are indistinguishable by the available epistemic means, as shown on the left. Also, no interaction of these elements with the available a, b will uncover any difference between them, as exemplified on the right:

	c	d	cd
$-$	abd	abc	ab
$\div$	ab	ab	ab
$\div-$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$-\div$	cd	cd	cd

	ac	ad	acd
$-$	bd	bc	b
$\div$	b	b	b
$\div-$	a	a	a
$-\div$	acd	acd	acd

Expanding the epistemic base with, say, recognition of the element c will, of course, lead to new distinctions, e.g., $\div c = ab \neq abc = \div d$.

Viewing the epistemic elements as the distinctions one is capable of recognizing, the modalities arise now from the interaction between such distinctions and the ontological ones which, epistemically, remain indistinct. One might even be tempted to read now $\div x$ as the impossibility to recognize x, with the consequences for:

– 'necessity' of $x = \div - x =$ impossibility to recognize negation of x and

– 'possibility' of $x = - \div x =$ the absence of impossibility of recognition of x.

5.7 Knowing versus knowing at most

In the study of epistemic logic, one is sometimes interested in stating not only that an agent knows something, but also that he does not know more than something. Propositional S4 gives a view of knowledge where it is difficult to address the issue of knowledge's limitations. One can state claims like 'the agent does not know a', $-\Box a$. But if we wanted to say that agent knows at most a, b, c, d, we would have to write infinitely many negations (in general, when the alphabet is infinite). A variety of logics of "only knowing" (as this aspect is termed in the literature) addresses this issue.

We write: $\div - x \to a \cup b \cup c \cup d$, i.e., $- \div -x \cup a \cup b \cup c \cup d$.

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