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# Cutwidth of split graphs, threshold graphs, and proper interval graphs\*

Pinar Heggenes†   Daniel Lokshtanov†   Rodica Mihal†   Charis Papadopoulos†

## Abstract

We give a linear-time algorithm to compute the cutwidth of threshold graphs, thereby resolving the computational complexity of cutwidth on this graph class. Although our algorithm is simple and intuitive, its correctness proof relies on a series of non-trivial structural results, and turns out to be surprisingly complex. Threshold graphs are a well-studied subclass of interval graphs and of split graphs, both of which are unrelated subclasses of chordal graphs. To complement our result, we show that cutwidth is NP-complete on split graphs, and consequently also on chordal graphs. In addition, we show that cutwidth is trivial on proper interval graphs, another subclass of interval graphs. The cutwidth of interval graphs is open, and only very few graph classes are known so far on which polynomial-time cutwidth algorithms exist. Thus we contribute to define the border between graph classes on which cutwidth is polynomially solvable and on which it remains NP-complete.

## 1 Introduction

The cutwidth problem asks, given a graph  $G$ , and a positive integer  $k$ , whether there exists a linear layout of the vertices of  $G$  so that any line inserted between two consecutive vertices of the layout cuts (intersects with) at most  $k$  edges. The cutwidth of the input graph is the smallest integer for which the question can be answered positively. This important graph layout problem was first proposed as a model to minimize the number of channels in a circuit [1, 19], and more recently it has found applications in areas like protein engineering [3], network reliability [16], automatic graph drawing [21], information retrieval [4], and as a subroutine in the cutting plane algorithm for TSP [14].

Like many other interesting graph problems, cutwidth is NP-complete [8], even when input graphs are restricted to planar graphs of maximum degree three [20], unit disk graphs, partial grids [9], and consequently bipartite graphs.

Coping with the NP-completeness of the problem has been mainly channeled via approximation algorithms and fixed parameter algorithms. There is a polynomial-time  $O(\log^2 n)$ -approximation algorithm for general graphs [17], and a polynomial-time constant factor approximation algorithm for dense graphs [2]. The best known parameterized algorithm for cutwidth so far runs in linear time (but of course exponential in the parameter  $k$ ) [22].

Polynomial-time algorithms for the exact computation of cutwidth are known only for very few graph classes. For certain trivial graph classes, like meshes or complete  $p$ -partite graphs, there exist closed formulas for their cutwidth (see [10]). The cutwidth of trees can be computed

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in  $O(n \log n)$  time by a sophisticated and technical algorithm [24] (see also [6]). The cutwidth of graphs both of whose treewidth *and* maximum degree can be bounded by constants, can be computed in polynomial time by advanced methods [23]. The computational complexity of cutwidth on threshold graphs has been open until now [10].

In this paper, we show that the cutwidth of threshold graphs can be computed in linear time. In fact we present an  $O(n)$ -time algorithm for computing the cutwidth of threshold graphs with  $n$  vertices. Threshold graphs are a well-studied graph class with a variety of theoretical applications [18], and are both split graphs and interval graphs [5, 12]. Split and interval graphs are two unrelated subclasses of the widely-known class of chordal graphs. Before presenting our algorithm for threshold graphs, we show that the cutwidth problem remains NP-complete on split graphs (even on a very restricted type of split graphs), and hence also on chordal graphs. As a complementary result, we show that for another subclass of interval graphs, proper interval graphs, the cutwidth problem has a trivial solution, which does not work for interval graphs in general or for threshold graphs. Our findings are summarized in Figure 1.

Although threshold graphs can be viewed as a quite restricted graph class, designing an algorithm for their cutwidth and proving its correctness proved to be a much more challenging task than expected. In fact, we present a simple and intuitive algorithm whose execution does not at all depend on properties of threshold graphs, thus it can also be run on general graphs as a heuristic. Interestingly, while the algorithm correctly computes the cutwidth of threshold graphs, it does not compute the cutwidth of any known superclass or closely related class, like chain graphs. Extending our results even to the class of trivially perfect graphs, which is a superset of threshold graphs and a subset of interval graphs, seems to be a non-trivial problem. For the proof of correctness of our algorithm on threshold graphs, we study the properties of a possible minimal counterexample through a series of structural results, and we show that the assumption of the existence of such a counterexample leads to a contradiction.

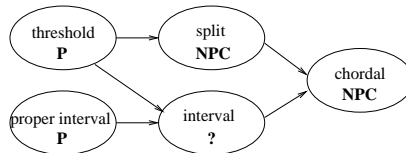


Figure 1: The graph classes studied in this paper, and the complexity of cutwidth on each class according to our results. The arrow represents the subset relation.

## 2 Preliminaries

We consider labeled undirected finite graphs with no loops or multiple edges. For a graph  $G = (V, E)$ , we denote its vertex and edge set by  $V$  and  $E$ , respectively, with  $n = |V|$ . Every vertex  $v \in V$  has a distinct label,  $label(v)$ , between 1 and  $n$ . We say that a vertex  $u$  is *smaller than*  $v$  if  $label(u) < label(v)$ . For a vertex subset  $S \subseteq V$ , the subgraph of  $G$  induced by  $S$  is denoted by  $G[S]$ . Moreover, we denote by  $G - S$  the graph  $G[V \setminus S]$  and by  $G - v$  the graph  $G[V \setminus \{v\}]$ . In this paper, we distinguish between *subgraphs* and *induced subgraphs*. By a *subgraph* of  $G$  we mean a graph  $G'$  on the same vertex set containing a subset of the edges of  $G$ , and we denote it by  $G' \subseteq G$ . If  $G'$  contains a proper subset of the edges of  $G$ , we write  $G' \subset G$ . We write  $G - uv$  to denote the graph  $(V, E \setminus \{uv\})$ .

The *neighborhood* of a vertex  $x$  of  $G$  is  $N_G(x) = \{v \mid xv \in E\}$ . The *closed neighborhood* of  $x$  is  $N_G[x] = N_G(x) \cup \{x\}$ . The *degree* of  $x$  is  $\Delta_G(x) = |N_G(x)|$ . If  $S \subseteq V$ , then  $N_G(S) = \bigcup_{x \in S} N_G(x) \setminus S$ . We define the *cut* of  $S$  to be  $\delta^G(S) = \{uv \in E \mid u \in S, v \notin S\}$ , and the *cut size* of  $S$  to be  $d^G(S) = |\delta^G(S)|$ . Vertex  $x$  is *universal* if  $N_G[x] = V$  and *isolated* if  $N_G(x) = \emptyset$ . We will omit the subscripts and superscripts when there is no ambiguity. A graph is *connected* if there is a path between any pair of vertices. A *connected component* of a disconnected graph is a maximal connected subgraph of it. A *clique* is a set of pairwise adjacent vertices, while an *independent set* is a set of pairwise non-adjacent vertices.

Given a graph  $G = (V, E)$ , a *layout*  $L$  is a one-to-one mapping  $L : V \rightarrow \{1, \dots, n\}$ . We will also denote a layout  $L$  by  $\langle v_1, v_2, \dots, v_n \rangle$  such that  $L(v_i) = i$ . For an integer  $i$  between 1 and  $n$  we define the set  $V_i$  to be  $\{v_1, \dots, v_i\}$ . We say that  $u$  is *before*  $v$  in  $L$ , or  $u <_L v$ , if  $L(u) < L(v)$ . The *cut of  $G$  at the  $i$ th gap* in a given layout  $L$  is defined as  $\delta_L(i) = \delta^G(V_i)$  and  $d_L(i) = d^G(V_i)$ . The *cutwidth* of a layout  $L$  of  $G$  is  $cw_L(G) = \max_{1 \leq i \leq n} d^G(V_i)$ . The *cutwidth of  $G$*  is  $cw(G) = \min_L \{cw_L(G)\}$  where  $L$  is any possible layout of  $G$ . In this paper, an *optimal layout* of  $G$  is a layout  $L$  such that  $cw(G) = cw_L(G)$ .

The *bisection width* of  $G$ , denoted by  $bw(G)$ , is the minimum cut size of any set  $S \subset V$  on  $\lfloor \frac{n}{2} \rfloor$  vertices. Since  $\delta(S) = \delta(V \setminus S)$  it follows that  $bw(G)$  is the minimum cut size of  $S$  of any set  $S$  on  $\lfloor \frac{n}{2} \rfloor$  or  $\lceil \frac{n}{2} \rceil$  vertices. It should be clear that  $bw(G)$  gives a lower bound for  $cw(G)$ , that is,  $bw(G) \leq cw(G)$  [10]. We will use the close connection between cutwidth and bisection width actively in some of our proofs. A useful observation is that the cutwidth of a subgraph  $G$  cannot exceed the cutwidth of  $G$  [10].

A graph is a *split graph* if its vertex set can be partitioned into a clique  $C$  and an independent set  $I$ , where  $(C, I)$  is called a *split partition*. Threshold graphs are a subset of split graphs, and for their original definition we refer to [12, 18]. We will use as definition the following characterization: A graph is a *threshold graph* if and only if it has a split partition  $(C, I)$  such that vertices of the  $I$  (and equivalently the vertices of  $C$ ) can be ordered by neighborhood inclusion [18].

In fact, if a split partition of  $G$  satisfies the above property, then all split partitions of  $G$  satisfy it [12, 18]. For a graph  $G$  with split partition  $(C, I)$ , if there is a vertex  $x$  of  $C$  which is not adjacent to any vertex of  $I$  then clearly  $(C \setminus \{x\}, I \cup \{x\})$  is also a split partition of  $G$ . For our purposes, we will always assume that every vertex of  $C$  has a neighbor in  $I$ . For a threshold graph  $G$ , we refine the sets  $I$  and  $C$  as follows, and call it a *threshold partition*:  $(I_0, I_1, I_2, \dots, I_\ell)$  is a partition of  $I$  such that  $I_0$  is the set of isolated vertices, and  $N(I_1) \subset N(I_2) \subset \dots \subset N(I_\ell)$ , where  $\ell$  is as large as possible and defined to be the number of *levels*. Thus all vertices in  $I_j$  have the same degree for  $0 \leq j \leq \ell$ . This also defines a partition  $(C_1, C_2, \dots, C_\ell)$  of  $C$ , where  $C_1 = N(I_1)$  and  $C_j = N(I_j) \setminus N(I_{j-1})$  for  $2 \leq j \leq \ell$ . Again, all vertices in  $C_j$  have the same degree for  $1 \leq j \leq \ell$ . We say that vertices of  $C_j$  and  $I_j$  belong to the  $i$ th level of the clique and of the independent set, respectively. Observe that by construction, the sets  $C_i$  and  $I_i$  are nonempty for every  $1 \leq i \leq \ell$ . For a vertex  $v$ , define  $level(v)$  to be the level that  $v$  belongs to.

**Lemma 2.1** ([13]). *Let  $G$  be a threshold graph with threshold partition  $((C_1, \dots, C_\ell), (I_1, \dots, I_\ell))$ . Let  $uv$  be an edge such that  $u \in I_j$  and  $v \in C_j$  for some  $j$ . Then  $G - uv$  is a threshold graph.*

### 3 Cutwidth of split graphs

In this section we show that the cutwidth problem is NP-complete on split graphs. In fact the proof of Theorem 3.1 shows that cutwidth is NP-complete even on split graphs where every

vertex of  $I$  in a split partition  $(C, I)$  has degree 2.

**Theorem 3.1.** *The cutwidth problem is NP-complete on split graphs.*

*Proof.* The reduction is from an arbitrary instance of the cutwidth problem. Given an arbitrary graph  $G = (V, E)$  with  $n$  vertices and  $m$  edges, we construct a split graph  $G'$  as follows. To start with,  $G'$  is a complete graph on  $V$ . Let  $k = n^2 + 1$ . For every edge  $uv \in E$  we add  $k$  more vertices to  $G'$ , making each new vertex adjacent to  $u$  and  $v$  in  $G'$ . We say that these vertices of  $G'$  *correspond* to the edge  $uv$  of  $G$ . Observe that  $G'$  has  $n + km$  vertices where the  $n$  vertices of  $V$  induce a clique in  $G'$ . The remaining  $km$  vertices are only adjacent to vertices of this clique. Hence  $G'$  is a split graph. Moreover the whole construction can be carried out in polynomial time. We now prove that for any  $1 \leq c \leq n^2$  we have  $cw(G) \leq c$  if and only if  $cw(G') < c(k + 1) + k$ . (Note that  $n^2$  is a trivial upper bound on the cutwidth of any graph on  $n$  vertices.)

If  $cw(G) \leq c$  then consider a layout  $L$  for which  $cw_L(G) \leq c$ . We create a layout  $L'$  of  $G'$  by ordering the vertices in  $V$  in the same order that they have in  $L$ . Every vertex  $x$  of  $G'$  that corresponds to an edge  $uv$  of  $G$  is placed in an arbitrary position between  $u$  and  $v$  in  $L'$ . Observe that since  $x$  has degree 2 and is placed between its neighbors,  $d_{L'}(L'(x) - 1) = d_{L'}(L'(x))$ , and thus, to compute  $cw_{L'}(G')$  it is sufficient to consider maximum  $d_{L'}(L'(v))$  over all  $v \in V$ . From the construction of  $L'$  it follows that for every vertex  $v$  in  $V$ ,  $\delta_{L'}(L'(v))$  contains at most  $k \cdot d_L(L(v))$  edges between vertices in  $V$  and vertices corresponding to edges of  $G$ , and at most  $n^2$  edges between pairs of vertices in  $V$ . Thus  $d_{L'}(L'(v)) \leq k \cdot d_L(L(v)) + n^2$  and  $k \cdot d_L(L(v)) + n^2 < kc + k = k(c + 1)$  and  $cw(G') \leq cw_{L'}(G') < k(c + 1)$  follows.

Let  $L'$  be a layout of  $G'$  for which  $cw_{L'}(G') < k(c + 1)$ . From  $L'$  we construct a layout  $L$  of  $G$  by ordering the vertices of  $V$  in the same order that  $L'$  orders them. We prove that  $cw_L(G) \leq c$ . For a given vertex  $x$  we observe that for every edge  $uv \in \delta_L(L(x))$  and vertex  $y$  of  $G'$  corresponding to the edge  $uv$ , either the edge  $yu$  or the edge  $yv$  must be in  $\delta_{L'}(L'(x))$ . Thus,  $k \cdot d_L(L(x)) \leq d_{L'}(L'(x)) < k(c + 1)$ . By dividing both sides by  $k$  we obtain  $d_L(L(x)) < c + 1$ . Since we chose  $x$  arbitrarily,  $cw_L(G) \leq c$  and the result follows.  $\square$

## 4 Cutwidth of threshold graphs

In this section we give an algorithm that computes the cutwidth of threshold graphs in linear time. This algorithm constructs a layout  $L$  by greedily appending at step  $i$  the vertex  $v_i$  that minimizes the cut  $\delta(V_{i-1} \cup \{v_i\})$ .

For a given graph  $G = (V, E)$  and a set  $S \subseteq V$ , we define the *rank* of a vertex  $v$  with respect to  $S$  to be  $rank_S^G(v) = |N_G(v) \cap (V \setminus S)| - |N_G(v) \cap S|$  (superscript  $G$  is omitted when not needed). Observe that if  $v \notin S$  then  $d(S \cup \{v\}) = d(S) + rank_S(v)$ . Thus, at step  $i$ , we greedily select the vertex  $v$  with the lowest rank with respect to  $V_{i-1}$  among the vertices that have not yet been laid out. If there is a tie, the algorithm picks the vertex with the highest degree among the tied vertices. If there still is a tie, the algorithm picks the vertex with the smallest label. The intuition behind the highest-degree tie-breaking strategy is as follows: when we add a vertex  $v$  to a set  $S$ , the ranks of all  $v$ 's neighbors with respect to  $S$  decrease by 2, while the ranks of  $v$ 's non-neighbors remain unchanged. Since we want the rank of the vertices we pick to be as small as possible, it is good to decrease the rank of as many vertices as possible, and that is why we choose a vertex of highest degree when there is a tie. The details of the algorithm called MinCut are given below.

**Algorithm: MinCut****Input:** A graph  $G = (V, E)$ .**Output:** A layout  $L = \langle v_1, v_2, \dots, v_n \rangle$  of  $G$  $V_0 := \emptyset;$ **for**  $i = 1$  to  $n$  **do**     $v_i :=$  the vertex in  $V \setminus V_{i-1}$  with smallest label;    **for** every vertex  $v$  in  $V \setminus V_{i-1}$  ordered by increasing label **do**        **if**  $\text{rank}_{V_{i-1}}(v) < \text{rank}_{V_{i-1}}(v_i)$  **then**  $v_i := v$         **else if**  $\text{rank}_{V_{i-1}}(v) = \text{rank}_{V_{i-1}}(v_i)$  and  $\Delta(v) > \Delta(v_i)$  **then**  $v_i := v$      $V_i := V_{i-1} \cup \{v_i\};$      $L(v_i) := i;$ 

Before reaching the details of why Algorithm MinCut produces optimal layouts when the input is a threshold graph, we need to study how the layouts produced by the algorithm look. Observe first that if  $G$  has isolated vertices, then these can be placed in arbitrary positions in any optimal cutwidth layout, and our algorithm places them in the beginning of the output layout. In the remainder of this section we let  $G$  be a threshold graph with threshold partition  $((C_1, \dots, C_\ell), (I_1, \dots, I_\ell))$ , and  $L$  be the layout computed by Algorithm MinCut when run on  $G$ .

One should notice that in a threshold graph, two vertices with the same degree have the same open neighborhood if they are nonadjacent and the same closed neighborhood if they are adjacent. Thus, in threshold graphs, the labels of the vertices do not really affect the layout produced by Algorithm MinCut, because whether or not there is an edge between  $v_i$  and  $v_j$  for  $1 \leq i < j \leq n$  is independent of the labels of the vertices, and depends only on to which sets of the threshold partition they belong to. The reason we include the labels in the description of the algorithm is that the labels simplify the discussions in the proofs. One should note that the labels do indeed affect the layout produced by the algorithm if the algorithm is run on a graph that is not a threshold graph.

Because of the discussion above, we can assume without loss of generality that the labels of  $G$  have a specific order. In particular, we will assume that  $G$  has been labeled in a manner such that every vertex in  $I$  has smaller label than every vertex in  $C$ , for every pair  $u$  and  $v$  of vertices in  $I$ ,  $\text{level}(u) < \text{level}(v)$  implies  $\text{label}(u) < \text{label}(v)$  and for every pair  $u$  and  $v$  of vertices in  $C$ ,  $\text{level}(u) < \text{level}(v)$  implies  $\text{label}(u) < \text{label}(v)$ . Note that this can also be done in an  $O(n)$ -time preprocessing step, using the threshold partition of  $G$ .

Given two vertices  $u$  and  $v$  of  $G$  such that  $u \in I$  and  $v \in C$ , we define the vertex set  $\text{over}(u, v)$  to contain all vertices  $x \in I$  such that  $\text{label}(x) < \text{label}(u)$  and all vertices  $y \in C$  such that  $\text{label}(y) < \text{label}(v)$ . The essence of the following lemma is that the algorithm picks vertices at lower levels before proceeding to higher levels, and that the first vertex of any level to be picked by the algorithm is an independent set vertex.

**Lemma 4.1.** *Let  $G$  be a threshold graph with threshold partition  $((C_1, \dots, C_\ell), (I_1, \dots, I_\ell))$  and let  $L$  be a layout computed by Algorithm MinCut. Then, for any  $i \in \{1, \dots, n\}$  and  $k \in \{1, \dots, \ell\}$ :*

- (a) *If  $V_i \cap I_k \neq \emptyset$  then  $I_{k'} \subseteq V_i$ , for every  $1 \leq k' < k$ .*
- (b) *If  $V_i \cap C_k \neq \emptyset$  then  $C_{k'} \subseteq V_i$ , for every  $1 \leq k' < k$ .*
- (c) *If  $V_i \cap I_k = \emptyset$  then  $V_i \cap C_k = \emptyset$ .*

*Proof.* We prove all the statements simultaneously by induction on  $i$ . Vertex  $v_1$  is of minimum degree and hence  $v_1 \in I_1$ , so for  $i = 1$  the statements are trivially true. Assume now that all

three statements hold whenever  $i < r$  and consider the  $r$ 'th step of the algorithm. We first prove that (a) must be true for  $i = r$ . It suffices to show that if  $v_r \in I$  then  $v_r$  is a vertex with the lowest degree out of the ones that are not in  $V_{r-1}$ . Indeed, consider two vertices  $u$  and  $v \in I \setminus V_{r-1}$  such that  $\Delta(u) < \Delta(v)$ . Since  $level(v) > level(u)$ , because (c) holds for  $V_{r-1}$  by the induction hypothesis, every neighbor of  $v$  that is not a neighbor of  $u$  is not in  $V_{r-1}$ . Thus  $rank_{V_{r-1}}(u) < rank_{V_{r-1}}(v)$  and (a) follows for  $i = r$ .

We prove that (b) is true for  $i = r$ ; that is, if  $v_r \in C$  then  $v_r$  is a vertex with the highest degree out of the ones that are not in  $V_{r-1}$ . Let  $t$  be the largest integer such that  $V_{r-1} \cap I_t \neq \emptyset$ . For a vertex  $u \in C_{t'}$  with  $t < t'$ , Let  $v$  be a vertex in  $I_{t'}$ . Because (c) holds for  $V_{r-1}$  by the induction hypothesis,  $v \notin V_{r-1}$  and  $rank_{V_{r-1}}(v) < rank_{V_{r-1}}(u)$  and so the algorithm would not pick  $v_r$  to be  $u$ . Furthermore, for every two vertices  $u$  and  $v$  in  $C \setminus V_{r-1}$  such that  $level(u) < level(v) \leq t$ , it follows that every neighbor of  $u$  that is a non-neighbor of  $v$  is in  $V_{r-1}$  and yielding  $rank_{V_{r-1}}(u) < rank_{V_{r-1}}(v)$  and completing the proof of (b) for  $i = r$ .

Finally, we prove that (c) is true for  $i = r$ . Consider a level  $t$  such that  $I_t \cap V_{r-1} = C_t \cap V_{r-1}$  and let  $u \in I_t$  and  $v \in C_t$ . Since (a) and (b) are true for  $r - 1$ , every neighbor of  $v$  that is not a neighbor of  $u$  is not in  $V_{r-1}$ . Unless  $t = \ell$  and  $|I_\ell| = 1$ ,  $\Delta(u) < \Delta(v)$  so  $rank_{V_{r-1}}(u) < rank_{V_{r-1}}(v)$ . If  $t = \ell$  and  $|I_\ell| = 1$  then  $u$  and  $v$  have the same closed neighborhood, but  $label(u) < label(v)$ . In both cases  $u <_L v$ , and so (c) must be true for  $i = r$ .  $\square$

One should notice that, as a direct consequence of Lemma 4.1 (a) and (b), for any  $i$  between 1 and  $n$ , if  $u \in I \cap V_i$  and  $v \in C \cap V_i$  then  $over(u, v) \subseteq V_i$ .

For the statements of the following results, let  $G = (V, E)$  be a threshold graph with threshold partition  $(C, I) = ((C_1, \dots, C_\ell), (I_1, \dots, I_\ell))$  and let  $L = \langle v_1, \dots, v_n \rangle$  be a layout computed by Algorithm MinCut on input  $G$ .

**Lemma 4.2.** *Let  $u \in I$  and  $v \in C$ . Then  $u <_L v$  if and only if  $u$  and  $v$  are non-adjacent or  $rank_{over(u,v)}(u) < rank_{over(u,v)}(v)$ .*

*Proof.* Let  $u$  and  $v$  be non-adjacent. Then, clearly  $level(u) < level(v)$ . Then by Lemma 4.1 (c),  $u <_L v$ . Let  $u$  and  $v$  be adjacent and  $u <_L v$ . We prove that  $rank_{over(u,v)}(u) < rank_{over(u,v)}(v)$ . Clearly,  $rank_{V_{L(u)-1}}(u) < rank_{V_{L(u)-1}}(v)$ . Furthermore, every vertex in  $over(u, v) \setminus V_{L(u)-1}$  is adjacent to both  $u$  and  $v$ . Thus  $rank_{over(u,v)}(u) < rank_{over(u,v)}(v)$ . Similarly, if  $u$  and  $v$  are adjacent and  $u > v$ , it follows that  $rank_{V_{L(v)-1}}(u) \geq rank_{V_{L(v)-1}}(v)$ . Now, every vertex in  $over(u, v) \setminus V_{L(v)-1}$  is nonadjacent to  $u$ . Hence  $rank_{over(u,v)}(u) \geq rank_{over(u,v)}(v)$ .  $\square$

We say that the algorithm *covers* a vertex set  $S$  if there is an index  $i$  such that  $V_i = S$ . If the algorithm covers  $S$  then we also say that  $S$  is covered.

**Lemma 4.3.** *Let  $u, u' \in I$  with  $label(u') = label(u) + 1$ , and let  $v, v' \in C$  with  $label(v') = label(v) + 1$ . Then  $over(u', v')$  is covered if and only if  $u <_L v'$  and  $v <_L u'$ .*

*Proof.* If  $over(u', v')$  is covered then  $over(u', v') = V_i$  with  $L(u) \leq i$ ,  $L(v) \leq i$ ,  $L(u') > i$  and  $L(v') > i$  so  $u <_L v'$  and  $v <_L u'$ . In the other direction, suppose  $u <_L v'$  and  $v <_L u'$ . Let  $S' = V_i$  be the smallest covered set that contains both  $u$  and  $v$ . By Lemma 4.1,  $over(u', v') = \{u, v\} \cup over(u, v) \subseteq S'$ . Furthermore, since  $S'$  is the smallest covered set that contains both  $u$  and  $v$ , either  $v_i = u$  or  $v_i = v$ . If  $v_i = u$  then  $v_i <_L u'$  by Lemma 4.1 and  $v_i <_L v'$  by assumption, while if  $v_i = v$  then  $v_i <_L u'$  by assumption and  $v_i <_L v'$  by Lemma 4.1. In both cases, neither  $u'$  nor  $v'$  can be in  $S'$  which means that  $S = S'$  and that  $S$  is covered.  $\square$



We now are equipped with most of the tools that are necessary to work with layouts produced by Algorithm MinCut. All that remains before we move on to proving the correctness of the algorithm are a couple of simple observations.

**Observation 4.4.** *For each integer  $i \leq n - 1$ ,  $\text{rank}_{V_i}(v_{i+1}) \geq \text{rank}_{V_{i-1}}(v_i) - 2$ . Furthermore, if  $\Delta(v_{i+1}) > \Delta(v_i)$  then  $\text{rank}_{V_i}(v_{i+1}) \geq \text{rank}_{V_{i-1}}(v_i) - 1$*

*Proof.* Observe that  $\text{rank}_{V_i}(v_{i+1}) = \text{rank}_{V_{i-1}}(v_{i+1}) - 2$  if  $v_{i+1}$  and  $v_i$  are adjacent and  $\text{rank}_{V_i}(v_{i+1}) = \text{rank}_{V_{i-1}}(v_{i+1})$  if not. Since the algorithm picked  $v_i$  and not  $v_{i+1}$  at step  $i$  the observation follows.  $\square$

**Observation 4.5.** *For every level  $k \leq \ell$  and every triple of vertices  $u, v \in C_k$  and  $w \notin C_k$ ,  $u <_L w$  if and only if  $v <_L w$ .*

*Proof.* If  $w \in C$  the observation follows from Lemma 4.1 (b). If  $w \in I$  and  $uw \notin E$  then  $vw \notin E$  and the observation follows from Lemma 4.2. If  $w \in I$  and  $uw \in E$  then  $vw \in E$ . Without loss of generality,  $\text{label}(u) < \text{label}(v)$ . Note that every vertex in  $\text{over}(w, v) \setminus \text{over}(w, u)$  is in  $C_k$  and so  $\text{rank}_{\text{over}(w, u)}(w) < \text{rank}_{\text{over}(w, u)}(u)$  if and only if  $\text{rank}_{\text{over}(w, v)}(w) < \text{rank}_{\text{over}(w, v)}(v)$ . Applying Lemma 4.2 completes the proof.  $\square$

## 4.1 Correctness of Algorithm MinCut

In this subsection, we show that Algorithm MinCut produces optimal layouts when the input is a threshold graph. We assume for contradiction that there is a threshold graph  $G = (V, E)$  with threshold partition  $((C_1, \dots, C_\ell), (I_1, \dots, I_\ell))$  on which Algorithm MinCut outputs layout  $L = \langle v_1, \dots, v_n \rangle$  such that  $cw_L(G) > cw(G)$ . We call such a threshold graph a *counterexample*, and we say that a counterexample is *minimal* if it has the smallest value of  $|V| + |E|$  among all counterexamples. A *bad set* of counterexample  $G$  is a set  $S \subseteq V$  that is covered by the algorithm and for which  $d(S) > cw(G)$ . A *locally worst* bad set is a bad set  $S = V_i$  such that  $d(S) \geq d(V_{i+1})$  and  $d(S) \geq d(V_{i-1})$ . Observe that  $\text{rank}_{V_{i-1}}(v_i)$  must be non-negative and  $\text{rank}_{V_i}(v_{i+1})$  must be non-positive for  $V_i$  to be locally worst. Observation 4.4 then implies that  $\text{rank}_{V_{i-1}}(v_i)$  is 2, 1 or 0. This means that if  $v_i \in C_1$  then  $i$  is  $\lfloor \frac{n}{2} \rfloor$ ,  $\frac{n}{2}$  or  $\lceil \frac{n}{2} \rceil$  respectively. Another thing to notice about locally worst bad sets is that if both  $V_i$  and  $V_{i-1}$  are bad sets with  $d(V_{i-1}) \geq d(V_i)$  then some locally worst bad set is a strict subset of  $V_i$ .

The main idea of the proof is to show that if there is a counterexample  $G$ , then there must be another counterexample  $G'$  that either has at most 2 levels, or exactly 3 levels and a very specific structure. We complement this result by showing that the algorithm produces optimal layouts on all graphs with at most 2 levels, and on all graphs with 3 levels and the mentioned structural properties. This yields that  $cw_L(G) = cw(G)$  for every threshold graph  $G$ .

**Lemma 4.6.** *Let  $G$  be a threshold graph on exactly 1 level. Then  $cw_L(G) = cw(G)$ .*

*Proof.* Let  $(C, I)$  be a threshold partition of  $G$ . Observe that every vertex of  $I$  is adjacent to every vertex of  $C$ . The algorithm lays out  $\lfloor \frac{|I|}{2} \rfloor$  vertices of  $I$ , then all of  $C$ , followed by the remaining vertices of  $I$ . By inspection,  $cw_L(G) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil - \lfloor \frac{|I|}{2} \rfloor \cdot \lceil \frac{|I|}{2} \rceil$ . Since all non-edges of  $G$  are between vertices in  $I$ ,  $bw(G) = \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil - \lfloor \frac{|I|}{2} \rfloor \cdot \lceil \frac{|I|}{2} \rceil$ . Thus  $cw_L(G) = bw(G) \leq cw(G) \leq cw_L(G)$  and  $cw_L(G) = cw(G)$  follows.  $\square$

Already for threshold graphs with 2 levels, the correctness proof for Algorithm MinCut is more complicated. Before we go on to this proof we need more tools to work with locally worst bad sets.

For the statements of all the remaining results and definitions in this section, whenever we mention a counterexample  $G$ ,  $((C_1, \dots, C_\ell), (I_1, \dots, I_\ell))$  is its threshold partition, and  $L = \langle v_1, \dots, v_n \rangle$  is the output of Algorithm MinCut on  $G$ .

**Lemma 4.7.** *Every locally worst bad set  $S$  of a counterexample  $G$  satisfies (i)  $C_1 \cap S \neq \emptyset$ , (ii)  $I_1 \subseteq S$ , (iii)  $C_\ell \cap S = \emptyset$ , and (iv)  $I_\ell \setminus S \neq \emptyset$ .*

*Proof.* (i) If  $S \cap C_1 = \emptyset$  and  $v_{i+1} \in I$  then  $\text{rank}_{V_i}(v_{i+1}) = \Delta(v_{i+1}) > 0$  contradicting that  $S$  is locally worst. If  $S \cap C_1 = \emptyset$ ,  $v_{i+1} \in C$  and  $\text{rank}_{V_i}(v_{i+1}) > 0$  then  $S$  is not locally worst, so  $\text{rank}_{V_i}(v_{i+1}) \leq 0$ . Since  $v_i \in I$  and  $\text{rank}_{V_{i-1}}(v_i) = \Delta(v_{i+1}) > 0$ , Observation 4.4 implies that  $\text{rank}_{V_i}(v_{i+1}) = 0$  and that  $\text{rank}_{V_{i-1}}(v_i) = 1 = \Delta(v_i)$ . This means  $n$  is odd, and  $i = \lfloor \frac{n}{2} \rfloor$ . Therefore  $d(S) = \lfloor \frac{n}{2} \rfloor \leq bw(G) \leq cw(G)$  contradicting that  $S$  is a bad set. We can conclude that  $S \cap C_1 \neq \emptyset$ .

(ii) Suppose for contradiction that  $I_1 \setminus S \neq \emptyset$ . By Lemma 4.1,  $S \subset I_1 \cup C_1$ . By the discussion in the previous paragraph,  $S \cap C_1 \neq \emptyset$ . If  $v_i \in I_1$  then Observation 4.5 implies  $C_1 \subseteq V_{i-1}$  so  $\text{rank}_{V_{i-1}}(v_i) = -\Delta(v_i) < 0$ , contradicting that  $S$  is locally worst. Thus  $v_i \in C_1$ .

Let  $I_f$  be the subset of  $I_1$  that  $L$  puts before  $C_1$ , and  $C_f$  be the  $\lceil \frac{|C_1|}{2} \rceil$  vertices of  $C_1$  with smallest labels. Observation 4.5 guarantees that the set  $S' = I_f \cup C_f$  is covered, so  $S' = V_j$ . We prove that  $S'$  is a bad set. Let  $x$  be the vertex of  $C_1$  with the smallest label and let  $k = L(x)$ . Since the algorithm chose  $x$  over a vertex in  $I_1$  in step  $k$ ,  $\text{rank}_{V_{k-1}}(x) \leq |C_1|$ . By Observation 4.4,  $\text{rank}_{V_{k-1}}(x) \geq |C_1| - 1$ . Now, for every  $k \leq t < k + |C_1|$  we have  $\text{rank}_{V_{t-1}}(v_t) = \text{rank}_{V_{k-1}}(x) - 2(t - k)$ . Note that since  $v_i \in C_1$ ,  $i < k + |C_1|$ . Since  $|C_f| = \lceil \frac{|C_1|}{2} \rceil$ , we have  $1 + (j - k) = \lceil \frac{|C_1|}{2} \rceil$ . Thus  $\text{rank}_{V_{j-1}}(v_j) = \text{rank}_{V_{k-1}}(x) - 2(j - k) = \text{rank}_{V_{k-1}}(x) - 2(\lceil \frac{|C_1|}{2} \rceil - 2)$ . This in turn implies  $|C_1| + 1 - 2\lceil \frac{|C_1|}{2} \rceil \leq \text{rank}_{V_{j-1}}(v_j) \leq |C_1| + 2 - 2\lceil \frac{|C_1|}{2} \rceil$ . Simplifying the bounds yields  $0 \leq \text{rank}_{V_{j-1}}(v_j) \leq 2$ . Therefore, for every  $t$  with  $k \leq t < j$ ,  $\text{rank}_{V_{t-1}}(v_t) \geq 0$ , while for every  $t$  such that  $j < t \leq i$ ,  $\text{rank}_{V_{t-1}}(v_t) \leq 0$ . Thus  $d(S') \geq d(S)$ .

Let  $G'$  be the graph obtained from  $G$  by removing all edges that do not have at least one endpoint in  $C_1$ . The threshold partition of  $G'$  is  $(C_1, V \setminus C_1)$  and every vertex of  $C_1$  is universal.  $S'$  contains  $\lceil \frac{|C_1|}{2} \rceil$  vertices of  $C_1$  and exactly  $\lfloor \frac{n - |C_1|}{2} \rfloor$  vertices of the independent set of the threshold partition of  $G'$ . Thus,  $d(S) \leq d(S') \leq bw(G') \leq cw(G') \leq cw(G)$  contradicting that  $S$  is a bad set. We can conclude that  $I_1 \subseteq S$ .

(iii) Suppose for contradiction that  $S \cap C_\ell \neq \emptyset$ . By Lemma 4.6  $G$  has at least two levels. If  $|I_\ell| = 1$  then all edges of  $\delta(S)$  are between vertices of  $C \cup I_\ell$  which induces a clique in  $G$ . In that case  $d(S) \leq cw(G[C \cup I_\ell]) \leq cw(G)$  contradicting that  $S$  is a bad set. Thus,  $|I_\ell| > 2$ . If  $v_i \in I_\ell$ , Lemma 4.5 implies that  $C_\ell \subseteq V_{i-1}$  which in turn implies  $C \subseteq V_{i-1}$ . Then  $\text{rank}_{V_{i-1}}(v_i) = -\Delta(v_i) < 0$ , contradicting that  $S$  is locally worst. Let  $I_f$  be the set of the  $\lfloor \frac{n}{2} \rfloor$  vertices of  $I_\ell$  with smallest labels. Let  $u$  be the vertex in  $I_f$  with the largest label,  $u'$  be the vertex in  $I_\ell$  with  $\text{label}(u') = \text{label}(u) + 1$  and  $v$  be the vertex in  $C_\ell$  with the smallest label.

We wish to show that  $u <_L v <_L u'$ . Notice that  $v$  is the only neighbor of  $u$  and  $u'$  that is a non-neighbor of  $v$  and that  $v \notin \text{over}(u, v)$  and  $v \notin \text{over}(u', v)$ . Among the neighbors of  $v$ , apart from  $u$ , that are non-neighbors of  $u$ , there are  $\lfloor \frac{|I_\ell|}{2} \rfloor - 1$  in  $\text{over}(u, v)$  and  $I_\ell - \lfloor \frac{|I_\ell|}{2} \rfloor \geq \lfloor \frac{|I_\ell|}{2} \rfloor$  outside. Thus by Lemma 4.2,  $u <_L v$ . Similarly, among the neighbors of  $v$ , apart from  $u'$ , that are non-neighbors of  $u'$ , there are  $\lfloor \frac{|I_\ell|}{2} \rfloor$  in  $\text{over}(u, v)$  and  $I_\ell - \lfloor \frac{|I_\ell|}{2} \rfloor - 1 \leq \lfloor \frac{|I_\ell|}{2} \rfloor$  outside. Since  $d(u') < d(v)$ , by Lemma 4.2,  $v <_L u'$ . Thus  $S \cap I_\ell = I_f$ .

Notice that all the edges of  $\delta(S)$  are between vertices in  $I_\ell \cup C$ . Thus, if we let  $S' = I_f \cup (C \cap S)$ , we have  $d(S) = d^{G[I_\ell \cup C]}(S')$ . Now  $G[I_\ell \cup C]$  has threshold partition  $(C, I_\ell)$  and every vertex of  $C$  is adjacent to every vertex of  $I_\ell$ .  $S'$  contains exactly  $\lfloor \frac{|I_\ell|}{2} \rfloor$  the vertices of  $I_\ell$  and thus  $d(S) \leq d^{G[I_\ell \cup C]}(S') \leq bw(G[I_\ell \cup C]) \leq cw(G[I_\ell \cup C]) \leq cw(G)$  contradicting that  $S$  is a bad set. Thus  $S \cap C_\ell = \emptyset$ .

(iv) Suppose for contradiction that  $I_\ell \subseteq S$ . If  $|I_\ell| = 1$ , let  $u$  be the vertex with the smallest label in  $I_\ell$  and let  $v$  be the vertex with the largest label in  $C_{\ell-1}$ . Every neighbor of  $v$  apart from  $u$  that is a non-neighbor of  $u$  lies in  $over(u, v)$ , thus by Lemma 4.2,  $v <_L u$  and  $C_1 \subset S$ . Then every edge of  $\delta(S)$  is between vertices of  $C \cup I_\ell$ . Since  $C \cup I_\ell$  induces a clique in  $G$  this contradicts that  $S$  is a bad set. Thus  $|I_2| > 1$ .

If  $|I_2| > 1$  let  $u$  be the vertex with the highest label in  $I_\ell$  and  $v$  be the vertex with the highest label in  $C_\ell$ . Every vertex that is Every neighbor of  $v$  apart from  $u$  that is a non-neighbor of  $u$  lies in  $over(u, v)$  thus, by Lemma 4.2  $v <_L u$  and  $C_2 \subset S$ . This contradicts that any locally worst bad set  $S$  of a counterexample  $G$  contains no vertices of  $C_\ell$ .  $\square$

**Lemma 4.8.** *Let  $G$  be a threshold graph on exactly 2 levels. Then  $cw_L(G) = cw(G)$ .*

*Proof.* Suppose for contradiction that  $G$  is a counterexample on two levels with minimum value of  $|V| + |E|$  among all counterexamples on two levels. Lemma 4.6 ensures that  $G$  is a counterexample with minimum value of  $|V| + |E|$  among all counterexamples on *at most* two levels, but  $G$  is *not* necessarily a minimal counterexample. Since  $G$  is a counterexample,  $G$  has a locally worst bad set  $S = V_i$ . Let  $S$  be the smallest locally worst bad set of  $G$ .

We first consider the case that  $|C_1| < |C_2|$  and  $S \cap I_2 = \emptyset$ . By Lemma 4.1,  $S \cap C_2 = \emptyset$ . By Lemma 4.7,  $S \cap C_1 \neq \emptyset$ . If  $v_i \in I_1$  then Observation 4.5 implies  $C_1 \subseteq V_{i-1}$ , so  $rank_{V_{i-1}}(v_i) = -\Delta(v_i) < 0$ , contradicting that  $S$  is locally worst. Thus  $v_i \in C_1$ . Furthermore, by Lemma 4.7,  $I_1 \subseteq S$ . If  $v_{i+1} \in I_2$ , by assumption, we have  $|C_1| < |C_2|$ . In this case  $rank_{V_i}(v_{i+1}) > 0$ , contradicting that  $S$  is locally worst. Therefore  $v_{i+1}$  must be in  $C_1$ .

Let  $u$  be the vertex of  $I_1$  with highest label,  $u'$  be the vertex of  $I_2$  with smallest label,  $v = v_i$  and  $v' = v_{i+1}$ . Observe that  $S = over(u', v')$ . Let  $x$  be the vertex in  $C_2$  with highest label. By Lemma 4.2,  $v <_L u'$  if and only if  $rank_{over(u', v)}(u') \geq rank_{over(u', v)}(v)$ . Now  $rank_{over(u', v)}(u') - rank_{over(u', v)}(v) = 1 + |I_1| - |I_2|$ .

Suppose  $1 + |I_1| > |I_2|$ . Then  $rank_{over(u', v)}(u') > rank_{over(u', v)}(v)$ . We make a new graph  $G'$  from  $G$  by deleting all edges between  $x$  and vertices in  $I_2$  and relabeling the vertices such that the labeling order of all vertices is the same as in  $G$ , with the exception of  $x$  which gets the highest label among the vertices the independent set of the threshold partition of  $G'$ . Observe that  $G'$  is a threshold graph on at most 2 levels, and that  $G'$  is a proper subgraph of  $G$ . In addition  $rank_{over(u, v')}(u)$ ,  $rank_{over(u, v')}(v')$  and  $rank_{over(u', v)}(v)$  remain unchanged from  $G$  to  $G'$  while  $rank_{over(u', v)}^{G'}(u') = rank_{over(u', v)}(u') - 1$ . In  $G$  we had  $rank_{over(u, v')}(u) < rank_{over(u, v')}(v')$  and  $rank_{over(u', v)}(u') > rank_{over(u', v)}(v)$ , so in  $G'$  we know that  $rank_{over(u, v')}^{G'}(u) < rank_{over(u, v')}^{G'}(v')$  and  $rank_{over(u', v)}^{G'}(u') \geq rank_{over(u', v)}^{G'}(v)$ . Lemma 4.2 yields  $u <_{L'} v'$  and  $v <_{L'} u'$ . By Lemma 4.3,  $S$  is covered also by  $L'$ . Since we did not delete any edges in  $\delta(S)$  it follows that  $S$  is a bad set also in  $G'$ . This contradicts that  $G$  is a counterexample on at most 2 levels with the smallest value of  $|V| + |E|$ .

To conclude the case that  $|C_1| < |C_2|$  and  $S \cap I_2 = \emptyset$ , suppose that  $1 + |I_1| = |I_2|$ . Then  $S \subseteq I_1 \cup C_1$ ,  $|I_1| < |I_2|$  and  $|C_1| < |C_2|$  together imply that  $|S| \leq |V \setminus S| + 2$ . But  $rank_{V_{i-1}}(v_i) \leq 2$  implies  $|S| \geq \lfloor \frac{n}{2} \rfloor$ , which is a contradiction.

Now, consider the case that  $|C_1| < |C_2|$  and  $S \cap I_2 \neq \emptyset$ . If  $|I_2| = 1$  then  $I_2 \subseteq S$ , contradicting Lemma 4.7. Thus  $|I_2| > 1$ . Suppose  $C_1 \setminus S \neq \emptyset$ . Let  $u$  be the vertex in  $I_2 \cap S$  with highest label and  $x$  be the vertex in  $C_1$  with lowest label. Lemmas 4.7 and 4.5 imply  $L(x) = L(u) + 1$  and  $v_{i+1} \in C_1$ . By Lemma 4.4,  $\text{rank}_{V_{L(x)-1}}(x) \geq |C| - 1$ . Thus  $\text{rank}_{V_i}(v_{i+1}) \geq |C| - 1 - 2(|C_1| - 1) = |C_2| + 1 - |C_1| > 0$ , contradicting that  $S$  is locally worst. Thus  $C_1 \subseteq S$ .

If  $v_{i+1} \in I_2$  then by Lemma 4.7,  $\text{rank}_{V_i}(v_{i+1}) \geq |C_2| - |C_1| > 0$  contradicting that  $S$  is locally worst. Thus  $v_{i+1} \in C_2$ . Furthermore as  $S \cap C_2 = \emptyset$  this means that  $v_{i+1}$  is the vertex of  $C_2$  with lowest rank and that  $|S \cap I_2| = \lfloor \frac{|I_2|}{2} \rfloor$ . Thus,  $d(S) \leq d^{G[I_2 \cup C]}((S \cap I_2) \cup (S \cap C)) \leq cw(G[I_2 \cup C]) \leq cw(G)$ , contradicting that  $S$  is a bad set. This concludes the case that  $|C_1| < |C_2|$ .

If  $|C_1| \geq |C_2|$ , then  $v_i \in C_1$  because otherwise  $C_1 \subseteq V_{i-1}$  and  $\text{rank}_{V_{i-1}}(v_i) \leq |C_2 - C_1| \leq 0$  contradicting that  $S$  is the *smallest* locally worst bad set. Let  $u$  be the vertex in  $I_1$  with the highest label, and let  $v = v_i$ . Lemma 4.7 implies  $u <_L v$ . By applying Lemma 4.2 we conclude that  $|I_2| + |C_2| - (|I_1| - 1) > 0$ , so  $|I_1| \leq |I_2| + |C_2|$ . Furthermore, since  $S$  is the smallest locally worst bad set,  $\text{rank}_{V_{i-1}}(v)$  is either 1 or 2, so  $|S| = \lfloor \frac{n}{2} \rfloor$ .

Now, let  $S'$  be a set of vertices in  $\overline{G}$  maximizing  $d^{\overline{G}}(S')$ . Notice that  $bw(G) \geq \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil - d^{\overline{G}}(S')$ . We will use this fact to show that  $S$  can not be a bad set and obtain a contradiction. Since  $\delta^{\overline{G}}(S') = \delta^{\overline{G}}(V \setminus S')$ , without loss of generality we assume  $|I_1 \cap S'| \geq |I_1 \setminus S'|$ . Observe that now  $d^{\overline{G}}(S' \setminus C_2) \geq d^{\overline{G}}(S')$ , so we assume that  $C_2 \cap S' = \emptyset$ . Also, if  $x \in I_1 \setminus S'$  and  $y \in I_2 \cap S'$ , observe that  $d^{\overline{G}}(S' \cup \{x\} \setminus \{y\}) > d^{\overline{G}}(S')$ , contradicting the choice of  $S'$ . Therefore, if  $I_2 \cap S' \neq \emptyset$  then  $I_1 \subseteq S'$ . Suppose now that  $x \in I_1 \setminus S'$ . Then  $I_2 \cap S' = \emptyset$  and since  $|I_1| \leq |I_2| + |C_2|$ ,  $d^{\overline{G}}(S' \cup \{x\}) > d^{\overline{G}}(S')$ , contradicting the choice of  $S'$ . Therefore  $I_1 \subseteq S'$ . We can conclude that  $I_1 \subseteq S'$  and that  $C_2 \cap S' = \emptyset$ .

If  $|I_2| \leq |I_1| + 1$ , by Lemma 4.2 applied to the vertex of  $I_2$  with lowest label and the vertex of  $C_1$  with highest label,  $S \subseteq I_1 \cup C_1$ . Furthermore if  $y \in S' \cap I_2$ ,  $d^{\overline{G}}(S' \setminus \{y\}) \geq d^{\overline{G}}(S)$ , so without loss of generality  $S' \cap I_2 = \emptyset$ . Since vertices in  $C_1$  are isolated in  $\overline{G}$  it follows that  $d^{\overline{G}}(S) = d^{\overline{G}}(S')$ , so  $bw(G) \geq \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil - d^{\overline{G}}(S)$ . In addition, since  $|S| = \lfloor \frac{n}{2} \rfloor$  we have  $\lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil - d^{\overline{G}}(S) = d(S)$ . This implies  $bw(G) \geq d(S)$ , contradicting that  $S$  is a bad set.

If  $|I_2| > |I_1| + 1$ , by Lemma 4.2 applied to the vertex of  $I_2$  with lowest label and the vertex of  $C_1$  with highest label,  $S$  contains at least one vertex of  $I_2$ . By Lemma 4.7  $S$  does not contain all of  $I_2$  so we can apply Lemma 4.2 again to obtain that  $|S \cap I| = \lfloor \frac{|I|}{2} \rfloor$ . Furthermore, if  $|S' \cap I| > \lfloor \frac{|I|}{2} \rfloor$ , let  $y \in S' \cap I_2$ . Then  $d^{\overline{G}}(S' \setminus \{y\}) \geq d^{\overline{G}}(S')$ , so without loss of generality  $|S' \cap I_2| \leq \lfloor \frac{|I|}{2} \rfloor$ . Similarly, if  $|S' \cap I| < \lfloor \frac{|I|}{2} \rfloor$ , let  $y \in I_2 \setminus S'$ . Then  $d^{\overline{G}}(S' \cup \{y\}) \geq d^{\overline{G}}(S')$ , so without loss of generality  $|S' \cap I_2| \geq \lfloor \frac{|I|}{2} \rfloor$ . Thus, without loss of generality  $|S' \cap I| = \lfloor \frac{|I|}{2} \rfloor$ . Since vertices in  $C_1$  are isolated in  $\overline{G}$  it follows that  $d^{\overline{G}}(S) = d^{\overline{G}}(S')$ . As in the previous paragraph, this implies  $bw(G) \geq d(S)$ , contradicting that  $S$  is a bad set.  $\square$

From this it follows that any counterexample has at least 3 levels. Over the next few lemmas, we show how any counterexample can be transformed into a counterexample with exactly 3 levels. The first observation is that in every minimal counterexample all parts of the graph participate in making the graph a counterexample.

**Definition 1.** A counterexample has the extremal property if it has a bad set  $S$  such that  $I \setminus I_\ell \subseteq S$  and  $S \cap C \subseteq C_1$ . In this case,  $S$  is called an extremal bad set.

**Lemma 4.9.** Every minimal counterexample  $G$  has the extremal property, and every locally worst bad set of  $G$  is an extremal bad set.

*Proof.* Suppose for contradiction that  $G$  has a locally worst bad set  $S$  that is not an extremal bad set. By Lemma 4.7 the sets  $I \cap S$ ,  $I \setminus S$ ,  $C \cap S$  and  $C \setminus S$  are non-empty. Let  $u$  and  $v$  be the vertices in  $I \cap S$  and  $C \cap S$  with the highest labels, and let  $u'$  and  $v'$  be the vertices in  $I \setminus S$  and  $C \setminus S$  with the lowest labels respectively. Now,  $S = \text{over}(u', v')$  so by Lemma 4.3  $u <_L v'$  and  $v <_L u'$

If  $S$  contains non-universal vertices in  $C$ , let  $x$  be the vertex of  $I$  with the smallest label. We show that  $G - x$  also is a counterexample. Furthermore, since  $S$  contains non-universal clique vertices, by Lemma 4.1  $S$  contains at least one vertex of  $I_2$ , so  $x$  is distinct from  $u, u', v$  and  $v'$ . Also, since  $x \in I_1$  and  $v \notin C_1$ , it follows that  $x$  is not adjacent to  $v$  or  $v'$ . Thus the rank of  $u, u', v$  and  $v'$  with respect to any set  $X$  in  $G$  is the same as the rank with respect to  $X \setminus \{x\}$  in  $G'$  for each of these vertices. Let  $L'$  be the layout output by the algorithm when executed on  $G - x$ . By Lemma 4.3  $S \setminus \{x\}$  is covered by  $L'$ . Thus, since  $\delta^{G-x}(S \setminus \{x\}) = \delta(S)$ ,  $S \setminus \{x\}$  is a bad set for  $G - x$  contradicting that  $G$  is a minimal counterexample. Thus  $S \cap C \subseteq C_1$  follows.

If not  $I \setminus I_\ell \subseteq S$ , then let  $x$  be the vertex in  $I_\ell$  with lowest label, and let  $y$  be the vertex in  $C$  with highest label. We show that  $G' = G - xy$  also is a counterexample. First, observe that by Lemma 2.1  $G'$  is a threshold graph. Furthermore, by the assumption that  $I \setminus (I_\ell \cup S) \neq \emptyset$ ,  $x$  is distinct from  $u$  and  $u'$ . Also, since  $S \cap C \subseteq C_1$  and  $G$  has at least three levels,  $y$  is distinct from  $v$  and  $v'$ . Let  $L'$  be the layout output by the algorithm when executed on  $G'$ . Since  $\text{over}(u, v')$  and  $\text{over}(u', v)$  are the same sets in  $G$  and  $G'$ , and the deleted edge  $xy$  is not incident to any of  $u, u', v$  or  $v'$ ,  $u <_{L'} v'$  and  $u' <_{L'} v$  and so  $S$  is covered by  $L'$ . But  $\delta^{G'}(S) = \delta(S)$  contradicting that  $G$  is a minimal counterexample.  $\square$

**Lemma 4.10.** *If there is a counterexample, then there is a counterexample with the extremal property and at most 3 levels.*

*Proof.* We start by showing that if there is a counterexample  $G$  on at least 4 levels with an extremal bad set  $S$  such that the sets  $I \cap S$ ,  $I \setminus S$ ,  $C \cap S$  and  $C \setminus S$  are non-empty, then there is a counterexample  $G'$  with  $G \subset G'$ , such that  $(C, I)$  is a threshold partition of  $G'$  and  $S$  is an extremal bad set of  $G'$ .

Let  $u$  and  $v$  be the vertices in  $I \cap S$  and  $C \cap S$  with the highest labels, and let  $u'$  and  $v'$  be the vertices in  $I \setminus S$  and  $C \setminus S$  with the lowest labels respectively. Now,  $S = \text{over}(u', v')$  so by Lemma 4.3  $u <_L v'$  and  $v <_L u'$ . We choose  $x$  to be the vertex of  $I_2$  with highest label and  $y$  to be the vertex of  $C_3$  with the lowest label. We add the edge  $xy$  to  $G$  to obtain a new threshold graph  $G'$ .  $(C, I)$  is a threshold partition of  $G'$  and  $G \subset G'$ . Furthermore  $\delta^{G'}(S) = \delta(S) \cup \{xy\}$  so  $d^{G'}(S) = d(S) + 1 > cw(G) + 1 \geq cw(G \cup \{xy\})$ . Also, in  $G'$   $u'$  is adjacent to all vertices of  $C$  and  $v$  is a universal vertex. Let  $L'$  be the layout  $L'$  produced by Algorithm MinCut when run on  $G'$ . To prove that  $S$  is an extremal bad set of  $G'$  it is sufficient to show that  $L'$  covers  $S$ . However,  $S = \text{over}(u', v')$  both in  $G$  and  $G'$  and none of  $u, u', v$  and  $v'$  are incident to the new edge  $xy$  so  $u <_{L'} v'$  and  $v <_{L'} u'$ . By Lemma 4.3  $L'$  covers  $S$ .

We can now proceed to prove the lemma. Without loss of generality,  $G$  is a minimal counterexample. By Lemma 4.9  $G$  has an extremal locally worst bad set  $S$ . By Lemma 4.7 the sets  $I \cap S$ ,  $I \setminus S$ ,  $C \cap S$  and  $C \setminus S$  are non-empty. Thus, if  $G$  has at most 3 levels we are done, otherwise by the discussion in the previous paragraph, there is a counterexample  $G'$  with  $G \subset G'$ , such that  $(C, I)$  is a threshold partition of  $G'$  and  $S$  is an extremal bad set of  $G'$ . If  $G'$  has at most 3 levels we are done, otherwise we can again apply the discussion above to  $G'$  and  $S$  to get yet another counterexample  $G''$  with  $G \subset G' \subset G''$ , such that  $(C, I)$  is a threshold partition of  $G''$  and  $S$  is an extremal bad set of  $G''$ . Reiterating this argument we can continue producing counterexamples on more and more edges. Since the clique is not a counterexample, this process

must stop at some point. The graph  $G^*$  we are considering at this step is a counterexample with at most 3 levels and with  $S$  as an extremal bad set.  $\square$

**Definition 2.** *A counterexample has the super extremal property if it has an extremal bad set  $S$ , such that either  $I_\ell \cap S \neq \emptyset$  or  $S \cap C \subset C_1$ . Then  $S$  is called a super extremal bad set.*

**Lemma 4.11.** *There are no counterexamples with the super extremal property.*

*Proof.* We show that if there is a counterexample  $G$  with the super extremal property then there is a counterexample  $G'$  with at most 2 levels. This would contradict Lemmas 4.6 and 4.8. The proof that if there is a counterexample  $G$  with the super extremal property then there is a counterexample  $G'$  with at most 2 levels is similar to the proof of Lemma 4.10.

We start by showing that if there is a counterexample  $G$  on at least 3 levels with a super extremal bad set  $S$  such that the sets  $I \cap S$ ,  $I \setminus S$  and  $C \setminus S$  are non-empty, then there is a counterexample  $G'$  with  $G \subset G'$ , such that  $(C, I)$  is a threshold partition of  $G'$  and  $S$  is a super extremal bad set of  $G'$ .

If  $C \cap S$  is nonempty, let  $u$  and  $v$  be the vertices in  $I \cap S$  and  $C \cap S$  with the highest labels, and let  $u'$  and  $v'$  be the vertices in  $I \setminus S$  and  $C \setminus S$  with the lowest labels respectively. Now,  $S = \text{over}(u', v')$  so by Lemma 4.3  $u <_L v'$  and  $v <_L u'$ . If  $S \cap C \subset C_1$  we choose  $x$  to be the vertex of  $I_1$  with highest label and  $y$  to be the vertex of  $C_2$  with the lowest label. If  $S \cap C \subset C_1$  does not hold then  $I_\ell \cap S \neq \emptyset$  and we choose  $x$  to be the vertex of  $I_2$  with highest label and  $y$  to be the vertex of  $C_3$  with the lowest label. We add the edge  $xy$  to  $G$  to obtain a new threshold graph  $G'$ .  $(C, I)$  is a threshold partition of  $G'$  and  $G \subset G'$ . Furthermore  $\delta^{G'}(S') = \delta(S) \cup \{xy\}$ , and in  $G'$   $u'$  is adjacent to all vertices of  $C$  and  $v$  is a universal vertex. Furthermore, if  $S \cap C \subset C_1$  then  $v'$  is a universal vertex both in  $G$  and  $G'$  while if  $I_\ell \cap S \neq \emptyset$  then  $u$  is adjacent to all vertices of  $C$  both in  $G$  and in  $G'$ . Let  $L'$  be the layout produced by Algorithm MinCut when run on  $G'$ . To prove that  $S$  is a super extremal bad set of  $G'$  it is sufficient to show that  $L'$  covers  $S$ . However,  $S = \text{over}(u', v')$  both in  $G$  and  $G'$  and none of  $u$ ,  $u'$ ,  $v$  and  $v'$  are incident to the new edge  $xy$  so  $u <_{L'} v'$  and  $v <_{L'} u'$ . By Lemma 4.3  $L'$  covers  $S$ .

If  $C \cap S = \emptyset$ , let  $u$  be the vertex in  $I \cap S$  with the highest label, and let  $v'$  be the vertex in  $C$  with the lowest label. Now,  $u <_L v'$  and  $S = \text{over}(u', v')$ . We choose  $x$  to be the vertex of  $I_1$  with highest label and  $y$  to be the vertex of  $C_2$  with the lowest label. We add the edge  $xy$  to  $G$  to obtain a new threshold graph  $G'$ .  $(C, I)$  is a threshold partition of  $G'$  and  $G \subset G'$ . Furthermore  $\delta^{G'}(S') = \delta(S) \cup \{xy\}$ , and in  $G'$ ,  $u$  is adjacent to all vertices of  $C$  and  $v'$  is a universal vertex. Let  $L'$  be the layout  $L'$  produced by Algorithm MinCut when run on  $G'$ . To prove that  $S$  is a super extremal bad set of  $G'$  it is sufficient to show that  $L'$  covers  $S$ . However,  $S = \text{over}(u', v')$  both in  $G$  and in  $G'$ , and none of  $u$ ,  $u'$  and  $v'$  are incident to the new edge  $xy$  so  $u <_{L'} v'$ . Thus  $L'$  covers  $S$ .

We can now proceed to show that if there is a counterexample  $G$  with the super extremal property then there is a counterexample  $G'$  with at most 2 levels. If  $G$  has at most 2 levels we are done, so assume that  $G$  has at least 3 levels. Let  $S = V_i$  be the largest super extremal bad set of  $G$ . By the definition of the extremal property  $I \cap S$  and  $C \setminus S$  are nonempty. Also, if  $a$  is the vertex of  $I$  with the largest label and  $b$  is the vertex of  $C_2$  with highest label then by Lemma 4.2  $b <_L a$ . Since  $b \notin S$  we have  $a \notin S$  which implies  $I \setminus S \neq \emptyset$ .

By the discussion in the first paragraphs of the proof, there is a counterexample  $G'$  with  $G \subset G'$ , such that  $(C, I)$  is a threshold partition of  $G'$  and  $S$  is a super extremal bad set of  $G'$ . If  $G'$  has at most 2 levels we are done, otherwise we can again apply the discussion above to  $G'$  and  $S$  to get yet another counterexample  $G''$  with  $G \subset G' \subset G''$ , such that  $(C, I)$  is a

threshold partition of  $G''$  and  $S$  is a super extremal bad set of  $G''$ . Reiterating this argument we can continue producing counterexamples on more and more edges. Since a clique is not a counterexample, this process must stop at some point. The graph  $G^*$  we are considering at this step is a counterexample with at most 2 levels, contradicting Lemmas 4.6 and 4.8.  $\square$

Lemmas 4.9, 4.10, and 4.11 allow us to concentrate on counterexamples on exactly 3 levels with the extremal property, but without the super extremal property.

**Definition 3.** *We say that a counterexample with 3 levels and the extremal property has the snake property if (i)  $u <_L v$ , where  $u$  is the vertex of  $I_2$  with highest label and  $v$  is the vertex of  $C_1$  with highest label, and (ii)  $u' <_L v'$ , where  $u'$  is the vertex of  $I_3$  with lowest label and  $v'$  is the vertex of  $C_2$  with lowest label.*

**Lemma 4.12.** *If there is a counterexample with 3 levels with the extremal property then there is a counterexample with 3 levels with the extremal and the snake properties.*

*Proof.* Let  $G$  be a counterexample with the extremal property, with 3 levels. Let  $S$  be an extremal bad set of  $G$ . By Lemma 4.11  $S$  is not super extremal. Thus  $S = I_1 \cup I_2 \cup C_1$ . Let  $u$  be the vertex of  $I_2$  with highest label and  $v$  be the vertex of  $C_1$  with highest label. Now, let  $u'$  and  $v'$  be the vertices of  $I_3$  and  $C_2$  with lowest labels respectively. If  $u <_L v$  and  $u' <_L v'$  we are done, so either  $v <_L u$  or  $v' <_L u'$  must hold. We choose  $x$  to be the vertex of  $I_1$  with lowest label, and let  $G' = G - x$ . Let  $L'$  be the layout constructed by the algorithm when run on  $G'$ . We show that  $L'$  covers  $S' = S \setminus \{x\}$ . Since  $x$  is nonadjacent to both  $u$  and  $v'$ , Lemma 4.2 implies  $u <_{L'} v'$ .

Suppose that  $v' <_L u'$ . Since  $x$  is nonadjacent to both  $u'$  and  $v'$ , Lemma 4.2 implies  $v' <_{L'} u'$ . Since  $v <_{L'} v'$  it follows that  $v <_{L'} u'$ . By Lemma 4.3,  $L'$  covers  $S'$ . Suppose now that  $v <_L u$ . Then in  $G$ ,  $\text{rank}_{\text{over}(u,v)}(v) \leq \text{rank}_{\text{over}(u,v)}(u)$ . Since  $u$  is adjacent to  $v$  but not to  $u'$  and all neighbors of  $u'$  that are non-neighbors of  $u$  are in  $C_3$ , we have that  $\text{rank}_{\text{over}(u',v)}(v) + 2 \leq \text{rank}_{\text{over}(u,v)}(v) \leq \text{rank}_{\text{over}(u,v)}(u) \leq \text{rank}_{\text{over}(u',v)}(u')$ . Deleting  $x$  increases the rank of  $v$  by one and does not decrease the rank of  $u'$ , so by Lemma 4.2,  $v <_{L'} u'$ . By Lemma 4.3,  $L'$  covers  $S'$ .

Now,  $\delta^{G'}(S') = \delta(S)$  so  $S'$  is a bad set of  $G'$ . By Lemma 4.8  $G'$  has 3 levels. Also  $S'$  is an extremal bad set of  $G'$ . If  $G'$  does not have the snake property, we can apply the argument above to get a counterexample  $G''$  on 3 levels with the extremal property and even fewer vertices than  $G'$ . Reiterating this argument we can continue producing counterexamples on fewer and fewer vertices. Since the single vertex graph is not a counterexample, this process must stop at some point. The graph  $G^*$  we are considering at this step is a counterexample with 3 levels and the extremal and snake properties.  $\square$

At this point all that remains is to analyze how a counterexample on 3 levels and the extremal and snake properties looks, and to show that in such a graph  $G$ ,  $cw_L(G) \leq bw(G) \leq cw(G)$ .

**Lemma 4.13.** *In a counterexample with 3 levels with the extremal and snake properties,  $n$  is even,  $|C|$  and  $|I|$  are odd,  $|C_1| = |C_2| + |C_3| + 1$ ,  $|I_3| = |I_1| + |I_2| + 1$ , and  $|I_1| + |I_2| + |C_1| = \frac{n}{2}$ .*

*Proof.* Let  $G$  be a counterexample with 3 levels with the extremal and snake properties. Let  $S$  be an extremal bad set of  $G$ . By Lemma 4.11,  $S$  is not super extremal, so  $S = V_i = I_1 \cup I_2 \cup C_1$ . Let  $u$  and  $v$  be the vertices of  $I_2$  and  $C_1$  with highest labels, and let  $u'$  and  $v'$  be the vertices of  $I_3$  and  $C_2$  with lowest labels respectively. Since  $S$  has the snake property,  $v = v_i$  and  $u' = v_{i+1}$ . If  $\text{rank}_{V_{i-1}}(v) \leq 0$  then  $V_{i-1}$  is a super extremal bad set, contradicting Lemma 4.11. Similarly, if  $\text{rank}_{V_i}(u') \geq 0$  then  $V_{i+1}$  is a super extremal bad set, contradicting Lemma 4.11. Thus

$rank_{V_{i-1}}(v) > 0$  and  $rank_{V_i}(u') < 0$ . By Lemma 4.4  $rank_{V_{i-1}}(v) = 1$  and  $rank_{V_i}(u') = -1$ . Since  $v$  is a universal vertex this implies that  $n$  is even and that  $|S| = |I_1| + |I_2| + |C_1| = \frac{n}{2}$ . Since  $u'$  is adjacent to all vertices of the clique,  $rank_{V_i}(u') = |C_2| + |C_3| - |C_1| = -1$  so  $|C_1| = |C_2| + |C_3| + 1$  and  $|C|$  is odd. Since  $|I| + |C| = n$  is even,  $|I|$  is odd. Finally, since  $|I_1| + |I_2| + |C_1| = |I_3| + |C_2| + |C_3|$  and  $|C_1| = |C_2| + |C_3| + 1$  we have  $|I_3| = |I_1| + |I_2| + 1$ .  $\square$

Now we are ready to show our main result.

**Theorem 4.14.** *For any threshold graph  $G$ ,  $cw_L(G) = cw(G)$ .*

*Proof.* Suppose for contradiction that there is a counterexample. Then, by Lemmas 4.6, 4.8, 4.9, 4.10 and 4.12 there is a counterexample  $G$  on 3 levels with the snake and extremal properties. Since Lemma 4.11 implies that  $G$  does not have the super extremal property,  $I_1 \cup I_2 \cup C_1$  is a bad set of  $G$ . By Lemma 4.13,  $n$  is even,  $|C|$  and  $|I|$  are odd,  $|C_1| = |C_2| + |C_3| + 1$ ,  $|I_3| = |I_1| + |I_2| + 1$ , and  $|I_1| + |I_2| + |C_1| = \frac{n}{2}$ .

Let  $S$  be a vertex set on  $\frac{n}{2}$  vertices that minimizes  $d(S)$ , that is, with  $d(S) = bw(G)$ . Notice that  $\delta(S) = \delta(V \setminus S)$ . Thus, without loss of generality  $|S \cap I| > \frac{|I|}{2}$ . We view the set  $S$  as a set of  $\frac{n}{2}$  pebbles that have been placed on distinct vertices of  $G$ . We can move pebbles from  $I_3$  to  $C_1$  and keep optimality of  $S$ , unless one of the following is true: (i) There are no pebbles on vertices of  $I_3$ , (ii) All vertices of  $C_1$  have pebbles on them, (iii) Moving a pebble from a vertex in  $I_3$  to a vertex in  $C_1$  increases  $d(S)$ .

Moving a pebble from a vertex  $x$  to a vertex  $y$  increases  $d(S)$  by  $rank_{S \setminus \{x\}}(y) - rank_{S \setminus \{x\}}(x)$ . Thus, if there is a pebble on a vertex  $x$  in  $I_3$  and a free spot on a vertex  $y$  in  $C_1$ , moving a pebble from  $x$  to  $y$  does not increase  $d(S)$  if and only if  $rank_{S \setminus \{x\}}(y) - rank_{S \setminus \{x\}}(x) = 1 - (|C \setminus S| - |C \cap S|) \leq 0$ . Rearranging terms yields that moving a pebble from  $x$  to  $y$  does not increase  $d(S)$  if and only if  $|C \cap S| < |C \setminus S|$ . In addition, one should notice that if there are no pebbles on vertices of  $I_3$  then  $|C \cap S| > |C \setminus S|$  because  $|I_3| = |I_1| + |I_2| + 1$ . Similarly, if all vertices of  $C_1$  have pebbles on them then  $|C \cap S| > |C \setminus S|$  because  $|C_1| = |C_2| + |C_3| + 1$ .

By our choice of  $S$ , before we start moving any pebbles,  $|I \cap S| > |I \setminus S|$ . Since  $|S| = \frac{n}{2}$  this means that  $|C \cap S| < |C \setminus S|$ . Therefore, by the discussion in the previous paragraph we can move pebbles from  $I_3$  to  $C_1$ , preserving minimality of  $d(S)$  until the inequality flips from  $|C \cap S| < |C \setminus S|$  to  $|C \cap S| > |C \setminus S|$ . At this point,  $|C \cap S| = \lceil \frac{|C|}{2} \rceil = |C_1|$  and  $|I \cap S| = |I_1| + |I_2|$ . Let  $\alpha$ ,  $\beta$  and  $\gamma$  be the ranks of a vertex in  $I_1$ ,  $I_2$  and  $I_3$  with respect to  $S$ .

If  $\alpha \leq \gamma$  and  $\beta \leq \gamma$  we can move pebbles from  $I_3$  to  $I_1$  and  $I_2$  keeping optimality of  $S$ . Since exactly  $|I_1| + |I_2|$  pebbles are placed on vertices of  $I$ , after the move every vertex of  $I_1 \cup I_2$  has a pebble on it, and no pebbles are on vertices in  $I_3$ . Now we can safely move all pebbles in  $C_2$  and  $C_3$  to  $C_1$ , keeping optimality of  $S$ . Since exactly  $|C_1|$  pebbles are placed on vertices of  $C$ , after the move every vertex of  $C_1$  has a pebble on it, and no pebbles are on vertices in  $C_2 \cup C_3$ . But this means that  $S = I_1 \cup I_2 \cup C_1$  and  $d(S) \leq bw(G) \leq cw(G)$  contradicting that  $I_1 \cup I_2 \cup C_1$  is a bad set of  $G$ .

If  $\alpha \geq \gamma$  and  $\beta \geq \gamma$  we can move all pebbles from  $I_1$  and  $I_2$  to  $I_3$  keeping optimality of  $S$ . Since exactly  $|I_1| + |I_2|$  pebbles are placed on vertices of  $I$ , after this move all but one vertex of  $I_3$  has a pebble on it, and no pebbles are on vertices in  $I_1 \cup I_2$ . Now we can safely move pebbles in  $C_1$  to  $C_2$  and  $C_3$ , keeping optimality of  $S$ . Since exactly  $|C_1|$  pebbles are placed on vertices of  $C$ , after this move exactly one vertex of  $C_1$  has a pebble on it, and all vertices of  $C_2 \cup C_3$  have pebbles on them. Let  $x$  be the vertex in  $I_3$  without a pebble and  $y$  be the vertex in  $C_1$  with a pebble. After the moves,  $rank_{S \setminus \{y\}}(x) = rank_{S \setminus \{y\}}(y) = 1$ , so we can move the pebble on  $y$  to



$x$ , keeping optimality of  $S$ . However  $d(I_1 \cup I_2 \cup C_1) = d(V \setminus S) = d(S) \leq bw(G)$  contradicting that  $I_1 \cup I_2 \cup C_1$  is a bad set of  $G$ .

If  $\alpha \leq \gamma \leq \beta$  we can move pebbles from  $I_2$  and  $I_3$  to  $I_1$  maintaining optimality until all vertices of  $I_1$  have a pebble on them. If any pebbles remain in  $I_2$  we can move these pebbles to  $I_3$ . Since there are  $|I_1| + |I_2|$  pebbles in  $I$  there are exactly  $|I_2|$  vertices in  $I_3$  that have pebbles on them. Now, we can move all pebbles in  $C_2$  to  $C_3$  and  $C_1$  maintaining optimality until there are no pebbles left in  $C_2$ . If  $|I_1| \geq |I_2|$  we can move all pebbles from  $C_3$  to  $C_1$  maintaining optimality. After this move, the set of vertices in  $C$  with pebbles on them is exactly  $C_1$ . Thus we can move all pebbles in  $I_3$  to  $I_2$  maintaining optimality. In this case,  $S = I_1 \cup I_2 \cup C_1$ , but  $d(S) \leq bw(G)$  contradicting that  $I_1 \cup I_2 \cup C_1$  is a bad set of  $G$ .

If  $|I_1| < |I_2|$  we can move pebbles from  $C_1$  to  $C_3$  until all vertices of  $C_3$  have pebbles on them. After this move there are exactly  $|C_1| - |C_3|$  pebbles on vertices in  $C_1$ . We consider  $d(S)$  and compare it to  $d(I_1 \cup I_2 \cup C_1) = |C_1|(|C_2| + |C_3|) + |I_2||C_2| + |I_3||C_1|$ . Counting the edges of  $d(S)$  we obtain  $d(S) = |C_1|(|C_2| + |C_3|) + |I_1||C_3| + |I_2|(|C_1| - |C_3|) + (|I_3| - |I_2|)|C_1| + |I_2|(|C_2| + |C_3|)$ . Simplifying yields  $d(S) = |C_1|(|C_2| + |C_3|) + |I_1||C_3| + |I_3||C_1| + |I_2||C_2|$  But this means that  $d(I_1 \cup I_2 \cup C_1) < d(S) \leq bw(G)$  contradicting that  $S$  is a bad set of  $G$ .

Finally, suppose  $\alpha \geq \gamma \geq \beta$ . Since  $d(S) = d(V \setminus S)$  we can move all pebbles over to vertices that do not have pebbles and preserve optimality. There are now exactly  $|I_3|$  pebbles in  $I$  and  $|C_1| + |C_2|$  pebbles in  $C$ . Since  $|I_3| > |I_1| + |I_2|$  there is a pebble on a vertex  $x$  in  $I_3$ . Also, since  $|C_1| > |C_2| + |C_3|$  there is a vertex  $y$  with no pebble in  $C_1$ . At this point,  $rank_{S \setminus \{x\}}(x) = rank_{S \setminus \{x\}}(y) = 1$ , so we can move a pebble from  $x$  to  $y$  and again obtain a set  $S$  with pebbles on  $|I_1| + |I_2|$  vertices in  $I$  and  $|C_1|$  vertices in  $C$ . In addition if  $\alpha'$ ,  $\beta'$  and  $\gamma'$  are the ranks of a vertex in  $I_1$ ,  $I_2$  and  $I_3$  with respect to  $S$ , we see that  $\alpha' = -\alpha - 2$ ,  $\beta' = -\beta - 2$  and  $\gamma' = -\gamma - 2$ , so at this point  $\alpha' \leq \gamma' \leq \beta'$  and the discussion in the previous paragraphs applies. This concludes the proof.  $\square$

With the correctness proved, we are ready to move to running time.

**Theorem 4.15.** *The cutwidth of a threshold graph on  $n$  vertices can be computed in  $O(n)$  time.*

*Proof.* We will show that Algorithm MinCut is exactly such an algorithm. Its correctness is already proved. Assuming that the input threshold graph is given by its threshold partition,  $(C, I) = ((C_1 \cup C_2 \cdots \cup C_\ell), (I_1 \cup I_2 \cdots \cup I_\ell))$  we will describe an implementation of Algorithm MinCut that runs in  $O(n)$  time. We assume that the vertices of  $G$  are labeled according to the threshold partition. If they are not we can relabel them in  $O(n)$  time and keep a copy of the old labels for output. By Lemma 4.1 we know that Algorithm MinCut picks vertices of  $I$  by increasing label and the vertices of  $C$  by increasing label. Therefore we keep track of the not yet picked vertices  $u \in I$  and  $v \in C$  with the lowest labels. We also keep track of the ranks of  $u$  and  $v$  with respect to  $over(u, v)$ , that is  $r_u = rank_{over(u, v)}(u)$  and  $r_v = rank_{over(u, v)}(v)$ . At each step of the algorithm we pick the one of  $u$  and  $v$  with the lowest rank (and highest degree if their rank is equal, lowest label if both rank and degree is equal). We now need to update the variables  $u$ ,  $v$ ,  $r_u$  and  $r_v$ . Lemma 4.1 guarantees that  $u$  and  $v$  are adjacent, so if we pick  $u$  we reduce  $r_v$  by 2 and if we pick  $v$  we reduce both  $r_u$  and  $r_v$  by 2. Finally, if we picked  $u$  we need to correct  $r_u$  for the fact that the next vertex in the independent set could be in a higher level, and similarly we need to correct  $r_v$ . If the algorithm picked  $u$ , let  $u'$  be the vertex in  $I$  with  $label(u) + 1 = label(u')$ . If  $level(u') > level(u)$  we increase  $r_u$  by  $|C_{level(u')}|$  because Lemma 4.1 guarantees that no vertices of  $C_{level(u')}$  have been picked yet. Similarly, if the algorithm picked  $v$ , let  $v'$  be the vertex in  $C$  with  $label(v) + 1 = label(v')$ . If  $level(v') > level(v)$  we increase  $r_v$

by  $|I_{level(v)}|$  because Lemma 4.1 guarantees that all vertices of  $I_{level(v)}$  have already been picked. For each new vertex to be picked the algorithm does  $O(1)$  work so the total time complexity is  $O(n)$ .  $\square$

Note that if only an adjacency list representation of  $G$  is given, then  $G$  has to be recognized as a threshold graph first, and its threshold partition can be computed during this recognition in  $O(n + m)$  time [11].

## 5 Concluding remarks: cutwidth of interval graphs

A natural open question and a future research direction is resolving the computational complexity of cutwidth on interval graphs. A graph is *interval* if sets of consecutive integers (intervals) can be assigned to its vertices such that two vertices are adjacent if and only if their intervals overlap. Some inherently difficult graph problems, like bandwidth, are polynomially solvable on interval graphs [15], whereas others, like optimal linear arrangement, are NP-complete [7]. Optimal linear arrangement can be seen as sum-bandwidth or sum-cutwidth, equivalently (see [10] for definitions). Simple examples exist that show that Algorithm MinCut can produce a layout with cutwidth that is a factor of  $O(n)$  larger than  $cw(G)$  when  $G$  is an interval graph, or even a proper interval graph. An interval graph is *proper interval* if it has an interval model where no interval properly contains another. Interestingly, for the class of proper interval graphs a trivial approach solves the cutwidth problem.

**Theorem 5.1.** *Let  $G$  be a proper interval graph and let  $L$  be an ordering of the vertices of  $G$  by increasing right endpoint of their corresponding intervals. Then  $cw_L(G) = cw(G)$ .*

*Proof.* For a contradiction, let  $G$  be a proper interval graph on fewest possible vertices such that  $cw_L(G) > cw(G)$  for some right endpoint ordering  $L$  of the vertices of  $G$ . Then  $G$  must have a bad set  $S$ , that is, a set that is covered by  $L$  and so that  $d(S) > cw(G)$ . If there is a vertex  $x \in S$  with no neighbors in  $V \setminus S$  then  $S \setminus \{x\}$  is a bad set of  $G - x$  contradicting that  $G$  is a smallest graph for which  $cw_L(G) > cw(G)$ . Similarly, if there is a vertex  $y \in V \setminus S$  with no neighbors in  $S$  then  $S$  is a bad set of  $G - y$  again yielding a contradiction. This means that every vertex in  $S$  has a neighbor in  $V \setminus S$  and that every vertex in  $V \setminus S$  has a neighbor in  $S$ . Since  $G$  is a proper interval graph and  $L$  is a right endpoint ordering of the vertices of  $G$  this implies that both  $S$  and  $V \setminus S$  induce cliques in  $G$ . Let  $S'$  be the set on  $|S|$  vertices that minimizes  $d(S')$  and define  $X = S \setminus S'$  and  $Y = S' \setminus S$ . Now,  $cw(G) \geq d(S')$  and  $|X| = |Y|$  so let  $k = |X| = |Y|$ . For two vertex sets  $A$  and  $B$ , let  $d(A, B)$  be the number of edges in  $G$  with one endpoint in  $A$  and the other in  $B$ . This gives the inequality  $d(S) \leq d(S') + d(X, V \setminus (S \cup Y)) + d(Y, S \setminus X) - d(X, S \setminus X) - d(Y, V \setminus (S \cup Y)) \leq d(S') + k(n - |S| - k) + k(|S| - k) - k(|S| - k) - k(n - |S| - k) = d(S')$  contradicting that  $S$  is a bad set of  $G$ .  $\square$

The above theorem can easily be converted into an  $O(n)$  time algorithm to compute the cutwidth of a proper interval graph. Observe that a right endpoint ordering is not necessarily an optimal layout of a threshold graph; a star is a simple counterexample.

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