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A complete characterisation of the linear clique-width of path powers*

Pinar Heggernes† Daniel Meister† Charis Papadopoulos‡

Abstract

A k-path power is the k-power graph of a simple path of arbitrary length. Path powers form a non-trivial subclass of proper interval graphs. Their clique-width is not bounded by a constant, and no polynomial-time algorithm is known for computing their clique-width or linear clique-width. We show that k-path powers above a certain size have linear clique-width exactly $k + 2$, providing the first complete characterisation of the linear clique-width of a graph class of unbounded clique-width. Our characterisation results in a simple linear-time algorithm for computing the linear clique-width of all path powers.

1 Introduction

Clique-width is a graph parameter that describes the structure of a graph and its behaviour with respect to hard problems [6]. Many NP-hard graph problems become solvable in polynomial time on graphs whose clique-width is bounded by a constant [21, 26]. If the problem, in addition, is expressible in a certain type of monadic second order logic, it becomes fixed parameter tractable when parameterised by clique-width and a corresponding clique-width expression is given [7]. Clique-width can be viewed as a generalisation of the more widely studied parameter treewidth, since there are graphs of bounded clique-width but unbounded treewidth (e.g., complete graphs), whereas graphs of bounded treewidth have bounded clique-width [9]. As pathwidth is a restriction on treewidth, *linear clique-width* is a restriction on clique-width, and hence graphs of bounded clique-width might have unbounded linear clique-width (e.g., cographs [16]). Both clique-width and linear clique-width are NP-hard to compute [11]. These two closely related graph parameters have received much attention recently, and the interest in them is increasing [4, 7, 9, 13, 8, 1, 10, 23, 24, 2, 5, 16, 3, 11, 14, 15, 22, 20, 17, 12].

In this paper, we give a complete characterisation of the linear clique-width of path powers, which form a subclass of proper interval graphs. Hereditary subclasses of proper interval graphs have bounded clique-width [22], however path powers are not hereditary, and they have unbounded clique-width [13] and thus unbounded linear clique-width. This is the first graph class of unbounded clique-width whose linear clique-width is hereby completely characterised. More precisely, we show that k-path powers above a certain size have linear clique-width exactly $k + 2$. A k-path power is the k-power graph of a simple path. We also characterise the linear

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clique-width of smaller k-path powers. Our characterisation results in a simple linear-time algorithm for computing the linear clique-width of path powers, making this the first graph class on which clique-width or linear clique-width is unbounded, and linear clique-width can be computed in polynomial time. In addition, we give a characterisation of the linear clique-width of path powers through forbidden induced subgraphs. The main difficulty to overcome in obtaining these results has been to prove a tight lower bound on the linear clique-width of path powers.

To review related results, we can mention that graphs of clique-width at most 2 [9] and at most 3 [4] can be recognised in polynomial time. Also graphs of linear clique-width at most 2 [14] and at most 3 [20] can be recognised in polynomial time. Several graph classes have been studied with respect to whether or not their clique-width is bounded by a constant [1, 2, 3, 10, 13, 17, 22, 23, 24]. For specific graph classes of unbounded clique-width and thus unbounded linear clique-width, little is known on the computation of their clique-width or linear clique-width. So far the only result that computes either of these parameters exactly is given by Golumbic and Rotics [13], who show that a $k \times k$ grid has clique-width $k+1$. (Notice that for fixed k, there are infinitely many k-path powers, but only one $k \times k$ grid.) Other than this, mainly some upper [11, 17] and lower [13, 5] bounds have been given some of which are mentioned below. Typical for lower bounds is that they are not tight, and therefore they do not lead to exact computation of the clique-width or the linear clique-width efficiently. For lower bounds, Golumbic and Rotics gave lower bounds on the clique-width of some subclasses of proper interval graphs and permutation graphs [13], and Corneil and Rotics showed an exponential gap between clique-width and treewidth [5].

Specifically for path powers, the results of Gurski and Wanke on the linear clique-width of power graphs imply that the linear clique-width of a k-path power is at most $(k+1)^2$ [17]. Fellows et al. showed that the linear clique-width of a graph is bounded by its pathwidth plus 2 [11], which gives $k+2$ as an upper bound on the linear clique-width of k-path powers. For lower bounds, Golumbic and Rotics showed that the clique-width and thus the linear clique-width of a k-path power on $(k+1)^2$ vertices is *at least* $k+1$ [13]. The authors conjecture that the clique-width of k-path powers on $(k+1)^2$ vertices is exactly $k+2$ [13]. This conjecture is still open. The same upper and lower bounds are still the best known bounds also on the linear clique-width of k-path powers on $(k+1)^2$ vertices. In this paper, we prove the conjecture to be true for linear clique-width.

The results that we present in this paper contribute to better understanding of linear clique-width and clique-width. The knowledge on these graph parameters is still limited, and there is no general intuition on what makes a graph structurally more complicated (larger clique-width) than other graphs. To prove the lower bound $k+2$ on the above mentioned k-path powers (in Section 5), the technique we apply is through identifying *maximal* k-path powers of linear clique-width *at most* $k+1$ (in Section 4).

2 Basic definitions, notation and linear clique-width

We consider undirected finite graphs with no loops or multiple edges. For a graph $G=(V,E)$, we denote its vertex and edge set by $V(G)=V$ and $E(G)=E$, respectively. Two vertices u and v of G are called *adjacent* if $uv \in E$; if $uv \notin E$ then u and v are *non-adjacent.* A *path* in G is a sequence of vertices $(v_1, v_2, \ldots, v_l)$ such that $v_i v_{i+1} \in E$ for $1 \leq i \leq l-1$. For a vertex set $S \subseteq V$, the *subgraph of G induced by S* is denoted by $G[S]$. Moreover, we denote by $G-v$

the graph $G[V \setminus \{v\}]$. The *neighbourhood* of a vertex x in G is $N_G(x) = \{v \mid xv \in E\}$ and its *degree* is $|N_G(x)|$. For two vertices x and y, if another vertex z is adjacent to exactly one of them then we say that z *distinguishes* x and y.

Let G and H be two vertex-disjoint graphs. The *disjoint union* of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The notion of clique-width was first introduced in [6]. The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations: create new vertex with label i, disjoint union, change all labels i to j, add all edges between vertices with label i and vertices with label j where $i \neq j$. The *linear clique-width* of a graph, denoted by $\mathrm{lcwd}(G)$, is introduced in [16] and defined by the same operations as above with the restriction that at least one of the operands of the disjoint union operation must be a graph on a single vertex. This results in a linear structure, and linear clique-width can be viewed as a graph layout problem [15, 19].

A *layout* for a graph $G = (V, E)$ is a linear ordering of its vertices, usually defined as a bijective mapping from the set $\{1, \ldots, |V|\}$ to V. For $A \subseteq V$, a *group* in A is a maximal set of vertices with the same neighbourhood in $V \setminus A$. Note that two groups in A are either equal or disjoint, implying that the group relation defines a partition of A. By $\nu_G(A)$, we denote the number of groups in A. Let β be a layout for G. Let x be a vertex of G and let p be the position of x in β, i.e., $p = \beta^{-1}(x)$. The *set of vertices to the left of x with respect to β* is $\{\beta(1), \ldots, \beta(p-1)\}$ and denoted as $L_\beta(x)$, and the *set of vertices to the right of x with respect to β* is $\{\beta(p+1), \ldots, \beta(|V|)\}$ and denoted as $R_\beta(x)$. We write $L_\beta[x]$ and $R_\beta[x]$ if x is included. Function ad_β is a $\{0,1\}$-valued function on the vertex set of G with respect to β. Given a vertex x of G, if one of the following conditions is satisfied then $\mathrm{ad}_\beta(x) = 1$; if none of the conditions is satisfied then $\mathrm{ad}_\beta(x) = 0$:

(1) all (other) vertices in the group in $L_\beta[x]$ that contains x are neighbours of x

(2) $\{x\}$ is not a group in $L_\beta[x]$, and there are a non-neighbour y of x in the group of $L_\beta[x]$ containing x and a neighbour z of x in $L_\beta(x)$ such that y and z are non-adjacent

The *groupwidth of a graph G with respect to a layout β for G*, denoted as $\mathrm{gw}(G, \beta)$, is the smallest number k such that $\nu_G(L_\beta(x)) + \mathrm{ad}_\beta(x) \leq k$ for all $x \in V(G)$. The *groupwidth* of a graph G, denoted as $\mathrm{gw}(G)$, is the smallest number k such that there is a layout β for G satisfying $\mathrm{gw}(G, \beta) \leq k$.

Theorem 2.1 ([19]) *For every graph G,* $\mathrm{lcwd}(G) = \mathrm{gw}(G)$.

For a given graph G, the *k-power graph* of G is the graph that has the same vertex set as G such that two vertices are adjacent if and only if the distance (length of a shortest path) between them is at most k in G. For a given $l \geq 1$, P_l is the graph with vertex set $\{x_1, x_2, \ldots, x_l\}$ and edge set $\{x_1x_2, x_2x_3, \ldots, x_{l-1}x_l\}$. A *$k$-path power* is a graph that is the k-power graph of P_l for some l. Notice that the k-power graph of P_l for any $k \geq l-1$ is a complete graph. Observe that for a k-path power that is not complete, a largest clique contains exactly $k+1$ vertices. A *path power* is a k-path power for some k. For a path power, a vertex of smallest degree is called *endvertex*. A path power that is not complete has exactly two endvertices, that are non-adjacent.

Lemma 2.2 *Let P be a path power and let β be a layout for P. If $\mathrm{ad}_\beta(x) = 0$ for a vertex x of P then x is an endvertex of P.*

Proof. Let P be the k-power graph of P_l for appropriate k and l. Let $(x_1, \ldots, x_n)$ be the underlying path of P. Let x be a vertex of P such that $\mathrm{ad}_\beta(x) = 0$. Let K be the group in $L_\beta[x]$ that contains x. By definition of function ad, $|K| \geq 2$. Since $K \setminus \{x\}$ does not contain only neighbours of x, there is a vertex y in $K \setminus \{x\}$ that is non-adjacent to x. Suppose that there is a vertex z of P that is adjacent to x and non-adjacent to y. If $z \notin L_\beta[x]$ then z distinguishes x and y and x and y cannot belong to the same group in $L_\beta[x]$, if $z \in L_\beta[x]$ then K contains a vertex that is non-adjacent to a neighbour of x in $L_\beta[x]$. Both cases contradict the assumption about $\mathrm{ad}_\beta(x) = 0$. Hence, all neighbours of x in P are neighbours of y. Let i, j be such that $x = x_i$ and $y = x_j$. If $1 < i < j$ then x_{i-1} is a neighbour of x and non-adjacent to y by the distance condition for k-power graphs and $xy \notin E(P)$, if $j < i < n$ then x_{i+1} is a neighbour of x and non-adjacent to y. Both cases contradict the neighbourhood inclusion property. Hence, $i = 1$ or $i = n$, and x is an endvertex of P. ∎

3 Groups in induced subgraphs of path powers

The linear clique-width bounds that we present in this paper are all proved by applying Theorem 2.1. The main technique is to count groups in subgraphs. As a main tool, we use a representation of path powers that arranges vertices into rows and columns of a 2-dimensional array.

Let G be a graph. A *bubble model* for G is a 2-dimensional structure $\mathcal{B} = \langle B_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ such that the following conditions are satisfied:

- for $1 \leq j \leq k$ and $1 \leq i \leq r_j$, $B_{i,j}$ is a (possibly empty) set of vertices of G
- the sets $B_{1,1}, \ldots, B_{r_k,k}$ are pairwise disjoint and cover $V(G)$
- two vertices u, v of G are adjacent if and only if there are $1 \leq j \leq j' \leq s$ and $1 \leq i \leq r_j$ and $1 \leq i' \leq r_{j'}$ such that $u, v \in B_{i,j} \cup B_{i',j'}$ and (a) $j = j'$ or (b) $j + 1 = j'$ and $i > i'$.

A similar structure is given by Golumbic and Rotics [13]. The sets $B_{i,j}$ are called *bubbles*. If every bubble $B_{i,j}$ contains exactly one vertex, we also write $\langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. A graph is a proper interval graph if and only if it has a bubble model [18]. For $1 \leq j \leq s$, we combine the sets $B_{1,j}, \ldots, B_{r_j,j}$ to the jth column, also denoted as $\mathcal{B}_j$. We say that $\mathcal{B}$ is a *bubble model on a columns and b rows* if $s = a$ and $r_1 = \cdots = r_{s-1} = \max\{r_1, \ldots, r_s\} = b$.

Theorem 3.1 *Let $k \geq 1$. A graph G is a k-path power if and only if there is $s \geq 1$ such that G has a bubble model on s columns and $k+1$ rows and all bubbles contain exactly one vertex.*

Proof. Let G be a k-path power. Let G be the k-power graph of P_l. We rename the vertices of the path as follows. For $1 \leq i \leq l$, let $b_{b,a} =_{\mathrm{def}} x_i$ where a and b are such that $i = a(k+1)+b$ and $1 \leq b \leq k+1$. Let s be smallest such that $l \leq s(k+1)$, and let $r_1 =_{\mathrm{def}} \cdots =_{\mathrm{def}} r_{s-1} =_{\mathrm{def}} k+1$ and $r_s =_{\mathrm{def}} n - (s-1)(k+1)$. Let $\mathcal{B} =_{\mathrm{def}} \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$. We show that $\mathcal{B}$ is a bubble model for G. Let u and v be two vertices of G. There are x_i and $x_{i'}$ such that $u = x_i$ and $v = x_{i'}$; without loss of generality, we may assume $i < i'$. Let a, a', b, b' be such that $1 \leq b, b' \leq k+1$ and $i = a(k+1) + b$ and $i' = a'(k+1) + b'$. Clearly, if $a = a'$ then $b' - b \leq k$ and therefore $i' - i \leq k$. If $a < a'$ then $i' - i \leq k$ if and only if $b > b'$. Hence, $\mathcal{B}$ is a bubble model for G. And by construction, $\mathcal{B}$ is a bubble model on $k+1$ rows and all bubbles contain exactly one vertex.

For the converse, let $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq s, 1 \leq i \leq r_j}$ be a bubble model for G of the assumed form. By definition, $r_1 = \cdots = r_{s-1} \geq r_s$. Let $k =_{\mathrm{def}} r_1 - 1$. We show that G is the k-power graph of the path with edge set $\{b_{1,1}b_{1,2},\ldots,b_{k,1}b_{k+1,1},b_{k+1,1}b_{1,2},\ldots,b_{k+1,s-1}b_{1,s},\ldots,b_{r_s-1,s}b_{r_s,s}\}$. Let $b_{i,j}$ and $b_{i',j'}$ be vertices of G where $j \le j'$. Observe that the distance of $b_{i,j}$ and $b_{i',j'}$ in the path is equal to $|(j'-j)(k+1)+i'-i|$. With the definition of bubble model, $b_{i,j}$ and $b_{i',j'}$ are adjacent in G if and only if $j = j'$ or $j+1 = j'$ and $i > i'$. This means that $b_{i,j}$ and $b_{i',j'}$ are adjacent in G if and only if they are at distance at most k in the path. ∎

We call the bubble model of a path power that is constructed in the proof of Theorem 3.1 *canonical.* Observe that the proof of Theorem 3.1 gives a simple linear-time algorithm for constructing a canonical bubble model for a given path power.

Lemma 3.2 *Let G be a graph. Let $A \subseteq B \subseteq V(G)$ and $C \subseteq V(G) \setminus B$. Then $\nu_{G[B]}(A) \le \nu_G(A \cup C)$.*

Proof. Let $u, v \in A$ be such that u and v are not in the same group in A with respect to $G[B]$. By definition of group, there is a vertex $w \in B \setminus A$ that distinguishes u and v. Since $w \notin A \cup C$, w distinguishes u and v in G, hence, u and v are not in the same group in $A \cup C$ with respect to G. ∎

In our lower bound proofs, we will heavily make use of Lemma 3.2. The main task is to identify appropriate sets B and A and determine the number of groups. Let G be a graph with bubble model $\mathcal{B} = \langle B_{i,j}\rangle_{1\le j\le s,1\le i\le r_j}$. Let $A \subseteq V(G)$ and let $1 \le \hat{\jmath} \le s$. The *$\hat{\jmath}$-boundary* of $\mathcal{B}[A]$ is the set $\Phi_{\hat{\jmath}}(\mathcal{B}[A])$ of pairs (i, t_i) that satisfy one of the following conditions:

- $t_i = \hat{\jmath}$ and $i < r_{\hat{\jmath}}$ and $B_{i,t_i} \subseteq A$ and $B_{i',\hat{\jmath}} \not\subseteq A$ for all $i < i' \le r_{\hat{\jmath}}$
- $t_i < \hat{\jmath}$ and $1 \le i \le \min\{r_{t_i},\ldots,r_{\hat{\jmath}}\}$ and $B_{i,t_i} \subseteq A$ and $B_{i,j} \not\subseteq A$ for all $t_i < j \le \hat{\jmath}$.

Lemma 3.3 *Let G be a graph with bubble model $\mathcal{B} = \langle B_{i,j}\rangle_{1\le j\le s,1\le i\le r_j}$ on $s \ge 2$ columns and $l \ge 2$ rows. Let $A \subseteq V(G)$ and let $1 \le \hat{\jmath} \le s$. The bubbles in $\Phi_{\hat{\jmath}}(\mathcal{B}[A])$ appear in pairwise different groups in A.*

Proof. For the following considerations, it is important to remember that all vertices in the bubbles in the $\hat{\jmath}$-boundary are contained in A. Let $(i, t_i) \in \Phi_{\hat{\jmath}}(\mathcal{B}[A])$. Let $b_{i',j'} \notin B_{i,t_i}$ be in the same group as a vertex b from B_{i,t_i} in A. By definition of group, all vertices of G that are adjacent to exactly one of b and $b_{i',j'}$ are contained in A. Let $t_i = \hat{\jmath}$. Since $B_{r_{\hat{\jmath}},\hat{\jmath}} \not\subseteq A$, $j' \ge \hat{\jmath}$. If $j' > \hat{\jmath}$ then $b_{i',j'}$ is not in the $\hat{\jmath}$-boundary, and if $j' = \hat{\jmath}$ then $i' = i$. Hence, no vertex from B_{i,t_i} for $t_i = \hat{\jmath}$ is in the same group as a vertex from another bubble of the $\hat{\jmath}$-boundary. Now, let $t_i < \hat{\jmath}$. If $t_i < j'$ then $B_{i,j'} \subseteq A$, which contradicts the definition of t_i and $(i, t_i) \in \Phi_{\hat{\jmath}}(\mathcal{B}[A])$. If $t_i = j'$ and $i < i'$ then $B_{i,t_i+1} \subseteq A$, in contradiction to the definition of t_i; if $t_i = j'$ and $i' < i$ then $B_{i',j'+1} \subseteq A$, and $b_{i',j'}$ is not a $\hat{\jmath}$-boundary vertex. If $j' < t_i$ then $B_{i',t_i} \subseteq A$, and $b_{i',j'}$ is not a $\hat{\jmath}$-boundary vertex. We conclude also for the case $t_i < \hat{\jmath}$ that no vertex from B_{i,t_i} is in the same group as a vertex from another bubble of the $\hat{\jmath}$-boundary. Hence, the bubbles in the $\hat{\jmath}$-boundary appear in pairwise different groups in A. ∎

4 Maximal k-path powers of linear clique-width $k+1$

In the next section we will show that the linear clique-width of a k-path power containing $k(k+1)+2$ vertices is at least $k+2$. In fact they will turn out to be the smallest k-path powers of maximum linear clique-width. This result is achieved by showing that a k-path power containing $k(k+1)+1$ vertices has layouts of groupwidth at most $k+1$ of only very restricted type. This is exactly what we prove in this section, through a series of results. More precisely, we concentrate on the beginning of a possible layout of groupwidth at most $k+1$, identify the earliest point where the maximum group number is reached, and we show that the two vertices on either side of this point are uniquely defined, hence the restriction on the layout. This restriction in the layouts is used in the next section to show that it is not possible to extend such a k-path power by even a single vertex without increasing the linear clique-width.

The main result of this section is given in Lemmas 4.5 and 4.6. To make the statements of the results shorter, we avoid repeating the following definitions. Throughout this section, let

- P be a k-path power on $k(k+1)+1$ vertices, with $k \geq 3$,
- β be a layout for P such that $\text{gw}(P, \beta) \leq k+1$,
- $\mathcal{B} = \langle b_{i,j} \rangle_{1 \leq j \leq k+1, 1 \leq i \leq r_j}$ be a canonical bubble model for P (Theorem 3.1) such that $b_{1,1} \prec_\beta b_{1,k+1}$.

Note that the restriction $b_{1,1} \prec_\beta b_{1,k+1}$ on the bubble model can be assumed without loss of generality, since it can always be achieved by renaming the vertices due to symmetry of path powers. Also note that β indeed exists, which is shown later (Lemma 5.3). Finally, note that $r_1 = \cdots = r_k = k+1$. For $A \subseteq V(P)$ and $1 \leq j \leq k$, we say that column $\mathcal{B}_j$ is *full* with respect to A if $b_{1,j}, \ldots, b_{k+1,j} \in A$. Let x_f be the leftmost vertex of P with respect to β such that there is an index j_f between 1 and k with $\mathcal{B}_{j_f}$ full with respect to $L_\beta[x_f]$. By the choice of x_f, j_f is uniquely defined. Let $L_f^- =_{\text{def}} L_\beta(x_f)$ and $L_f =_{\text{def}} L_\beta[x_f]$. Denote by $x_{f+1}, x_{f+2}, x_{f+3}$ the three vertices that follow x_f in β. When we use these vertices, they always exist.

By Lemma 2.2 and the fact that non-complete path powers have at most two endvertices, there are at most two vertices for which function ad can have value 0. We can be even more specific.

Lemma 4.1 *1) If $\text{ad}_\beta(b_{1,1}) = 0$ then $b_{1,1}$ and $b_{1,2}$ are in the same group in $L_\beta[b_{1,1}]$.*
2) If $\text{ad}_\beta(b_{1,k+1}) = 0$ then $b_{1,k+1}$ and $b_{1,k}$ are in the same group in $L_\beta[b_{1,k+1}]$.

Proof. For the first statement, assume that $\text{ad}_\beta(b_{1,1}) = 0$. Let K be the group containing $b_{1,1}$ in $L_\beta[b_{1,1}]$. By the first condition of the definition of function ad, K contains a non-neighbour y of $b_{1,1}$. Suppose that there is $z \in N_P(b_{1,1}) \setminus N_P(y)$. If $z \in R_\beta(b_{1,1})$ then $b_{1,1}$ and y are distinguished by z and thus cannot be in the same group in $L_\beta[b_{1,1}]$, if $z \in L_\beta(b_{1,1})$ then $\text{ad}_\beta(b_{1,1}) = 1$ according to the second condition of the definition of function ad. Thus, $N_P(b_{1,1}) \setminus N_P(y) = \emptyset$, i.e., $N_P(b_{1,1}) = \{b_{2,1}, \ldots, b_{k+1,1}\} \subseteq N_P(y)$. This only holds for $y = b_{1,2}$.

The second statement holds by symmetry. ■

Lemma 4.2 *Let $u \in V(P)$. Let K be a group in $L_\beta[u]$. Let $b_{i,j}$ and $b_{i',j'}$ be two vertices in K. Then, $N_P(b_{i,j}) \triangle N_P(b_{i',j'}) \subseteq L_\beta[u]$. In particular, if $|j-j'| \geq 2$ then $\mathcal{B}_j$ and $\mathcal{B}_{j'}$ are full with respect to $L_\beta[u]$.*

Proof. Let $y \in N_P(b_{i,j}) \triangle N_P(b_{i',j'})$. If $y \notin L_\beta[u]$ then y distinguishes $b_{i,j}$ and $b_{i',j'}$, so that they cannot be in the same group in $L_\beta[u]$. Since this contradicts the assumption, $y \in L_\beta[u]$. If $|j-j'| \geq 2$ then $b_{i,j}$ is non-adjacent to every vertex in $\mathcal{B}_{j'}$ and $b_{i',j'}$ is non-adjacent to every vertex in $\mathcal{B}_j$, so that $(V(\mathcal{B}_j) \cup V(\mathcal{B}_{j'})) \setminus \{b_{i,j}, b_{i',j'}\} \subseteq N_P(b_{i,j}) \triangle N_P(b_{i',j'})$. ■

From Lemma 4.2, it follows that a group in L_f can contain vertices from only the same column or from two consecutive columns, since exactly one column is full with respect to L_f.

Lemma 4.3 *There is no $1 \le j \le k$ such that $b_{1,j}, \ldots, b_{k+1,j} \notin L_f$.*

Proof. Suppose for a contradiction that there is $1 \le j \le k$ with $b_{1,j}, \ldots, b_{k+1,j} \notin L_f$. As the first case, let $j_f < j$. We apply Lemma 3.3 and obtain that L_f contains at least $k+1$ groups. According to Lemma 2.2 and the groupwidth assumption for β, x_{f+1} is an endvertex of P. Since x_{f+1} is not vertex from $\mathcal{B}_j$, $L_\beta[x_{f+1}]$ contains $k+1$ groups. Then, x_{f+2} is an endvertex of P and $L_\beta[x_{f+2}]$ contains $k+1$ groups. Then, $\mathrm{ad}_\beta(x_{f+3}) = 1$, since P has only two endvertices, which gives $\mathrm{gw}(P, \beta) \ge \nu_P(L_\beta(x_{f+3})) + \mathrm{ad}_\beta(x_{f+3}) > k+1$. This contradicts the groupwidth assumption for β.

As the second case, let $j < j_f$. Let $\mathcal{B}'$ be the bubble model that is obtained from $\mathcal{B}$ by reversing the columns and turning each column upside down. It is not hard to see that also $\mathcal{B}'$ is a bubble model for P, however not canonical, since the first k bubbles in the first column of $\mathcal{B}'$ are empty. We apply Lemma 3.3 to $\mathcal{B}'$ and obtain $\nu_P(L_f) \ge k+1$. By assumption, x_{f+1} is an endvertex of P. If $j \ge 2$, which also includes the case $x_{f+1} = b_{1,k+1}$, we can continue as above and obtain a contradiction. So, $j = 1$ and $x_{f+1} = b_{1,1}$. Since $\mathrm{ad}_\beta(b_{1,1}) = 0$, we can apply Lemma 4.1 and conclude that $b_{1,1}$ and $b_{1,2}$ are in the same group in $L_\beta[b_{1,1}]$, and due to Lemma 4.2, $\mathcal{B}_2$ is full with respect to $L_\beta[b_{1,1}]$. Thus, $j_f = 2$, and $L_\beta[x_{f+1}]$ has $k+1$ groups. Since x_{f+2} not endvertex contradicts the groupwidth assumption for β, so $x_{f+2} = b_{1,k+1}$ and $L_\beta[x_{f+2}]$ has $k+1$ groups: k groups with vertices from $\mathcal{B}_1$ and $\mathcal{B}_2$ only and one group that contains other vertices and possibly $b_{k+1,2}$. Then, x_{f+3} is not an endvertex of P and yields a contradiction to the groupwidth assumption for β. This completes the proof. ∎

Let $A \subseteq V(P)$. For every $1 \le j \le k$, we denote by $g_j(A)$ the number of groups in A that contain a vertex from column $\mathcal{B}_j$ but not from any of the columns $\mathcal{B}_{j+1}, \ldots, \mathcal{B}_{k+1}$. Note that if there is at most one column that is full with respect to A then it suffices to forbid vertices from $\mathcal{B}_{j+1}$ due to Lemma 4.2.

Lemma 4.4 *Let $u \in L_f^-$ be such that for every $1 \le j \le k$, there is $1 \le i \le k+1$ with $b_{i,j} \in L_\beta[u]$. Then, $g_1(L_\beta[u]), \ldots, g_{k-1}(L_\beta[u]) \ge 1$.*

Proof. Note that no group in $L_\beta[u]$ contains vertices from non-consecutive columns. Let $1 < j \le k$ be such that there are i, i' with $b_{i,j-1}$ and $b_{i',j}$ are in the same group in $L_\beta[u]$. Due to Lemma 4.2, $b_{1,j-1}, b_{i,j}, \ldots, b_{k+1,j} \in L_\beta[u]$. If $b_{1,j-1}$ is in the same group as a vertex from $\mathcal{B}_j$ then $\mathcal{B}_j$ is full with respect to $L_\beta[u]$, which contradicts $u \prec_\beta x_f$ and the choice of x_f. Now, suppose that there is $1 \le j \le k-1$ such that $g_j(L_\beta[u]) = 0$. Since $L_\beta[u]$ contains a vertex from $\mathcal{B}_j$ by assumption, this vertex is in the same group as a vertex from $\mathcal{B}_{j+1}$ according to the definition of $g_j(L_\beta[u])$. Then, $b_{1,j} \in L_\beta[u]$ and the group containing $b_{1,j}$ contains no vertex from $\mathcal{B}_{j+1}$. Hence, $g_j(L_\beta[u]) \ge 1$, a contradiction, and the lemma follows. ∎

Lemma 4.5 *The vertices $b_{1,k}$ and $b_{1,k+1}$ are not in L_f.*

Proof. By definition of x_f, we know that $b_{1,k+1} \ne x_f$. We have two possibilities: $b_{1,k+1} \in L_f$ and $b_{1,k+1} \notin L_f$.

First case: $b_{1,k+1} \in L_f$
Let b be the leftmost vertex with respect to β such that there is a vertex from every column (including $\mathcal{B}_{k+1}$) in $L_\beta[b]$. Let $L_b =_{\text{def}} L_\beta[b]$. Let b' be the vertex following b in β. Since $b_{1,1} \prec_\beta b_{1,k+1}$ by definition and since $b_{1,k+1} \in L_b$, b' is not endvertex of P. Lemma 2.2 shows that $\nu_P(L_b) \geq k+1$ implies $\nu_P(L_\beta(b')) + \text{ad}_\beta(b') > k+1$. Thus, $\nu_P(L_b) \leq k$. We show that this assumption leads to a contradiction.

Consider $g_1(L_b), \ldots, g_k(L_b)$. Since there is no full column with respect to L_b, $b_{1,k+1}$ is not in the same group as any vertex from $\mathcal{B}_1, \ldots, \mathcal{B}_{k-1}$ due to Lemma 4.2. We apply Lemma 4.4 and obtain $\nu_P(L_b) \geq g_1(L_b) + \cdots + g_{k-1}(L_b) + 1 \geq k$. Hence, $g_1(L_b) = \cdots = g_{k-1}(L_b) = 1$ and $g_k(L_b) = 0$. Let K be the group in L_b that contains $b_{1,k+1}$. By $g_k(L_b) = 0$, all vertices from $\mathcal{B}_k$ in L_b are contained in K. If $b_{1,k} \notin L_b$ then $b_{1,k}$ distinguishes the vertices of $\mathcal{B}_k$ in L_b from $b_{1,k+1}$, so that K cannot contain a vertex from $\mathcal{B}_k$. Since this contradicts Lemma 4.3, $b_{1,k} \in L_b$, in particular, $b_{1,k} \in K$. Due to Lemma 4.2, $b_{2,k-1}, \ldots, b_{k+1,k-1} \in L_b$. Since $\mathcal{B}_{k-1}$ is not full with respect to L_b, $b_{1,k-1} \notin L_b$. By assumption $g_{k-1}(L_b) = 1$ and the fact that K contains all vertices from $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$ that are in L_b, $b_{2,k-1}, \ldots, b_{k+1,k-1}$ are in the same group in L_b. For an induction, assume the following for $1 < j < k$:

(1) $b_{k-j+1,j}, \ldots, b_{k+1,j} \in L_b$

(2) the vertices from $\mathcal{B}_j$ in L_b form a group in L_b.

Note that the statements hold for $j = k-1$ by the above considerations. We consider $\mathcal{B}_{j-1}$. Since $b_{k-j+1,j}$ and $b_{k+1,j}$ are in the same group in L_b, $b_{k-j+2,j-1}, \ldots, b_{k+1,j-1} \in L_b$ due to Lemma 4.2. And since $b_{k+1,j}$ is not adjacent to any of the vertices in $\mathcal{B}_{j-1}$ and since $b_{i,j-1} \notin L_b$ for some $1 \leq i < k-j+2$, no vertex from $\mathcal{B}_{j-1}$ is in the same group as a vertex from $\mathcal{B}_j$. Thus, the statements hold for $j-1$. Note that all vertices from $\mathcal{B}_{j-1}$ in L_b are in the same group by the assumption $g_{j-1}(L_b) = 1$. It follows that $b_{k+1,1} \in L_b$ and that $b_{k+1,1}$ is not in the same group as any vertex from another column. By assumption $b_{1,1} \prec_\beta b_{1,k+1}$ and the choices of x_f and b, $b_{1,1} \in L_b$. However, $b_{1,1}$ and $b_{k+1,1}$ are distinguished by all vertices from $\mathcal{B}_2$ except for $b_{k+1,2}$, so that $b_{1,1}$ and $b_{k+1,1}$ cannot be in the same group in L_b. Since we have shown that no vertex from $\mathcal{B}_1$ in L_b is in the same group as a vertex from another column, we finally conclude $g_1(L_b) \geq 2$. This means $\nu_P(L_b) > k$, which yields the contradiction. Hence, $b_{1,k+1} \in L_f$ cannot hold.

Second case: $b_{1,k+1} \notin L_f$
Suppose for a contradiction that $b_{1,k} \in L_f$. Suppose that $j_f = k$. Let $1 \leq i \leq k+1$ be such that $x_f = b_{i,k}$. We consider L_f^-. With Lemma 4.4 and $b_{1,k+1} \notin L_f^-$, $g_1(L_f^-), \ldots, g_k(L_f^-) \geq 1$. If there is $1 \leq j \leq k$ such that $g_j(L_f^-) \geq 2$ then $\nu_P(L_f^-) \geq k+1$, and since $b_{i,k}$ is not endvertex of P, $\text{gw}(P, \beta) > k+1$. Therefore, $g_1(L_f^-) = \cdots = g_k(L_f^-) = 1$. If $i \geq 2$ then L_f^- contains $b_{1,k}$ and another vertex from $\mathcal{B}_k$. Since they are distinguished by $b_{1,k+1}$, this gives $g_k(L_f^-) \geq 2$, which is a contradiction. Thus, $i = 1$, i.e., $x_f = b_{1,k}$. In particular, $b_{1,k} \notin L_f^-$. As an auxiliary result, we show the following by induction. For every $2 \leq j \leq k$:

(1) $b_{k+2-j,j}, \ldots, b_{k+1,j} \in L_f^-$ and $b_{1,j} \notin L_f^-$

(2) the vertices from $\mathcal{B}_j$ in L_f^- form a group in L_f^-.

The two statements are correct for the case $j = k$: $b_{2,k}, \ldots, b_{k+1,k} \in L_f^-$ and $\{b_{2,k}, \ldots, b_{k+1,k}\}$ is a group in L_f^- by the considerations above and $b_{1,k+1} \notin L_f^-$. Now, consider $j < k$. No vertex from $\mathcal{B}_j$ in L_f^- is in the same group as a vertex from $\mathcal{B}_{j+1}$ by induction hypothesis, so $g_j(L_f^-) = 1$ implies that all vertices from $\mathcal{B}_j$ in L_f^- are in the same group. Since $b_{k+2-(j+1),j+1}$

and $b_{k+1,j+1}$ are in the same group in L_f^- by induction hypothesis, $b_{k+2-(j+1)+1,j}, \ldots, b_{k+1,j} \in L_f^-$ due to Lemma 4.2. Note that $b_{1,j+1}$ is a common neighbour of $b_{k+2-j,j}, \ldots, b_{k+1,j}$. Since $b_{1,j+1} \notin L_f^-$ by induction hypothesis, no vertex that is non-adjacent to $b_{1,j+1}$ is in the same group as $b_{k+2-j,j}, \ldots, b_{k+1,j}$. In particular, $b_{k+2-j,j}, \ldots, b_{k+1,j}$ are not in the same group as $b_{1,j}$ or a vertex from any of the columns $\mathcal{B}_1, \ldots, \mathcal{B}_{j-1}$. This also means $b_{1,j} \notin L_f^-$. This completes the proof of the auxiliary result.

Now, consider the groups in L_f. Observe that $b_{1,k}$ is in a singleton group in L_f, since it cannot be in the same group as vertices $b_{2,k}, \ldots, b_{k+1,k}$ because of $b_{1,k+1}$, or with the vertices from $\mathcal{B}_{k-1}$ because of $b_{1,k-1}$, or with vertices from any other column because of a missing vertex. Furthermore, every group in L_f^- is a group in L_f: if there are two groups in L_f^- that are distinguished by only $b_{1,k}$ then one group contains the vertices from $\mathcal{B}_{k-1}$ and the other group contains the vertices from a column $\mathcal{B}_j$ with $j < k-1$. However, since a vertex from $\mathcal{B}_j$ is not contained in L_f by the definition of x_f, $b_{k+1,k-1}$ and the vertices from $\mathcal{B}_j$ in L_f are distinguished by a second vertex. We conclude $\nu_P(L_f) = k+1$ and $g_1(L_f) = \cdots = g_{k-1}(L_f) = 1$ and $g_k(L_f) = 2$. Then, $\mathrm{ad}_\beta(x_{f+1}) = 0$, i.e., x_{f+1} is an endvertex of P. Suppose $x_{f+1} = b_{1,1}$. By Lemma 4.1, $\mathrm{ad}_\beta(b_{1,1}) = 0$ requires $b_{1,2} \in L_f$, which is a contradiction to the above auxiliary result. Thus, $x_{f+1} = b_{1,k+1}$, and therefore, $b_{1,1} \in L_f$. If there are two vertices from $\mathcal{B}_1$ in L_f then there is $2 \leq i \leq k+1$ such that $b_{1,1}, b_{i,1} \in L_f^-$. Since these two vertices are distinguished by $b_{1,2}$, we obtain $g_1(L_f^-) \geq 2$, which is a contradiction. Hence, $b_{1,1}$ is the only vertex from $\mathcal{B}_1$ in L_f. We show that $\nu_{P-b_{1,k+1}}(L_f) = k+1$. Consider $\mathcal{B}[L_f \setminus \{b_{1,1}\}]$. Let $\mathcal{B}'$ be defined as in the proof of Lemma 4.3. We apply Lemma 3.3 to $\mathcal{B}'[L_f \setminus \{b_{1,1}\}]$ and its $(k+1)$-boundary: there are (at least) k boundary vertices. Suppose that $b_{1,1}$ is in the same group as a boundary vertex in L_f. Since no other vertex from $\mathcal{B}_1$ is in L_f, $b_{1,1}$ can be in group only with $b_{1,2}$. This, however, contradicts $b_{1,2} \notin L_f$ due to the auxiliary result. Hence, $b_{1,1}$ is not in the same group as any vertex from the boundary, and therefore $\nu_{P-b_{1,k+1}}(L_f) = k+1$. Applying Lemma 3.2, we obtain $\nu_P(L_\beta[x_{f+1}]) = k+1$, and since x_{f+2} is not an endvertex of P, we conclude a contradiction to the groupwidth assumption for β. Since we have constructed contradictions for all cases, we conclude $j_f < k$.

Suppose that there is $2 \leq i \leq k+1$ such that $b_{i,k} \in L_f^-$. Since $b_{1,k}$ and $b_{i,k}$ are distinguished by $b_{1,k+1}$, $g_k(L_f^-) \geq 2$. Due to Lemma 4.3, we can apply Lemma 4.4 and obtain $g_1(L_f^-), \ldots, g_{k-1}(L_f^-) \geq 1$, which yields $\nu_P(L_f^-) \geq k+1$. Thus, $\mathrm{ad}_\beta(x_f) = 0$, i.e., x_f is an endvertex of P, i.e., $x_f = b_{1,1}$. Since no column except for $\mathcal{B}_1$ is full with respect to L_f and since all neighbours of $b_{1,1}$ are in $\mathcal{B}_1$, the group in L_f containing $b_{1,1}$ contains only vertices from $\mathcal{B}_1$. Since these vertices are adjacent to $b_{1,1}$, $\mathrm{ad}_\beta(x_f) = 1$, which gives a contradiction. Hence, $b_{1,k}$ is the only vertex from $\mathcal{B}_k$ in L_f^-. The k-boundary of $\mathcal{B}[L_f]$ contains k vertices, and due to Lemma 3.3, they are in pairwise different groups in L_f. Since no vertex in columns $\mathcal{B}_1, \ldots, \mathcal{B}_{k-1}$ is adjacent to $b_{k+1,k}$, which is not contained in L_f, $b_{1,k}$ is vertex in a singleton group, and therefore $\nu_P(L_f) \geq k+1$ and $\mathrm{ad}_\beta(x_{f+1}) = 0$. According to Lemma 2.2, x_{f+1} is endvertex of P. If $x_{f+1} = b_{1,1}$ then $j_f > 1$, and no vertex from $\mathcal{B}_1$ is in the same group as a vertex from another column in L_f because of $b_{1,1}$. Then, the above arguments show that L_f has at least $k+2 > k+1$ groups, which is a contradiction to the groupwidth assumption for β. Thus, $x_{f+1} = b_{1,k+1}$. Then, $\nu_P(L_\beta[x_{f+1}]) \geq k+1$, since no vertex in L_f is adjacent to $b_{1,k+1}$. However, x_{f+2} is no endvertex of P, which yields $\nu_P(L_\beta(x_{f+2})) + \mathrm{ad}_\beta(x_{f+2}) > k+1$, a contradiction to the groupwidth assumption for β. Hence, the assumption $b_{1,k} \in L_f$ is false, and we conclude the lemma. ■

Lemma 4.6 *The following holds for layout β:*

- $\nu_P(L_f) = k+1$ *and* $x_f = b_{1,2}$ *and* $x_{f+1} = b_{1,1}$
- $b_{3,1}, \ldots, b_{k+1,1} \in L_f$ *and* $b_{2,k}, b_{3,k} \in L_f$
- *the vertices from* $\mathcal{B}_1$ *in* L_f *are in the same group and the vertices from* $\mathcal{B}_k$ *in* L_f *are in the same group in* L_f.

Proof. Suppose that $\nu_P(L_f^-) = k+1$. By our groupwidth assumption for β, $\text{ad}_\beta(x_f) = 0$, i.e., x_f is an endvertex of P due to Lemma 2.2. Since $b_{1,k+1} \notin L_f$ due to Lemma 4.5, $x_f = b_{1,1}$. Due to Lemma 4.1, $b_{1,1}$ and $b_{1,2}$ are in the same group in L_f, and due to Lemma 4.2, $\mathcal{B}_2$ is full with respect to L_f. This means $j_f = 2$, which contradicts $x_f = b_{1,1}$. Hence, $\nu_P(L_f^-) \leq k$.

We show by induction that $b_{1,1}, \ldots, b_{1,k+1} \notin L_f^-$ and that every group in L_f^- corresponds to the vertices in L_f^- from a single column. According to Lemma 4.5, $b_{1,k}, b_{1,k+1} \notin L_f^-$. Therefore, no vertex from $\mathcal{B}_k$ in L_f^- is in the same group as a vertex from the columns $\mathcal{B}_1, \ldots, \mathcal{B}_{k-1}$. Hence, $g_k(L_f^-) \geq 1$, and thus $g_1(L_f^-), \ldots, g_k(L_k^-) \geq 1$ by application of Lemma 4.4. With $\nu_P(L_f^-) \leq k$, we conclude $g_1(L_f^-) = \cdots = g_k(L_k^-) = 1$. Assume for some $1 \leq j < k$ that $b_{1,j+1} \notin L_f^-$ and the vertices from $\mathcal{B}_{j+1}$ in L_f^- form a group in L_f^-. If $b_{1,j}, b_{i,j} \in L_f^-$ for some $2 \leq i \leq k+1$ then $b_{1,j}$ and $b_{i,j}$ are not in the same group in L_f^- since they are distinguished by $b_{1,j+1}$. Since neither $b_{1,j}$ nor $b_{i,j}$ is in the same group as a vertex from $\mathcal{B}_{j+1}$, $g_j(L_f^-) \geq 2$, which is a contradiction. We consider two cases. Suppose that $b_{1,j} \in L_f^-$. Then, $b_{2,j}, \ldots, b_{k+1,j} \notin L_f^-$. We have two possibilities: $j_f < j$ and $j < j_f$. Note that $j_f \neq j$, since $\mathcal{B}_j$ misses at least three vertices in L_f^-. Let $j_f < j$. We consider L_f. The j-boundary of $\mathcal{B}[L_f]$ contains $k+1$ vertices, that are in pairwise different groups in L_f due to Lemma 3.3. By induction hypothesis, the vertices from $\mathcal{B}_{j+1}$ form a group in L_f, so that we obtain $\nu_P(L_f) \geq k+2$, which is a contradiction to the groupwidth assumption for β. Now, let $j < j_f$. Note that the induction hypothesis shows that $x_f = b_{1,j_f}$. Let $\mathcal{B}'$ be defined as in the proof of Lemma 4.3. We consider the $(k+2-j)$-boundary in $\mathcal{B}'[L_f^-]$. Note that the boundary contains k vertices, among which is not $b_{1,j}$. Because of $b_{1,j+1}, \ldots, b_{1,k} \notin L_f^-$, $b_{1,j}$ is not in the same group as any vertex from the boundary. Furthermore, $j_f < k$ due to $b_{1,k} \notin L_f$, so that no vertex from $\mathcal{B}_k$ is in the same group as $b_{1,j}$ or a vertex from the boundary. Hence, $\nu_P(L_f^-) \geq k+1$, which yields a contradiction with $\text{ad}_\beta(x_f) = 1$ by x_f not endvertex of P. Thus, $b_{1,j} \notin L_f^-$. And because of $b_{1,j}, b_{1,j+1} \notin L_f^-$, no vertex from $\mathcal{B}_j$ in L_f^- is in the same group as a vertex from another column, and by assumption $g_j(L_f^-) = 1$, the vertices from $\mathcal{B}_j$ in L_f^- form a group in L_f^-.

We determine j_f. Observe that $x_f = b_{1,j_f}$ and x_f is vertex in a singleton group in L_f. The latter is true since every other vertex in L_f is adjacent to a vertex $b_{1,j}$ that is not in L_f. And since pairs of groups in L_f^- are distinguished by at least two vertices, no group in L_f contains vertices from two different columns. Hence, $\nu_P(L_f) = k+1$, and $\text{ad}_\beta(x_{f+1}) = 0$. Suppose that $j_f = 1$, i.e., $x_f = b_{1,1}$. Then, $x_{f+1} = b_{1,k+1}$, and Lemma 4.1 implies $b_{1,k} \in L_f$, which is a contradiction to Lemma 4.5. Suppose $j_f \geq 3$, i.e., $3 \leq j_f \leq k-1$. Then, $x_{f+1} = b_{1,1}$, and Lemma 4.1 implies $b_{1,2} \in L_f$, which is a contradiction to the auxiliary result. Hence, $j_f = 2$ and $x_f = b_{1,2}$ and $\nu_P(L_f) = k+1$ and $x_{f+1} = b_{1,1}$. And since $b_{2,2}$ and $b_{k+1,2}$ are in the same group in L_f, $b_{3,1}, \ldots, b_{k+1,1} \in L_f$ due to Lemma 4.2. Finally, we apply the auxiliary result and Lemma 4.2 and conclude from $b_{2,2}$ and $b_{k+1,2}$ in the same group in L_f^- that $b_{2,3}, \ldots, b_{k,3} \in L_f^-$. By induction, we obtain $b_{2,k}, b_{3,k} \in L_f^-$, which concludes the proof. ∎

5 The linear clique-width of path powers

In this section, we are finally ready to give a complete characterisation of the linear clique-width of path powers of all sizes. We start with the previously mentioned lower bound.

Lemma 5.1 *Let G be a k-path power on $k(k+1)+2$ vertices, with $k \geq 1$. Then,* $\mathrm{lcwd}(G) \geq k+2$.

Proof. For $k = 1$, G is a 1-path power on four vertices, i.e., $G = P_4$. It holds that $\mathrm{lcwd}(P_4) = 3$. For $k = 2$, G is a 2-path power on eight vertices. It can be checked that $\mathrm{lcwd}(G) = 4$. So, let $k \geq 3$. Suppose for a contradiction that there is a layout β for G such that $\mathrm{gw}(G, \beta) \leq k + 1$. Let a be an endvertex of G. Then, $G{-}a$ is a k-path power on $k(k + 1) + 1$ vertices. Let β' be obtained from β by deleting a. Then, $\mathrm{gw}(G{-}a, \beta') \leq k + 1$, and the results of Section 4 can be applied to $G{-}a$ and β'. Let $\mathcal{B} = \langle b_{i,j}\rangle_{1 \leq j \leq k+1, 1 \leq i \leq r_i}$ be a canonical bubble model for $G{-}a$ such that $b_{1,1} \prec_{\beta'} b_{1,k+1}$. Let x_f and L_f and L_f^- for $G{-}a$ and β' be defined as in Section 4. Due to Lemma 4.6, $b_{k,1}, b_{k+1,1} \in L_f$, and $b_{2,k}, b_{3,k} \in L_f$, and $b_{k,1}$ and $b_{k+1,1}$ are in the same group in L_f, and $b_{2,k}$ and $b_{3,k}$ are in the same group in L_f. Furthermore, $b_{1,1}, b_{1,2} \notin L_f^-$ and $b_{1,k}, b_{1,k+1} \notin L_f$ (Lemma 4.5). By the choice of a as an endvertex of G, a is adjacent to $b_{k,1}$ and non-adjacent to $b_{k+1,1}$ or a is adjacent to $b_{3,k}$ and non-adjacent to $b_{2,k}$ in G. If $x_f \prec_\beta a$ then a distinguishes $b_{k,1}$ and $b_{k+1,1}$ in the former case, and $b_{2,k}$ and $b_{3,k}$ in the latter case. With $\nu_{G-a}(L_f) = k+1$ due to Lemma 4.6, it follows that $\nu_G(L_f) \geq k + 2$, which is a contradiction to our assumption. Hence, $a \prec_\beta x_f$. Since $\nu_{G-a}(L_f) = k + 1$ and $\mathrm{ad}_{\beta'}(x_f) = 1$, $\nu_{G-a}(L_f^-) = k$. Note also that $\mathrm{ad}_\beta(x_f) = 1$ due to Lemmata 2.2 and 4.6, so that $\nu_G(L_\beta(x_f)) = k$ by our assumptions. Remember that there is a vertex for every column of $\mathcal{B}$ that is not in L_f^-. If the neighbours of a are in $\mathcal{B}_1$ then a is vertex in a singleton group in $L_\beta(x_f)$, particularly because of $b_{1,1}, b_{1,2} \notin L_\beta(x_f)$. If the neighbours of a are in $\mathcal{B}_k$ and $\mathcal{B}_{k+1}$ then a is vertex in a singleton group in $L_\beta(x_f)$, particularly because of $b_{1,k}, b_{1,k+1} \notin L_\beta(x_f)$. Hence, $\nu_G(L_\beta(x_f)) > k$, which yields a contradiction to our assumption together with $x_f = b_{1,2}$ and $\mathrm{ad}_\beta(b_{1,2}) = 1$. Therefore, $\mathrm{gw}(G) \geq k + 2$. ∎

Now we give the upper bounds. It is known that $\mathrm{lcwd}(G) \leq \mathrm{pw}(G) + 2$ for G an arbitrary graph [11], where $\mathrm{pw}(G)$ is the pathwidth of G. For path powers, the pathwidth is equal to the maximum clique size minus 1, which implies the next result.

Lemma 5.2 ([11]) *Let G be a k-path power, with $k \geq 1$. Then,* $\mathrm{lcwd}(G) \leq k + 2$.

For path powers on few vertices, we can show an even better bound.

Lemma 5.3 *Let G be a k-path power on $l(k+1)+1$ vertices, with $2 \leq l \leq k$. Then,* $\mathrm{lcwd}(G) \leq l + 1$.

Proof. Let $\mathcal{B} = \langle b_{i,j}\rangle_{1 \leq j \leq l+1, 1 \leq i \leq r_j}$ be a canonical bubble model for G. Note that $r_1 = \cdots = r_l = k + 1$ and $r_{l+1} = 1$. Let

$$\beta = \langle b_{k+1,l}, \ldots, b_{k+1,1}, b_{k,l}, \ldots, b_{2,1}, b_{1,2}, b_{1,1}, b_{1,3}, \ldots, b_{1,l+1}\rangle\,,$$

i.e., the vertices in $\mathcal{B}$ appear in β row by row, starting from the bottom row, and within a row, from right to left, except for the first row. We show that $\mathrm{gw}(G, \beta) \leq l + 1$. Let $x = b_{i,j}$ be a vertex of G. If $i \geq 2$ then $\nu_G(L_\beta[x]) \leq l$. To see this, observe that $b_{i+1,j'}, \ldots, b_{k+1,j'} \in L_\beta[x]$ and $b_{1,j'}, \ldots, b_{i,j'} \notin L_\beta[x]$ for all $j' < j$ and $b_{i,j'}, \ldots, b_{k+1,j'} \in L_\beta[x]$ and $b_{1,j'}, \ldots, b_{i-1,j'} \notin L_\beta[x]$ for all $j' \geq j$. Hence, the vertices of every column that are in $L_\beta[x]$ are in the same group. Since there are l columns in $\mathcal{B}$ with vertices in $L_\beta[x]$, the claim holds. Now, let $i = 1$. It holds that $b_{2,j'}, \ldots, b_{k+1,j'} \in L_\beta[x]$ for all $1 \leq j' \leq l$. If $x = b_{1,2}$ then $L_\beta[x]$ has exactly $l+1$ groups, since $b_{1,2}$ is not in the same group as any other vertex. It holds that $\mathrm{ad}_\beta(b_{1,1}) = 0$, which is easy to check with the definition of function ad. Thus, $\nu_G(L_\beta(b_{1,1})) + \mathrm{ad}_\beta(b_{1,1}) = l+1+0 \leq l+1$. Now, let $j \geq 3$. Then, $b_{1,1}, \ldots, b_{k+1,1}, b_{1,2}, \ldots, b_{k+1,j-2}, b_{1,j-1}$ are in the same group, and in $L_\beta[x]$, and $\nu_G(L_\beta[x]) \leq l$. We conclude that $\mathrm{gw}(G, \beta) \leq l+1$. ∎

With the lower and upper linear clique-width bounds, we are ready to give the complete characterisation.

Theorem 5.4 *Let G be a k-path power on n vertices, with $k \geq 1$ and $n \geq k+2$.*

- *If $n \geq k(k+1)+2$ then* $\mathrm{lcwd}(G) = k+2$.
- *If $k+2 \leq n \leq k(k+1)+1$ then* $\mathrm{lcwd}(G) = \lceil \frac{n-1}{k+1} \rceil + 1$.

Proof. If $n \geq k(k+1)+2$ then $\mathrm{lcwd}(G) \leq k+2$ due to Lemma 5.2 and $\mathrm{lcwd}(G) \geq k+2$ due to Lemma 5.1 and since the k-path power on $k(k+1)+1$ vertices is an induced subgraph of G. Let $k+2 \leq n \leq k(k+1)+1$. Let $\mathcal{B}$ be a canonical bubble model for G. Note that $\lceil \frac{n-1}{k+1} \rceil$ is equal to the number of columns of $\mathcal{B}$ with at least two vertices. Let d be the number of (non-empty) columns of $\mathcal{B}$. If the rightmost column of $\mathcal{B}$ contains exactly one vertex then G is a k-path power on $(d-1)(k+1)+1$ vertices. Then, $\lceil \frac{n-1}{k+1} \rceil = d-1$, and $\mathrm{lcwd}(G) \leq d$ due to Lemma 5.3 and $\mathrm{lcwd}(G) \geq d$ since for the case $d = 2$, G contains two adjacent vertices, and for the case $d \geq 3$, the $(d-2)$-path power on $(d-2)(d-1)+2$ vertices is an induced subgraph of G and due to Lemma 5.1. If the rightmost column of $\mathcal{B}$ contains at least two vertices then $(d-1)(k+1)+2 \leq n \leq d(k+1)$ and G contains a $(d-1)$-path power on $(d-1)d+2$ vertices as induced subgraph. Then, $\lceil \frac{n-1}{k+1} \rceil = d$, and $\mathrm{lcwd}(G) \leq d+1$ due to Lemma 5.3 and $\mathrm{lcwd}(G) \geq d+1$ due to Lemma 5.1. ∎

Note that k-path powers on at most $k+1$ vertices are complete graphs and therefore have linear clique-width at most 2.

Corollary 5.5 *Let $k \geq 1$ and let G be a path power on at least two vertices. Then,* $\mathrm{lcwd}(G) \leq k+1$ *if and only if G does not contain the k-path power on $k(k+1)+2$ vertices as induced subgraph.*

Proof. If G has a k-path power on $k(k+1)+2$ vertices as induced subgraph then $\mathrm{lcwd}(G) \geq k+2$ due to Theorem 5.4. Let G not have a k-path power on $k(k+1)+2$ vertices as induced subgraph. If G is a k'-path power for some $k' < k$ then $\mathrm{lcwd}(G) \leq k+1$ due to Lemma 5.2. Now, let G be a k''-path power for some $k'' \geq k$. Then, G contains at most $k(k+1)+1$ vertices. Since $\lceil \frac{k(k+1)}{k''+1} \rceil \leq k$, $\mathrm{lcwd}(G) \leq k+1$ due to Theorem 5.4. ∎

With the characterisation in Corollary 5.5, we can construct a simple algorithm that computes the linear clique-width of path powers.

Theorem 5.6 *There is a linear-time algorithm that computes the linear clique-width of path powers.*

Proof. Let G be a path power. A canonical bubble model for G can be computed in linear time. Applying Corollary 5.5, $\mathrm{lcwd}(G) = l+1$ where l is the smallest number such that G does not contain an l-path power on $l(l+1)+2$ vertices as induced subgraph. This number is easy to determine from the computed bubble model. ∎

6 Conclusions

We have given a complete characterisation of the linear clique-width of path powers. We have seen that the linear clique-width of a path power is a function of the size of the largest clique and the number of vertices. In a second result, we have characterised the path powers of bounded linear clique-width by forbidden induced subgraphs. In fact, there is exactly one minimal such forbidden induced subgraph. Note that every class of path powers of bounded linear clique-width contains infinitely many graphs, if the bound is larger than 1. All results are based on a thorough analysis of layouts of bounded groupwidth.

Path powers are an interesting graph class to study properties of linear clique-width. As mentioned in the Introduction, the linear clique-width of path powers is between the known upper and lower bound on the clique-width of path powers. Does equality hold, at least for path powers on sufficiently many vertices?

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