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Weight hierarchies of extremal non-chain ternary codes of dimension 4*

Wende Chen† and Torleiv Kløve ‡

Abstract

The weight hierarchy of a linear $[n,k;q]$ code C over $GF(q)$ is the sequence $(d_1, d_2, \dots, d_k)$ where d_r is the smallest support of an r–dimensional subcode of C. An $[n,k;q]$ code is extremal non-chain if for any r and s, where $1 \le r < s \le k$, there are no subspaces D and E such that $D \subset E$, $\dim D = r$, $\dim E = s$, $w_S(D) = d_r$, and $w_S(E) = d_s$. The possible weight hierarchies of such ternary codes of dimension 4 are determined.

I Introduction

The weight hierarchy of linear codes has been studied by a number of researchers. For a code of dimension k, it is a sequence of parameters $(d_1, d_2, \dots, d_k)$. In particular, d_1 is the minimum distance of the code. The parameters were first introduced in [11]. In [17] it was shown that these parameters are important in the analysis of an application of linear codes to the wiretap channel of type II. Later, the weight hierarchy has been shown to be important in the analysis of the trellis complexity of linear codes, see e.g. [9], [13], [16]; and analysis of linear codes for error detection on the local binomial channel, see [15]. The possible weight hierarchies of binary linear codes of dimension up to 4 were determined in [14]. The chain condition was introduced in [18]. Codes satisfying this condition have been studied in e.g. [1], [9], [10], [13], [16], [18]. For small lengths and dimensions, the codes with largest values of of the minimum support weights satisfies the chain conditions and this is possibly a general phenomenon. This is the main reason for studying codes satisfying the chain condition. Also, the analysis of the weight hierarchies of product codes is simpler if both codes satisfy the chain condition, see [10], [18]. The possible weight hierarchies of binary linear codes of dimensions up to 5 satisfying the chain condition were determined in [8]. In [2]–[6] we studied the possible weight hierarchies of linear codes of dimension 4 or less over arbitrary finite fields. The chain condition is a statement that subspaces of smallest support are related in a particular way. To get a better understanding of how weight hierarchies behave in general, it is interesting to study how the subspaces of smallest support are related. One extreme are codes satisfying the chain condition. The other extreme are what we call extremal non-chain codes. In [3], [4] we determined the possible weight hierarchies of extremal non-chain codes of dimension 3. It turns out the the complexity

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of doing such a classification increases dramatically with the dimension. In [5] we gave bounds on the weight hierarchies of extremal non-chain codes of dimension 4. In this paper we determine exactly the possible weight hierarchies of ternary extremal non-chain codes of dimension 4.

II Notations and problem formulation

Throughout this paper, unless otherwise stated, C will be a linear $[n,4;q]$ code, that is a code of length n and dimension 4 over $GF(q)$. Mainly, we consider $q=3$. For convenience we give all definitions below for 4-dimensional codes, rather than codes of general dimension, since we concentrate on 4-dimensional codes.

For any subcode D of C, we define the *support* of D to be the set of positions where not all the codewords of D are zero, and we denote it by $\chi(D)$. Further, we define the *support weight* of D to be the size of $\chi(D)$, and we denote it by $w_S(D)$.

For $1 \le r \le 4$, the *the r-th minimum support weight* (or Generalized Hamming weight) of C is defined by

$$d_r(C) = \min\{w_S(D) \mid D \text{ is an } [n,r] \text{ subcode of } C\}.$$

The sequence (d_1,d_2,d_3,d_4) is the *weight hierarchy* of C.

We note that if we add a zero-position to an $[n,4;q]$ code C we get an $[n+1,4;q]$ code

$$C' = \{(\mathbf{c}|0) \mid \mathbf{c} \in C\}.$$

The codes C and C' have the same weight hierarchy. Therefore, without loss of generality, we can restrict ourselves to codes without zero-positions, that is, we will assume that $n = d_4$. Our problems can then be reformulated in terms of projective geometry and we do this next.

The *difference sequence* (DS) (i_0,i_1,i_2,i_3) of a $[d_4,4;q]$ code is defined by

$$i_0 = d_4 - d_3, \quad i_1 = d_3 - d_2, \quad i_2 = d_2 - d_1, \quad i_3 = d_1.$$

The difference sequence can easily be computed from the weight hierarchy and vice versa.

Let G be a generator matrix for C. For any $\mathbf{x} \in GF(q)^4$, $m(\mathbf{x})$, the *value* of $\mathbf{x}$, will denote the number of occurrences of $\mathbf{x}$ as a column in G. In [12] it was shown that there is a one-one correspondence between the subspaces of C of dimension r and the subspaces of $GF(q)^4$ of dimension $4-r$ such that if D corresponds to U, then

$$d_4 - w_S(D) = \sum_{\mathbf{x}\in U} m(\mathbf{x}).$$

We find it convenient to look at the vectors as projective points.

From now on we assume that $q=3$. Let V_3 be the projective space $PG(3,3)$. A *value assignment* is a function

$$m : V_3 \to \mathbb{N} = \{0,1,2,\ldots\}.$$

For $p \in V_3$ we call $m(p)$ the *value* of p. A value assignment defines a generator matrix and a code (up to equivalence). We define the value of a subset S of V_3 as follows:

$$m(S) = \sum_{p\in S} m(p).$$

Let C_1 be the set of lines and C_2 the set of planes in V_3. The existence of a code with weight hierarchy (d_1,d_2,d_3,d_4) is equivalent to the existence of a value assignment m such that

$$\begin{aligned}&\max\{m(p) \mid p \in V_3\} = i_0,\\ &\max\{m(l) \mid l \in C_1\} = i_0 + i_1,\\ &\max\{m(P) \mid P \in C_2\} = i_0 + i_1 + i_2,\\ &m(V_3) = i_0 + i_1 + i_2 + i_3.\end{aligned}$$

Let

$$\begin{aligned}M_0 &= \{p \mid p \in V_3 \text{ and } m(p) = i_0\},\\ M_1 &= \{l \mid l \in C_1 \text{ and } m(l) = i_0 + i_1\},\\ M_2 &= \{P \mid P \in C_2 \text{ and } m(P) = i_0 + i_1 + i_2\}.\end{aligned}$$

The research reported in this paper is part of a project to describe the possible weight hierarchies, or equivalently, the possible value assignments of dimension 4. It is convenient to split the analysis into cases. The *chain condition*:

"there exist $p \in M_0$, $l \in M_1$, and $P \in M_2$
such that $p \in l \subset P$"

was introduced by Wei and Yang (1993). Codes of dimension 4 over $GF(q)$ satisfying this condition were considered in [2]. The other extreme, namely *extremal non-chain codes*, was considered in [5] and [7]. The corresponding DS are called *extremal non-chain difference sequence* (ENDS). These are DS of value assignments which satisfy the following three conditions:

(C1): "There do not exist $p \in M_0$ and $l \in M_1$ such that $p \in l$".
(C2): "There do not exist $p \in M_0$ and $P \in M_2$ such that $p \in P$".
(C3): "There do not exist $l \in M_1$ and $P \in M_2$ such that $l \subset P$".

In [7] we determined exactly the possible weight hierarchies of binary extremal non-chain codes of dimension 4. In this paper we determine exactly the possible weight hierarchies of extremal non-chain ternary codes of dimension 4. The result is given in Theorem 1 below. First we need a couple of definitions. Let

$$\vartheta = \begin{cases} 0, & \text{if } 3i_0 - i_1 \equiv 3 \pmod 4,\\ 3i_0 - i_1 \pmod 4, & \vartheta \in \{0,1,2\}, \text{otherwise,}\end{cases}$$
$$\vartheta_1 = \begin{cases} 1, & \text{if } i_0 \text{ is odd,}\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 1 *For $q = 3$, (i_0, i_1, i_2, i_3) is an ENDS if and only if*

i) $\frac{1}{3}(i_0 + 4) \le i_1 \le 3i_0 - 4$,

ii) $i_0 + 1 \le i_2$,
$i_2 \le \min\left(\frac{1}{4}(9(i_0 + i_1) - 13) - \vartheta, \frac{3}{2}i_0 + 3i_1 - \frac{9}{2} - \vartheta_1, 6i_1 - 9, i_0 + 4i_1 - 6\right)$,

iii) $\max(i_0 + i_1 + 1, 2i_0 + 1, i_0 + \frac{1}{3}(i_2 + 4), i_0 - 3i_1 + i_2 + 6) \le i_3$,
$i_3 \le \min(12i_1 - i_2 - 13, 3i_2 - 4)$.

The proof of this theorem is broken down into two parts. In the next section we prove the "only if" part and in the following section we prove the "if" part by a number of explicit constructions.

We denote the line containing the points p_1, p_2 by $\overline{p_1p_2}$, and the plane containing the points p_1, p_2, p_3 by $\widehat{p_1p_2p_3}$.

Proof of the bounds

Lemma 1 *For $q = 3$, if (i_0, i_1, i_2, i_3) is an ENDS. then*

i) $\frac{1}{3}(i_0 + 4) \le i_1 \le 3i_0 - 4$,

ii) $i_0 + 1 \le i_2$

$i_2 \le \min\Big(\frac{1}{4}\big(9(i_0 + i_1) - 13\big) - \vartheta, \frac{3}{2}i_0 + 3i_1 - \frac{9}{2} - \vartheta_1, 6i_1 - 9, i_0 + 4i_1 - 6\Big)$,

iii) $\max(i_0 + i_1 + 1, 2i_0 + 1, i_0 + \frac{1}{3}(i_2 + 4), i_0 - 3i_1 + i_2 + 6) \le i_3$

$i_3 \le \min(12i_1 - i_2 - 13, 3i_2 - 4)$.

Proof: i) We proved $i_1 \le 3i_0 - 4$ for $q = 3$ in [5].
iii) From [5] we have

$$m(V_3) = i_0 + i_1 + i_2 + i_3 \le i_0 + 13(i_1 - 1).$$

Hence

$$i_3 \le 12i_1 - i_2 - 13. \tag{1}$$

Let $l^* \in M_2$. Since (C3) is false,

$$i_0 + i_1 + i_2 + i_3 = m(V_3) = m(l^*) + \sum_{\substack{P \\ l^* \subset P}} m(P \setminus l^*) \le (i_0 + i_1) + 4(i_2 - 1).$$

Combined with (1) this proves the upper bounds of iii).

Let $P^* \in M_3$, $p \in P^*$. Since (C3) is false, $m(l) \le i_0 + i_1 - 1$ for all lines $l \subset P^*$. Hence

$$i_0 + i_1 + i_2 = m(P^*) = m(p) + \sum_{\substack{l \\ p \in l \subset P^*}} \big(m(l) - m(p)\big) \le m(p) + 4\big(i_0 + i_1 - 1 - m(p)\big),$$

and so

$$m(p) \le i_0 + i_1 - \frac{i_2}{3} - \frac{4}{3}. \tag{2}$$

Let $p^* \in M_0$. We denote the plane containing l^* and p^* by P_1. Let $l_1 = P_1 \cap P^*$, $p_1 = l^* \cap P^*$. Since

$$\begin{aligned} m(P^*) &= m(p_1) + \sum_{\substack{l \\ p_1 \in l \subset P^*}} \big(m(l) - m(p_1)\big) \\ &= \big(m(l_1) - m(p_1)\big) - 2m(p_1) + \sum_{\substack{l \ne l_1 \\ p_1 \in l \subset P^*}} m(l) \\ &\le m(l_1 \setminus \{p_1\}) - 2m(p_1) + 3(i_0 + i_1 - 1) \end{aligned}$$

we have

$$m(l_1 \setminus \{p_1\}) \ge 2m(p_1) - 2i_0 - 2i_1 + i_2 + 3. \tag{3}$$

Since $m(l^* \setminus \{p_1\}) = i_0 + i_1 - m(p_1)$, this implies

$$m(l_1 \setminus \{p_1\}) + m(l^* \setminus \{p_1\}) \ge m(p_1) - i_0 - i_1 + i_2 + 3. \tag{4}$$

Since (C1) is false,

$$m(l_1 \setminus \{p_1\}) + m(l^* \setminus \{p_1\}) \le \sum_{\substack{l \\ p_1 \notin l \\ p^* \in l \subset P_1}} m(l) - 3i_0 \le 3(i_0 + i_1 - 1) - 3i_0 = 3i_1 - 3, \tag{5}$$

Combined with (4) this implies

$$m(p_1) \le i_0 + 4i_1 - i_2 - 6 \tag{6}$$

Since $m(\overline{p^*p_1}) \leq i_0 + i_1 - 1$,

$$m(p_1) \leq i_1 - 1 \tag{7}$$

In particular, this proves that $m(l^* \setminus \{p_1\}) \geq i_0 + 1$. Hence there exists a $\hat{p} \in l^* \setminus \{p_1\}$ such that

$$m(\hat{p}) \geq \frac{1}{3}(i_0 + 1)$$

and so

$$i_0 + i_1 - 1 \geq m(\overline{\hat{p}p^*}) \geq i_0 + \frac{1}{3}(i_0 + 1)$$

which proves that $i_1 \geq \frac{1}{3}(i_0 + 4)$ in i).

Since (C1) is false we have

$$m(p_1) \leq i_0 - 1, \tag{8}$$

$$i_3 \geq i_0 + m(l^* \setminus \{p_1\}) = 2i_0 + i_1 - m(p_1), \tag{9}$$

Combining (2), (6), (7), (8), (9) we get the lower bounds of iii).

ii) From (6) we have

$$m(p_1) = i_0 + 4i_1 - i_2 - 6 - d \tag{10}$$

where $d \geq 0$. Substituting (10) in (3) we get

$$m(l_1 \setminus \{p_1\}) \geq 6i_1 - i_2 - 9 - 2d.$$

Hence there exists a point $p_2 \in l_1 \setminus \{p_1\}$ such that

$$m(p_2) \geq \frac{1}{3}(6i_1 - i_2 - 9 - 2d). \tag{11}$$

By (10) we get

$$m(l^* \setminus \{p_1\}) = i_0 + i_1 - m(p_1) = i_2 - 3i_1 + 6 + d.$$

Hence from (5) we have

$$\begin{aligned} m(l_1 \setminus \{p_1\}) &\leq 3i_1 - 3 - (i_2 - 3i_1 + 6 + d) \\ &= 6i_1 - i_2 - 9 - d. \end{aligned}$$

So by (10) we get

$$\begin{aligned} m(P^* \setminus l_1) &= (i_0 + i_1 + i_2) - m(p_1) - m(l_1 \setminus \{p_1\}) \\ &\geq 3i_2 - 9i_1 + 15 + 2d, \end{aligned} \tag{12}$$

Therefore, from

$$m(P^* \setminus l_1) = \sum_{\substack{l \\ l \neq l_1, p_2 \in l \subset P^*}} \big(m(l) - m(p_2)\big),$$

there exists a line l such that $l \neq l_1, p_2 \in l \subset P^*$ and

$$m(l) - m(p_2) \geq i_2 - 3i_1 + 5 + \frac{2}{3}d.$$

Hence, by (11) we have

$$\begin{aligned} m(l) &\geq i_2 - 3i_1 + 5 + \frac{2}{3}d + \frac{1}{3}(6i_1 - i_2 - 9 - 2d) \\ &= \frac{2}{3}i_2 - i_1 + 2 \end{aligned} \tag{13}$$

Since (C3) is false, $m(l) \le i_0 + i_1 - 1$, Combined with (13) this implies

$$i_2 \le \frac{3}{2} i_0 + 3i_1 - \frac{9}{2}.$$

By (12) there exists a point $p_3 \in P^* \setminus l_1$, such that

$$m(p_3) \ge \frac{1}{3} i_2 - i_1 + \frac{5}{3} + \frac{2d}{9}.$$

Hence if $i_2 = \frac{3}{2} i_0 + 3i_1 - \frac{9}{2}$ then

$$m(p_3) \ge \frac{1}{3}\Big(\frac{3}{2} i_0 + 3i_1 - \frac{9}{2}\Big) - i_1 + \frac{5}{3} + \frac{2d}{9} \ge \frac{1}{2} i_0 + \frac{1}{6}$$

and by (2) $m(p_3) \le i_0 + i_1 - \frac{1}{3}(\frac{3}{2} i_0 + 3i_1 - \frac{9}{2}) - \frac{4}{3} = \frac{1}{2} i_0 + \frac{1}{6}$, but this is impossible for i_0 odd since $m(p_3)$ is an integer. Hence

$$i_2 \le \frac{3}{2} i_0 + 3i_1 - \frac{9}{2} - \vartheta_1. \tag{14}$$

Let $p_0 \in P^*$, $m(p_0) = \max\{m(p) \mid p \in P^*\}$. We have

$$m(p_0) \ge \frac{1}{13} m(P^*) = \frac{1}{13}(i_0 + i_1 + i_2), \tag{15}$$

Combined with (2) this implies

$$i_2 \le \frac{9}{4}(i_0 + i_1) - \frac{13}{4}. \tag{16}$$

If $\vartheta = 1$, then

$$i_1 = 3i_0 - 4\alpha - 1, \tag{17}$$

where α is an integer, hence by (16)

$$i_2 \le \frac{9}{4}(4i_0 - 4\alpha - 1) - \frac{13}{4} = 9i_0 - 9\alpha - \frac{22}{4}.$$

Suppose $i_2 = 9i_0 - 9\alpha - 6$. Then (15), (17) imply

$$m(p_0) \ge \frac{1}{13}(i_0 + i_1 + i_2) = i_0 - \alpha - \frac{7}{13}$$

and (2)(17) imply

$$m(p_0) \le i_0 + i_1 - \frac{i_2}{3} - \frac{4}{3} = i_0 - \alpha - \frac{1}{3}$$

but this is impossible since $m(p_0)$ is an integer. Hence $i_2 \le 9i_0 - 9\alpha - 7$. By (17)

$$i_2 \le 9i_0 - 9\alpha - 7 = \frac{9}{4}(i_0 + i_1) - \frac{13}{4} - \vartheta - \frac{1}{2} < \frac{9}{4}(i_0 + i_1) - \frac{13}{4} - \vartheta. \tag{18}$$

If $\vartheta = 2$, then

$$i_1 = 3i_0 - 4\alpha - 2, \tag{19}$$

Hence by (14)

$$i_2 \le 9i_0 - 9\alpha - \frac{31}{4}.$$

Suppose $i_2 = 9i_0 - 9\alpha - 8 - \theta$, where $\theta = 0$ or 1. Then (15) and (19) imply

$$m(p_0) \ge i_0 - \alpha - \frac{10 + \theta}{13}.$$

and (2) and (19) imply

$$m(p_0) \leq i_0 - \alpha - \frac{2-\theta}{3}$$

but this is impossible since $m(p_0)$ is an integer. Hence

$$i_2 \leq 9i_0 - 9\alpha - 10 < \frac{9}{4}(i_0 + i_1) - \frac{13}{4} - \vartheta. \tag{20}$$

By (5) and $m(p_1) \geq 0$ we have

$$i_2 \leq i_0 + 4i_1 - 6 \tag{21}$$

By (6) there exists a point $p \in l^* \setminus \{p_1\}$ such that

$$m(p) \geq \frac{1}{3}\big(i_0 + i_1 - m(p_1)\big) \geq \frac{1}{3}i_2 - i_1 + 2.$$

Hence $i_0 + i_1 - 1 \geq m(\overline{pp^*}) \geq i_0 + \Big(\frac{1}{3}i_2 - i_1 + 2\Big)$, i.e.

$$i_2 \leq 6i_1 - 9,$$

Combined with (14), (16), (18), (20), (21) this proves

$$i_2 \leq \min\Big(\frac{(9(i_0+i_1) - 13}{4} - \vartheta, \frac{3i_0 - 9}{2} + 3i_1 - \vartheta_1, 6i_1 - 9, i_2 + 4i_1 - 6\Big)$$

From

$$\begin{array}{rcl} i_0 + i_1 + i_2 - 1 & \geq & m(P_1) \geq m(l^*) + m(p^*) \\ & = & 2i_0 + i_1 \end{array}$$

we get

$$i_2 \geq i_0 + 1.$$

■

III Constructions

For the constructions, we consider four main cases, some with a number of subcases. The main cases are determined by which of the following four numbers are maximal:

$$2i_0 + 1,\ i_0 + i_1 + 1,\ i_0 - 3i_1 + i_2 + 6,\ i_0 + (i_2 + 4)/3.$$

More precisely, we consider the following four mutually exclusive cases:

- Case I.

$$\begin{array}{rcl} i_0 + (i_2+4)/3 & > & \max\big(2i_0 + 1, i_0 + i_1 + 1\big), \\ i_0 + (i_2+4)/3 & \geq & i_0 - 3i_1 + i_2 + 6. \end{array}$$

- Case II.

$$i_0 - 3i_1 + i_2 + 6 > \max\big(2i_0 + 1, i_0 + i_1 + 1, i_0 + (i_2+4)/3\big).$$

- Case III.

$$i_0 + i_1 + 1 \geq \max\big(2i_0 + 1, i_0 - 3i_1 + i_2 + 6, i_0 + (i_2+4)/3\big).$$

- Case IV.

$$\begin{aligned} 2i_0+1 &\geq \max\bigl(i_0-3i_1+i_2+6, i_0+(i_2+4)/3\bigr), \\ 2i_0+1 &> i_0+i_1+1. \end{aligned}$$

A given (i_0, i_1, i_2) satisfying Theorem 1 i) and ii) belongs to one of the cases above. For this case we construct a value assignment m such that $i_3 = \min\bigl(12i_1 - i_2 - 13, 3i_2 - 4\bigr)$. In each case, value assignments for other i_3 satisfying Theorem 1 iii) are obtained by reducing the values of the points, except one point of maximal value, one line of maximal value and one plane of maximal value.

The space PG(3,3) contains 40 points, 130 lines and 40 planes. To each point $\bar{b} = (b_0, b_1, b_2, b_3) \in PG(3,3)$, where the first nonzero b_j from right to left is 1, we associate the integer

$$\beta = \beta_{\bar{b}} = b_0 + 3b_1 + 9b_2 + 27b_3.$$

It is illustrated in Fig. 1, where some of the points and lines are included.

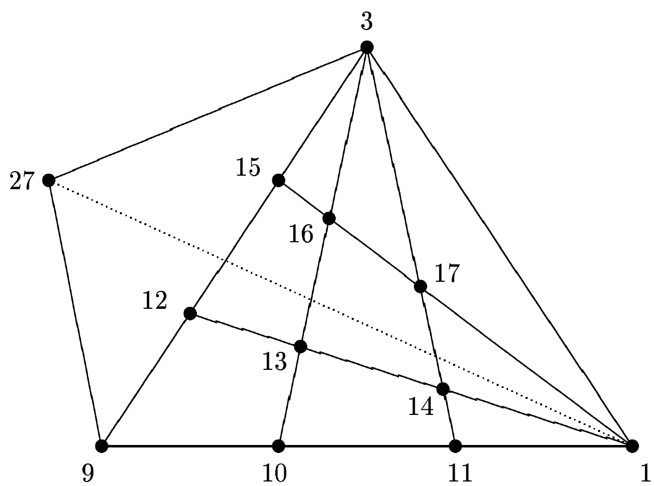

Figure 1: Representation of $PG(3,3)$

Case I

The conditions of Case I and Theorem 1 i) and ii) easily imply the following relations.

$$i_1 \leq 3i_0 - 4, \tag{22}$$

$$i_2 \geq 3i_0, \tag{23}$$

$$i_2 \geq 3i_1, \tag{24}$$

$$i_2 \leq 9i_1/2 - 7, \tag{25}$$

$$i_2 \leq (9(i_0+i_1) - 13)/4 - \vartheta. \tag{26}$$

By (22) we have

$$i_0 \geq 2, \tag{27}$$

By (24) we have

$$12i_1 - i_2 - 13 < 3i_2 - 4.$$

Let

$$\theta \equiv -(i_2+4) \pmod 3 \quad \text{where} \quad \theta \in \{0,1,2\},$$
$$\delta \equiv 1-(i_2+4+\theta)/3 \pmod 3 \quad \text{where} \quad \delta \in \{0,1,2\},$$
$$\delta_1 \equiv i_0-(i_2+1+\theta)/3 \pmod 2 \quad \text{where} \quad \delta_1 \in \{0,1\}.$$

Define $\varepsilon_1(p)$ by

$$\varepsilon_1(p) = -1 \quad \text{if } p=4 \text{ and } \delta_1 = 1,$$
$$\varepsilon_1(p) = 0 \qquad \text{otherwise.}$$

Define $\varepsilon(p)$ and $m(p)$ by the following tables.

$\varepsilon(p) = \varepsilon(\theta, \delta, p)$

δ	11,15,30,33,39,42	14	17	27,36	29,38	32,41,49	35,44,52
0	0	1	0	0	−1	0	−1
1	0	0	0	−1	−1	−1	−1
2	−1	0	0	0	−2	−1	−1

θ	δ	9	12	45	46	48	51
0	0	0	0	0	0	0	0
	1	−1	0	−1	0	0	0
	2	0	−1	−2	−1	0	0
1	0	0	−1	0	0	0	0
	1	−1	−1	−1	0	0	0
	2	−1	−1	−1	−1	0	−1
2	0	0	−1	0	−1	−1	0
	1	−1	−1	−1	−1	−1	0
	2	−1	−1	−1	−2	−1	−1

For the points $p \in \{10,13,16,28,31,34,37,40,43,47,50,53\}$ we define

$$\varepsilon(p) = \varepsilon(p_1) + \varepsilon(p_2),$$

where p_1, p_2 are the point colinear with 1 and p. For example, $\varepsilon(53) = \varepsilon(51) + \varepsilon(52)$ (we use the notation p_1, p_2 with the same meaning also in the proof below).

$m(p) =$	for
i_0	$p = 1$
$i_0 + i_1 - \frac{1}{3}(i_2+4+\theta)$	$p = 3$
$\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta) - 1 + \delta\Big) + \varepsilon(p)$	$p \in \{9,11,12,14,15,17, 27,29,30,32,33,35, 36,38,39,41,42,44, 45,46,48,49,51,52\}$
$\frac{1}{2}(\frac{1}{3}(i_2+1+\theta) - i_0 + \delta_1) + \varepsilon_1(p)$	$p \in \{4,5\}$
$i_1 - \frac{2}{3}\Big(\frac{1}{3}(i_2+4+\theta) - 1 + \delta\Big) - 1 - \varepsilon(p)$	otherwise

The corresponding DS is the ENDS

$$(i_0, i_1, i_2, 12i_1 - i_2 - 13).$$

Proof: First consider the points. By (24), (26), (27) we have

$$i_0 - 1 \geq m(3) \geq \frac{1}{4}(i_0+i_1) + \frac{13}{12} - \frac{4+\theta}{3} > -1.$$

For $m(p) = \frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta) - 1 + \delta\Big) + \varepsilon(p)$ by (23), (27) we have

$$\begin{aligned} m(p) &= \frac{1}{9}(i_2+4+\theta) - \frac{1}{3} + \Big(\frac{\delta}{3} + \varepsilon(p)\Big) \\ &\geq \frac{10}{9} - \frac{1}{3} + \Big(\frac{\delta}{3} + \varepsilon(p)\Big) \geq \frac{7}{9} - \frac{4}{3} > -1, \end{aligned}$$

and by (22), (26) we have

$$\begin{aligned} m(p) &\le \frac{1}{9}\Big(\frac{9}{4}(i_0+i_1)-\frac{13}{4}+4+\theta\Big)-\frac{1}{3}+\Big(\frac{\delta}{3}+\varepsilon(p)\Big) \\ &\le \frac{1}{4}(i_0+3i_0-4)+\frac{1}{12}+\frac{\theta}{9}-\frac{1}{3}+1=i_0-\frac{1}{36}<i_0. \end{aligned} \tag{28}$$

For these point p, except point 14 and $\delta=0$, we have $\varepsilon(p)\le 0$, Hence by (26) we have

$$\begin{aligned} m(p)-m(3) &= \frac{4}{9}(i_2+4+\theta)-\frac{1}{3}+\Big(\frac{\delta}{3}+\varepsilon(p)\Big)-(i_0+i_1) \\ &\le \frac{4}{9}\theta+\frac{\delta}{3}+\varepsilon(p)-\frac{4}{9}\vartheta\le\frac{4}{9}\theta+\frac{\delta}{3}-\frac{4}{9}\vartheta \end{aligned} \tag{29}$$

If $\vartheta=0$, then $3i_0-i_1\equiv 0$ or $3 \pmod 4$ If $3i_0-i_1\equiv 3 \pmod 4$, we have $i_1=3i_0-4\alpha-3$, By (23) we get $i_2\le 9i_0-9\alpha-10$, If $i_2=9i_0-9\alpha-10$, we get $i_2\equiv 8 \pmod 9$, $\theta=0$, Hence

$$\frac{4}{9}\theta+\frac{\delta}{3}-\frac{4}{9}\vartheta<1 \tag{30}$$

If $i_2=9i_0-9\alpha-11$, we get $i_2\equiv 7 \pmod 9$, $\theta=1$, $\delta=0$, (30) is true. If $i_2\le 9i_0-9\alpha-12$, then since i_2 decreases by 2,

$$m(p)-m(3)\le\frac{4}{9}\theta-\frac{8}{9}+\frac{\delta}{3}-\frac{4}{9}\vartheta<1.$$

Similarly, for $3i_0-i_1\equiv 0$ we have $m(p)-m(3)<1$. If $\vartheta>0$, then by (18) we also have $m(p)-m(3)\le\frac{4}{9}\theta+\frac{\delta}{3}-\frac{4}{9}(1+\frac{1}{2})<1$. Therefore we have

$$m(p)\le m(3) \tag{31}$$

For $m(p)=i_1-\frac{2}{3}\big(\frac{1}{3}(i_2+4+\theta)-1+\delta\big)-\varepsilon(p_1)-\varepsilon(p_2)-1$ by (22) we have

$$\begin{aligned} m(p) &\ge i_1-\frac{2}{3}\Big(\frac{1}{3}\Big(\frac{9}{2}i_1-7+4+\theta\Big)-1+\delta\Big)-\varepsilon(p_1)-\varepsilon(p_2)-1 \\ &= \frac{1}{3}-\frac{2}{9}(\theta+3\delta)-\varepsilon(p_1)-\varepsilon(p_2) \end{aligned} \tag{32}$$

If $\theta=\delta=0$, then $-\varepsilon(p_1)-\varepsilon(p_2)\ge -1$,

$$i_2\equiv 8 \quad (mod \quad 9), \tag{33}$$

Combined with (25), (33) implies

$$i_2\le\frac{9}{2}i_1-10.$$

Since i_2 decreases by 3, similar to (32) we get

$$m(p)\ge\frac{1}{3}+\frac{2}{9}\cdot 3-1=0.$$

For the remained cases, if $\delta\ne 2$, then $-\varepsilon(p_1)-\varepsilon(p_2)\ge 0$, hence by (32)

$$m(p)\ge\frac{1}{3}-\frac{2}{9}(2+3)=-\frac{7}{9}>-1;$$

if $\delta=2$, then $-\varepsilon(p_1)-\varepsilon(p_2)\ge 1$, by (32)

$$m(p)\ge\frac{1}{3}-\frac{2}{9}(2+6)+1=-\frac{4}{9}>-1.$$

By (24) we have

$$\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta)-1+\delta\Big)-m(p) \;=\; \frac{1}{3}i_2-i_1+\frac{1}{3}(4+\theta)+\delta+\varepsilon(p_1)+\varepsilon(p_2)$$
$$\geq \frac{1}{3}(4+\theta)+\delta+\varepsilon(p_1)+\varepsilon(p_2), \tag{34}$$

Hence from the following table:

θ	0			1			2		
δ	0	1	2	0	1	2	0	1	2
$\frac{1}{3}(4+\theta)+\delta+\varepsilon(p_1)+\varepsilon(p_2)\geq$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{3}$	1	1	1

and since $\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta)-1+\delta\Big)$ is an integer, we get

$$m(p)\leq\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta)-1+\delta\Big)-1. \tag{35}$$

Similar to (28) we have

$$\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta)-1+\delta\Big)\leq\frac{1}{4}(4i_0-4)+\frac{\theta}{9}-\frac{1}{4}+\frac{\delta}{3}\leq i_0-\frac{13}{36}, \tag{36}$$

Hence

$$m(p)\leq i_0-2. \tag{37}$$

For points 4 and 5 by (23) we have

$$m(p)\geq\frac{1}{2}\Big(\frac{1}{3}(1+\theta)+\delta_1\Big)+\varepsilon_1(p)\geq\frac{1}{6}-1>-1$$

and by (22), (26) we get

$$\begin{aligned}
&\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta)-1+\delta\Big)-m(p)\\
&= \frac{1}{2}i_0-\frac{1}{18}i_2+\frac{1}{9}(4+\theta)-\frac{1}{3}+\frac{1}{3}\delta-\frac{1}{6}-\frac{1}{6}\theta-\frac{1}{2}\delta_1-\varepsilon_1(p)\\
&\geq \frac{1}{2}i_0-\frac{1}{18}\Big(\frac{9}{4}(i_0+3i_0-4)-\frac{13}{4}\Big)-\frac{1}{18}(1+\theta)+\frac{1}{3}\delta-\frac{1}{2}\delta_1\\
&= \frac{1}{2}+\frac{13}{72}-\frac{1}{18}(1+\theta)+\frac{1}{3}\delta-\frac{1}{2}\delta_1\geq\frac{1}{72}.
\end{aligned}$$

Hence by (36) we get

$$m(p)\leq\frac{1}{3}\Big(\frac{1}{3}(i_2+4+\theta)-1+\delta\Big)-1\leq i_0-2. \tag{38}$$

Next consider the lines. For each line l containing the point 1 we have

$$m(l)=i_0+i_1-1. \tag{39}$$

For any line l we denote the four points on l by $p_1[l],p_2[l],p_3[l],p_4[l]$ where $p_4[l]$ is the point on l with maximal value. For the lines containing point 3 we have the following table (where $\overline{p_1,p_2}$ denotes the line determined by p_1 and p_2):

For the remaining four lines containing point 3, by (35) we have

$$m(l)\leq m(3)+\frac{1}{3}(i_2+4+\theta)-1+\delta-3\leq i_0+i_1-2. \tag{40}$$

Table I: The values of lines.

$m(l) =$	for
$i_0 + i_1$	$l = \overline{11, 17}$
$i_0 + i_1 - 1$	$l \in \{\overline{27, 33}, \overline{36, 42}\}$
$i_0 + i_1 - 1 - \theta(\theta - 1)/2$	$l = \overline{45, 51}$
$i_0 + i_1 - 1 - \operatorname{sign}\theta$	$l = \overline{9, 15}$
$i_0 + i_1 - 2 - \theta(\theta - 1)/2$	$l = \overline{46, 52}$
$i_0 + i_1 - 3$	$l \in \{\overline{38, 44}, \overline{29, 35}\}$

For the remaining nine lines in the plane $\widehat{3, 9, 27}$, by (31) we have

$$\begin{aligned} m(l) &= \sum_{j=1}^{4} m(p_j[l]) \le m(3) + \frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta + \sum_{j=1}^{3} \varepsilon(p_j[l]) \\ &= i_0 + i_1 - 1 + \delta + \sum_{j=1}^{3} \varepsilon(p_j[l]). \end{aligned}$$

If $\delta = 0$, then $\sum_{j=1}^{3} \varepsilon(p_j[l]) \le 0$. For $\delta = 1$, since each l contains a point p on the line $\overline{9, 45}$ for which $\varepsilon(p) = -1$, we have $\delta + \sum_{j=1}^{3} \varepsilon(p_j[l]) \le 0$. If $\delta = 2$, it is not to hard to check that $\sum_{j=1}^{3} \varepsilon(p_j[l]) \le -2$. Hence we get

$$m(l) \le i_0 + i_1 - 1.$$

Similarly, consider the remaining nine lines in plane $\widehat{3, 11, 27}$. For $\delta = 0$ and the lines containing point 14, by (26) we have

$$\begin{aligned} m(l) &= \frac{4}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) + \sum_{j=1}^{4} \varepsilon(p_j[l]) \\ &\le i_0 + i_1 - 1 + \frac{4}{9}\theta + \frac{4}{3}\delta + \sum_{j=1}^{4} \varepsilon(p_j[l]) \\ &\le i_0 + i_1 - 1 + \frac{8}{9} + \sum_{j=1}^{4} \varepsilon(p_j[l]). \end{aligned}$$

Since $\varepsilon(38) = \varepsilon(44) = \varepsilon(52) = -1$, we have $\sum_{j=1}^{4} \varepsilon(p_j[l]) \le 0$. Hence $m(l) < i_0 + i_1$. For the other cases by (31) we have

$$m(l) \le i_0 + i_1 - 1 + \delta + \sum_{j=1}^{3} \varepsilon(p_j[l]).$$

It is easy to see that

$$\sum_{j=1}^{3} \varepsilon(p_j[l]) \le \begin{cases} 0, & \text{if} \quad \delta = 0, \\ -2, & \text{if} \quad \delta = 2; \end{cases}$$

For $\delta = 1$ each l contains a point p in $\{38, 41, 44\}$ for which $\varepsilon(p) = -1$, and so

$$\delta + \sum_{j=1}^{3} \varepsilon(p_j[l]) \leq 0.$$

Hence we get

$$m(l) \leq i_0 + i_1 - 1.$$

For the remaining six lines in the plane $\widehat{1,3,27}$, by (26), (35), (38) we have

$$\begin{aligned} m(l) &\leq \frac{4}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) - 2 + \sum_{j=1}^{2} \varepsilon(p_j[l]) \\ &\leq i_0 + i_1 - 3 + \frac{4}{9}\theta + \frac{4}{3}\delta + \sum_{j=1}^{2} \varepsilon(p_j[l]), \end{aligned}$$

where $p_1[l] \in \{27, 30, 33\}$, $p_2[l] \in \{29, 32, 35\}$. Since

$$\sum_{j=1}^{2} \varepsilon(p_j[l]) \leq \begin{cases} 0, & \text{if} \quad \delta = 0, \\ -1, & \text{if} \quad \delta = 0, \end{cases}$$

we get

$$m(l) \leq i_0 + i_1 - \frac{4}{9} < i_0 + i_1.$$

For the remaining nine lines in plane $\widehat{3,10,27}$ by (26) and (35) we have

$$\begin{aligned} m(l) &\leq \frac{4}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) - 3 + \varepsilon(p) \\ &\leq i_0 + i_1 - 4 + \frac{4}{9}\theta + \frac{4}{3}\delta + \varepsilon(p) \leq i_0 + i_1 - \frac{4}{9}. \end{aligned}$$

where $p \in \{27, 30, 33\}$.

Each other line must have four joint points with $\widehat{3,9,27} \setminus \overline{3,27}$, $\widehat{3,10,27} \setminus \overline{3,27}$, $\widehat{3,11,27} \setminus \overline{3,27}$, and $\widehat{3,1,27} \setminus \{3, 27, 30, 33, 1\}$. By (35) and (38) we have the following upper bounds on $m(p) - \frac{1}{3}(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta)$:

	$m(p) - \frac{1}{3}(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta) \leq$			
p *in*	$\widehat{3,9,27}$ $\setminus\overline{3,27}$	$\widehat{3,10,27}$ $\setminus\overline{3,27}$	$\widehat{3,11,27}$ $\setminus\overline{3,27}$	$\widehat{3,1,27}$ $\setminus\{\overline{3,27} \cup \{1\}\}$
$\delta = 0$	0	-1	1	0
$\delta = 1$	0	-1	0	-1
$\delta = 2$	0	-1	0	-1

Let $p_4 \in \widehat{3,9,27} \setminus \overline{3,27}$, $l = \{p_1, p_2, p_3, p_4\}$. By (31) we have $m(p_4) \leq m(3)$. Hence we get

$$\begin{aligned} m(l) &= \sum_{j=1}^{4} m(p_j) \leq m(3) + \frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta \\ &\quad + \sum_{j=1}^{3} \Big(m(p_j) - \frac{1}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big)\Big) \\ &\leq i_0 + i_1 - 1. \end{aligned}$$

Finally, consider the planes. For each plane containing point 1 by (24) and (39) we have

$$m(P) = i_0 + 4(i_1 - 1) \leq i_0 + i_1 + i_2 - 4.$$

For the remaining nine planes containing point 3 we have

$$m(P) = m(3) + \sum_{j=1}^{4}(m(l_j[P]) - m(3)) = \sum_{j=1}^{4} m(l_j[P]) - 3m(3),$$

where the point $3 \in l_j[P]$ and $\bigcup_{j=1}^{4} l_j[P] = P$. Hence, by Table 1, we get

$$m(\widehat{3,9,27}) = 4(i_0 + i_1) - 3m(3) - 4 - \theta(\theta - 1)/2 - \operatorname{sign}\theta = i_0 + i_1 + i_2.$$

From (34) we have

$$\begin{aligned} m(\overline{3,28}) &\leq m(3) + \frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta - (4 + \theta) - 3\delta \\ &\quad -\varepsilon(27) - \varepsilon(29) - \varepsilon(30) - \varepsilon(32) - \varepsilon(33) - \varepsilon(35) \\ &\leq i_0 + i_1 - 3. \end{aligned}$$

Similarly we get

$$\begin{aligned} m(\overline{3,47}) &\leq i_0 + i_1 - 5 - 2\delta - \theta - \varepsilon(45) - \varepsilon(46) - \varepsilon(48) - \varepsilon(49) - \varepsilon(51) - \varepsilon(52) \\ &\leq i_0 + i_1 - 4. \end{aligned}$$

Hence, by Table 1, (40) we get:

$m(P) - (4(i_0 + i_1) - 3m(3)) \leq$	$P =$
-9	$\widehat{3,10,27}$
$-6 - \theta(\theta - 1)/2$	$\widehat{3,11,27}$
$-8 - \theta(\theta - 1)/2 - \operatorname{sign}\theta$	$\widehat{3,9,28}$
$-11 - \operatorname{sign}\theta$	$\widehat{3,9,29}$
$-9 - \theta(\theta - 1)/2$	$\widehat{3,10,28}$
$-8 - \theta(\theta - 1)/2$	$\widehat{3,10,29}$
-8	$\widehat{3,11,28}$
$-6 - \theta(\theta - 1)/2$	$\widehat{3,11,29}$

For the remaining planes, each plane contains 13 joint points with the 13 lines containing point 3. By (35) and (38) we have

$$m(P) \leq 13 \cdot \frac{1}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) - 5 + \sum_{j=1}^{8} \varepsilon(p_j[P]), \tag{41}$$

where $p_j[P] \in P$ for $1 \leq j \leq 8$ and $p_8[P] \in \{38, 41, 44\}$. Since

$$\begin{aligned} m(\widehat{3,9,27}) &= m(3) + 12 \cdot \frac{1}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) + \sum_{j=1}^{12} \varepsilon(p_j^*) \\ &= i_0 + i_1 + i_2, \end{aligned}$$

where $p_j^* \in \widehat{3,9,27}$, $1 \leq j \leq 12$, and we have the table:

θ	0			1			2		
δ	0	1	2	0	1	2	0	1	2
$\sum_{j=1}^{12} \varepsilon(p_j^*) =$	0	−4	−8	−1	−5	−9	−2	−6	−10
$\sum_{j=1}^{7} \varepsilon(p_j[P]) \leq$	1	−1	−2	1	−1	−3	1	−2	−4

so by (29), (31), (41), and $\varepsilon(\theta, 2, p_8[P]) \leq -1$, we get

$$\begin{aligned} m(P) &\leq m(p_8[P]) + 12 \cdot \frac{1}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) - 5 + \sum_{j=1}^{7} \varepsilon(p_j[P]) \\ &\leq m(3) + \varepsilon(\theta, 2, p_8[P]) + 12 \cdot \frac{1}{3}\Big(\frac{1}{3}(i_2 + 4 + \theta) - 1 + \delta\Big) - 5 + \sum_{j=1}^{7} \varepsilon(p_j[P]) \\ &\leq i_0 + i_1 + i_2, \end{aligned}$$

By (39) we have

$$m(V_3) = i_0 + 13(i_0 + i_1 - 1 - i_0) = i_0 + i_1 + i_2 + (12i_1 - i_2 - 13).$$

Hence $i_3 = 12i_1 - i_2 - 13$. ■

Case II

The conditions of Case II and Theorem 1 i) and ii) easily imply the following relations.

$$i_2 > i_0 + 3i_1 - 5, \tag{42}$$

$$i_2 > 4i_1 - 5, \tag{43}$$

$$i_2 > \frac{9}{2}i_1 - 7, \tag{44}$$

$$i_2 \leq i_0 + 4i_1 - 6, \tag{45}$$

$$i_2 \leq \frac{3}{2}i_0 + 3i_1 - \frac{9}{2} - \vartheta_1, \tag{46}$$

$$i_2 \leq \frac{9}{4}(i_0 + i_1) - \frac{13}{4} - \vartheta, \tag{47}$$

$$i_2 \leq 6i_1 - 9, \tag{48}$$

$$i_1 \leq 3i_0 - 4. \tag{49}$$

By (44) and (47) we have

$$i_1 \leq i_0 + 1. \tag{50}$$

By (42) and (48) we have

$$i_1 > \frac{1}{3}i_0 + \frac{4}{3}. \tag{51}$$

By (50) we get $i_0 \geq 2$. If $i_0 = 2$ then by (49) and (51) we have $2 \geq i_1 > 2$, but this is impossible. Hence

$$i_0 \geq 3. \tag{52}$$

From (44), (45) we get

$$i_1 \geq 3. \tag{53}$$

Let

$$\delta \equiv -i_2 \pmod 3, \qquad \delta \in \{0,1,2\},$$
$$\delta_1 \equiv i_0 - i_1 - i_2 + 1 \pmod 2, \quad \delta_1 \in \{0,1\}.$$

$\varepsilon_1(\delta_1, p)$ is defined as in Case I. Define $\varepsilon(p)$, $m(p)$ by the following tables.

$\varepsilon(p) = \varepsilon(\delta, p)$									
							$i_2 =$ $6i_1 - 9 - \delta$	$i_2 \le$ $6i_1 - 12$	
δ	9 11 33,35 42,44	15 17	12 14,30 32,39 41,51	48	27 29 36 38	45	46 49, 52	46	49 52
0	0	0	0	0	−1	−1	0	0	1
1	−1	0	0	−2	−1	0	0	1	0
2	−1	0	−1	−1	−1	−1	0	0	0

$m(p) =$	for
i_0	$p = 1$
$i_0 + 4i_1 - i_2 - 6$	$p = 3$
$\frac{1}{3}(i_2 + \delta) - i_1 + 2 + \varepsilon(p)$	$p \in \{11, 14, 17, 27, 30, 33, 36, 39, 42, 45, 48, 51\}$
$2i_1 - \frac{1}{3}(i_2 + \delta) - 3 - \varepsilon(p)$	$p \in \{9, 12, 15, 29, 32, 35, 38, 41, 44, 46, 49, 52\}$
$\frac{1}{2}(i_1 - i_0 + i_2 - 1 + \delta_1) - 2i_1 + 3 + \varepsilon_1(p)$	$p \in \{4, 5\}$
2	$p = 50$ if $\delta = 1$
1	if $i_2 = 6i_1 - 9 - \delta$, then $p = 47$ if $\delta = 0$, $p \in \{47, 50, 53\}$ if $\delta = 2$; if $i_2 \le 6i_1 - 12$, then $p = 47$ if $\delta = 1$, $p \in \{47, 50, 53\}$ if $\delta \in \{0, 2\}$
0	otherwise

The corresponding DS is ENDS $(i_0, i_1, i_2, 12i_1 - i_2 - 13)$.

Case III

The conditions of Case III and Theorem 1 i) and ii) easily imply the following relations.

$$i_1 \ge i_0 \ge 2, \tag{54}$$

$$i_2 \le 3i_1 - 1, \tag{55}$$

$$i_2 \le 4i_1 - 5. \tag{56}$$

Let

$$i_3 = \min(12i_1 - i_2 - 13, 3i_2 - 4). \tag{57}$$

We start with two special subcase. Next we give one of the main constructions.

Subcase IIIa

For this subcase, $i_2 = 3i_1 - \theta$ where $\theta \in \{1, 2\}$. We give the following construction in this case. First, $\varepsilon_1(p) = 1$ if $p = 4$ and $\delta_1 = 1$; $\varepsilon_1(p) = 0$ in all other cases. Define $\varepsilon(p)$ and $m(p)$ by following tables.

$\varepsilon(p) = \varepsilon(\delta, p)$								
δ	27, 31, 34 36, 40, 43 45, 49, 52	13	9, 28 32, 35, 37, 41 44, 46, 50, 53	14	10	11	16	15
0	δ	1	1	0	1	-1	$3-\theta$	$\theta-2$
1	δ	2	1	0	1	0	$3-\theta$	$\theta-1$
2	δ	2	1	1	2	0	$3-\theta$	θ

For the other points $\varepsilon(p) = 0$.

p	$m(p)$
1	i_0
3	$i_0 - 1$
4,5	$\frac{1}{2}(i_1 - i_0 + \delta_1) - \varepsilon_1(p)$
otherwise	$\frac{1}{3}(i_1 + \delta) - \varepsilon(p)$

The corresponding DS is ENDS $(i_0, i_1, i_2, 12i_1 - i_2 - 13)$.

Subcase IIIb

For $(i_0, i_1, i_2) = (3, 3, 5)$, a value assignment is given by the following table:

p	$m(p)$
11	3
1,3	2
4,5,15,27,29,30,33,36,39,40,42,45,47,48,49	1
otherwise	0

Subcase IIIc

To give the precise characterization of this subcase, we need a number of definitions. First we will give an informal description of the idea behind the construction. We start with a construction for $i_2 = 3i_1 - 3$ which is not too hard to find. The construction has one point of maximal value (namely 11), one line of maximal value $(\overline{1,3})$, and one plane of maximal value $(\widehat{3,9,27})$. Next, we reduce the value of i_2 by one repeatedly. In each step the construction is modified. The value of 11 and $\overline{1,3}$ is not changed. The value of each plane is reduced by at least one (and $\widehat{3,9,27}$ by exactly one) by reducing the values of each of the four points on a suitable line by one; this is obtained by increasing $\varepsilon(u,p)$ as explained below. The process is repeated until $i_2 = L_1$ (defined below) at which point $m(14) = m(17) = 0$. The process is further repeated, now by increasing $\zeta(v,p)$ by one for the four points on a suitable line in each step. This goes on until $i_2 = L_1 - L_2$ (defined below) at which point also $m(10) = m(13) = m(16) = 0$. For $i_2 < L_1 - L_2$ we use a new construction (Case IIId). We can show that if $i_2 = L_1 - L_2$, then $i_2 \leq i_0 + i_1 + \delta - 2$. Hence for $i_2 < L_1 - L_2$ the conditions for CASE IIId applies.

Let

$$\delta_1 \equiv i_0 - i_1 \pmod 2, \quad \delta_1 \in \{0, 1\}.$$

Define $\varepsilon_1(p)$ by the following table.

$\varepsilon_1(p) = \varepsilon_1(\delta_1, p)$		
	14	17
$\delta_1 = 0$	0	0
$\delta_1 = 1$, $i_1 \neq 4$,	0	1
$\delta_1 = 1$, $i_1 = 4$,	1	0

Further, for $p \in \{14, 17\}$, let

$$\beta_p = \frac{1}{2}(i_1 - i_0 + \delta_1) - \varepsilon_1(p),$$

$$b_p = \beta_p - 4\left\lfloor \frac{\min\{\beta_{14}, \beta_{17}\}}{4} \right\rfloor,$$

$$a = \begin{cases} 16, & \text{if} \quad b_{14} = b_{17} = 0 \text{ and } \min\{\beta_{14}, \beta_{17}\} \geq 4, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$L_1 = 3i_1 - 3 - 24\left\lfloor \frac{i_1 - i_0 - \delta_1}{8} \right\rfloor + a - b_{14} - b_{17}. \tag{58}$$

Define $K, c, u, d, v, \delta, t, \eta$ by

$$\begin{array}{lll} K = \max(i_2, L_1), & & \\ 3i_1 - 3 - K = 24c + u, & \text{where} & 0 \leq u \leq 23, \\ K - i_2 = 24d + v, & \text{where} & 0 \leq v \leq 23, \\ \delta \equiv -i_1 \pmod 3 & \text{where} & \delta \in \{0, 1, 2\}, \end{array}$$

$$t = \begin{cases} 1, & c \equiv 2 \pmod 3, \\ 0, & \text{otherwise,} \end{cases}$$
$$\eta = \begin{cases} 1, & c \not\equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

We now define $\varepsilon(\omega, p) = \varepsilon(\delta, \omega, p)$ and $\zeta(p) = \zeta(\sigma, p)$. We have $\zeta(0, p) = 0$ for all p. Further, $\varepsilon(\delta, 0, p)$ is defined by the following table.

	$\varepsilon(\delta, 0, p)$		
points	$\delta \in \{0, 1\}$	$\delta = 2$, $i_1 = 4$	$\delta = 2$, $i_1 \neq 4$
1	$\delta - 1$	0	0
4	0	1	1
9, 13	1	2	2
10	$t + 1$	$t + 1$	$t + 1$
12, 28, 29, 32, 35, 37, 38			
41, 44, 46, 47, 50, 53	1	1	1
15	δ	1	1
16	$\eta + \delta$	$\eta + 1$	$\eta + 1$
27, 31 34, 40, 43, 45, 49, 52	δ	2	2
36	δ	1	2
42	0	1	0
other points	0	0	0

For a given $\omega > 0$, $\varepsilon(\delta, \omega, p) - \varepsilon(\delta, \omega - 1, p)$ does not depend on δ. For convenience, we will omit δ from the notation and write just $\varepsilon(\omega, p)$. Further, for any given ω, $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p)$ for all points except four, which four will depend on ω and they are given in the tables below. In the same way, for $\sigma > 0$ we have $\zeta(\sigma, p) = \zeta(\sigma - 1, p)$ except in four points. We consider the following condition which may or may not be true:

$$i_1 \neq 4. \tag{59}$$

$$c = 0, \; d = 0, \; v = 4. \tag{60}$$

$\varepsilon(p) = \varepsilon(\omega, p)$					
ω	conditions	$\varepsilon(\omega,p) - \varepsilon(\omega-1,p) = 1$ iff for the following p			
1	(59), (60)	14,	29,	40,	51
1	(59), not (60)	14,	35,	37,	48
1	not (59),	17,	32,	37,	51
2	(59)	17,	47,	34,	39
3	(59), (60)	14,	35,	37,	48
3	(59), not (60)	14,	29,	40,	51
4	(59)	17,	38,	52,	30
5	(59)	14,	32,	43,	45
6	(59)	17,	50,	28,	42
7	(59)	14,	44,	46,	30
8	(59)	17,	53,	31,	36

	(mod 3)			$\varepsilon(\omega,p) - \varepsilon(\omega-1,p) = 1$ and $\zeta(\sigma,p) - \zeta(\sigma-1,p) = 1$ for the following p			
	$c \equiv 0$	$c \equiv 1$	$c \equiv 2$				
ω	σ	σ	σ				
9	9, 18	2, 11	1, 10	13,	29,	52,	39
10	1, 10	3, 12	2, 11	16,	44,	28,	48
11	2, 11	4, 13	3, 12	10,	53,	43,	33*
12	3, 12	5, 14	4, 13	13,	32,	46,	42
13	4, 13	6, 15	5, 14	16,	38,	31,	51
14	5, 14	7, 16	6, 15	10,	47,	37,	27
15	6, 15	8, 17	7, 16	13,	35,	49,	36
16	7, 16	9, 18	8, 17	16,	41,	34,	45
17	8, 17	1, 10	9, 18	10,	50,	40,	30
18	19	19	19	15,	40,	46,	34
19	20	20	20	15,	31,	37,	52
20	21	21	21	9,	50,	41,	32
21	22	22	22	9,	53,	44,	35
22	23	23	23	12,	41,	53,	29
23				12,	49,	34,	37

* if $i_1 = 4$ and $\sigma = 2$, then this row is changed to 10, 29, 46, 36.

For $p \in \{10, 13, 16\}$, let

$$\beta_p = \frac{1}{3}(i_1 + \delta) - \left\lfloor \frac{10c}{3} \right\rfloor - \varepsilon(u, p)$$

and

$$b_p = \beta_p - 6\left\lfloor \frac{\min\{\beta_{10}, \beta_{13}, \beta_{16}\}}{6} \right\rfloor,$$

$$e = \begin{cases} 6, & \text{if} \quad b_p = 0 \text{ and } \beta_p \geq 6 \text{ for } p = 10, 13, 16, \\ 0, & \text{otherwise,} \end{cases}$$

Let

$$L_2 = 24\left\lfloor \frac{\min\{\beta_{10}, \beta_{13}, \beta_{16}\}}{6} \right\rfloor - e + b_{10} + b_{13} + b_{16}. \tag{61}$$

Subcase IIIc is determined by

$$3i_1 - 3 \geq i_2 \geq L_1 - L_2$$

and $(i_0, i_1, i_2) \neq (3, 4, 4)$. In this subcase $m(p)$ is defined by the following table.

p	$m(p)$
11	i_0
3	$i_0 - 1$
1, 4, 5	$\frac{1}{3}(i_1+\delta) - \varepsilon(u,p)$
14, 17	$\frac{1}{2}(i_1 - i_0 + \delta_1) - 4c - \varepsilon_1(p) - \varepsilon(u,p)$
10, 13, 16	$\beta_p - 6d - \zeta(v,p)$
9, 12, 15, 27, 30, 33 36, 39, 42, 45, 48, 51	$\frac{1}{3}(i_1+\delta) - 2c - \varepsilon(u,p) - 2d - \zeta(v,p)$
otherwise	$\frac{1}{3}(i_1+\delta) - 3c - \varepsilon(u,p) - 3d - \zeta(v,p)$

The corresponding DS is ENDS $(i_0, i_1, i_2, 3i_2 - 4)$.

Proof (for Subcase IIIc). For $i_2 = K$ and $u = 0$, the proof is similar to Case I, but point 9 is similar to point 3 of Case I and

$$m(\widehat{1,3,9}) = m(\widehat{1,3,45}) = m(\widehat{1,3,36}) = m(\widehat{1,3,27}) = i_0 + i_1 + i_2 - 1;$$

the details of the proof are omitted.

Consider $i_2 = K$ and $u > 0$. First we show that

$$0 \le m(p) < i_0, \quad \text{for} \quad p \ne 11.$$

Since if $1 \le u \le 23$ then

$$m(i_0, i_1, i_2, p) \le m(i_0, i_1, i_2 + 1, p)$$

and $m(i_0, i_1, i_2, p) < i_0$, for $p \ne 11$ and $u = 0$, so $m(p) < i_0$, for $p \ne 11$. We only need to prove $m(p) \ge 0$, for the smallest value of i_2, namely

$$i_2 = L_1,$$

and for $u = 23$ and $c \ge 0$.

For $u = c = 0$ by $i_1 \le 3i_0 - 4$ we have

$$\begin{aligned}
&\frac{1}{3}(i_1+\delta) - \varepsilon(p) - \Big(\frac{1}{2}(i_1 - i_0 + \delta_1) - \hat{\varepsilon}_1(17)\Big) \\
\ge\ & \frac{2}{3} - \frac{1}{2}\delta_1 + \frac{1}{3}\delta + \hat{\varepsilon}_1(17) - \varepsilon(p) \\
=\ & \frac{2}{3} + \big(\frac{1}{3}\delta - \varepsilon(p)\big) + \big(\hat{\varepsilon}_1(17) - \frac{1}{2}\delta_1\big) \\
\ge\ & \frac{2}{3} - \frac{4}{3} + 0 > -1
\end{aligned}$$

Hence

$$m(p) \ge \hat{m}(17) \triangleq \min\big(m(17), m(14)\big), \quad \text{for} \quad p \in V_3 \tag{62}$$

For $u = 0$, $c \ge 1$, by $4c > 3c > 2c$, $4c \ge \lfloor \frac{10}{3}c \rfloor + 1, t, \eta \le 1$, we also get (62).

For $u = 23$, $c \ge 0$ by definition of $\varepsilon(\omega, p)$, $1 \le \omega \le 23$, we have

$$\begin{aligned}
&\varepsilon(23,p) - \varepsilon(0,p) = 4, \quad \text{for} \quad p \in \{37, 53, 34\} \quad \text{and} \quad p \in \{14, 17\} \\
&\varepsilon(23,p) - \varepsilon(0,p) \le 3, \quad \text{for other} \quad p.
\end{aligned}$$

Hence by (62) we get $m(p) \ge \hat{m}(17)$ for all p, $u = 23$.

It is easy to see that

$$\hat{m}(17) \ge 0, \quad \text{for} \quad i_2 \ge L_1.$$

If $i_2 = L_1$, then $\hat{m}(17) = 0$, By the definition of $\varepsilon(\omega, p)$, $1 \le \omega \le 8$, we get $m(p) \ge \hat{m}(17) = 0$, for all p, $i_2 = L_1$. Therefore we have $m(p) \ge 0$ for all p, u, c.

For the lines, it is sufficient to consider $u = 0$.

For the planes, by definition of $\varepsilon(\omega, p)$ the four points which satisfy $\varepsilon(\omega, p) \ne \varepsilon(\omega - 1, p)$ are always in a line l. Since each plane has a joint point with l, the corresponding DS is ENDS $(i_0, i_1, i_2, 3i_2 - 4)$.

For $i_2 < K$, the proofs for the lines and the planes are similar to the proofs for $i_2 = K$ and are omited. We only consider the points.

For $i_2 < K$, $v = 0$, $d \ge 0$. Since if $d = v = 0$, then $i_2 = K$ and so we already proved that the corresponding DS is the ENDS $(i_0, i_1, i_2, 3i_2 - 4)$, so it is easy to get all proof for $d > 0$ and we omit it.

Consider $v > 0$, $i_2 < K$. Similar to the proof for $u > 0$, $i_2 = K$, we only need prove $m(p) \ge 0$ for the smallest value of i_2, namely $i_2 = L_1 - L_2$ for $v = 23$, $d \ge 0$. We denote all points in $\{10, 13, 16\}$ by p_2, and denote all points in $V_3 \setminus \{3, 11, 14, 17, 10, 13, 16\}$ by p_1.

For $v = d = 0$, $u = 0$ we have

$$\begin{aligned} &\min\{m(p_1) | p_1 \in V_3 \setminus \{3, 11, 14, 17, 10, 13, 16\}\} - m(13) \\ \ge\ & \lfloor \frac{c}{3} \rfloor + \varepsilon(13) - \max_{p_1} \varepsilon(p_1) \\ \ge\ & \lfloor \frac{c}{3} \rfloor \ge 0. \end{aligned} \tag{63}$$

For $i_2 = L_1$, by the definition of $\varepsilon(\omega, p)$, $0 \le \omega \le 8$, and (63) we have

$$\min_{p_1} m(p_1) - m(13) \ge -1 + \lfloor \frac{c}{3} \rfloor, \quad i_2 = L_1 \tag{64}$$

where we use

$$\begin{aligned} &\varepsilon(8, 30) - \varepsilon(0, 30) = 2, \\ &\varepsilon(8, p) - \varepsilon(0, p) \le 1, \qquad \text{for other} \quad p \notin \{14, 17\}. \end{aligned}$$

and $\varepsilon(0, 30) + 1 \le \varepsilon(0, 13)$.

For $i_2 < l_1$, $v = 0$, $d \ge 1$ by (64) and $6d > 3d + 2 > 2d + 2$, we have

$$\min_{p_1} m(p_1) > m(13) + 1, \quad v = 0. \tag{65}$$

For $v = 23$, $d \ge 0$ by the definition of $\zeta(v, p)$ we have

$$\zeta(23, p) - \zeta(0, p) = \begin{cases} 2, & \text{if} \quad p \in \widehat{3, 9, 27}, \\ 6, & \text{if} \quad p \in \{10, 13, 16\}, \\ 4, & \text{if} \quad p \in \{41, 53\}, \\ \le 3, & \text{otherwise}. \end{cases} \tag{66}$$

Hence by (64) and (65) we get

$$m(p_1) \ge m(13) \ge 0, \quad \text{for all} \quad p_1 \text{ when } v = 23,$$

where it is easy to check that

$$m(13) \ge 0, \quad \text{for} \quad i_2 \ge L_1 - L_2.$$

If $i_2 = L_1 - L_2$, then $m(p_2) = 0$. By the definition of $\zeta(v, p)$, $1 \le v \le 18$, and (65) we get

$$m(p_1) \ge m(13) = 0, \quad \text{for} \quad d \ge 1, i_2 = L_1 - L_2.$$

If $d = 0$, $i_2 = L_1 - L_2$, $v > 3a'$, $a' = 1$ or 2, then by the definition of $\zeta(v,p)$, $1 \le v \le 18$, we have

$$\max_{p_2}\big(\zeta(v,p_2) - \zeta(0,p_2)\big) - \max_{p_1}\big(\zeta(v,p_1) - \zeta(0,p_1)\big) \ge a'. \tag{67}$$

Note that

$$\max_{p_2} \varepsilon(p_2) - \min_{p_2} \varepsilon(p_2) \le 1, \quad \text{for same} \quad \delta, c'. \tag{68}$$

Hence by (64) (67) we get

$$m(p_1) \ge m(13) - 1 + a' - \big(m(p_2) - m(13)\big),$$

i.e. $m(p_1) \ge m(p_2) = 0$, for $v > 6$, $d = 0$, $i_2 = L_1 - L_2$.

If $d = 0$, $i_2 = L_1 - L_2$, $c \ge 6$ or $c \ge 3$, $v > 3$, then by (64), (67), (68) we get

$$m(p_1) \ge m(13) - 1 + \lfloor \frac{c}{3} \rfloor + a' - \big(m(p_2) - m(13)\big),$$

i.e.

$$m(p_1) \ge m(p_2) = 0. \tag{69}$$

If $d = 0$, $i_2 = L_1 - L_2$, $c < 3$, $4 \le v \le 6$, or $1 \le v \le 3$, $c < 6$, then it is complicated and we discuss as following. We have following table:

$\varepsilon(p) = \varepsilon(\delta, c', p)$ for $i_2 = L_1$							
	δ	0		1		2	
c'	p	$\varepsilon(p)$	v'	$\varepsilon(p)$	v'	$\varepsilon(p)$	v'
3	13	1	1	1	3	2	2
	16	0		1		1	
	10	1		1		1	
1	13	1	3	1	2	2	1
	16	1		2		2	
	10	1		1		1	
2	13	1	2	1	1	2	3
	16	1		2		2	
	10	2		2		2	

Table 1

where $v' \equiv v \equiv L_2 \pmod 3$ and v' takes the values in the table if $d = 0$ and $i_2 = L_1 - L_2$.

For $i_2 = L_1$ we have

$$\varepsilon(13) - \min_{p_2} \varepsilon(p_2) = \begin{cases} 1, & \text{if} \quad c' = 3, \delta = 1, 0 \quad \text{or} \quad c' = 1, \delta = 2, \\ 0, & \text{otherwise} \end{cases}$$

Hence similar to (69) we get

$$m(p_1) \ge m(p_2) = 0, \qquad \text{for} \quad d = 0, i_2 = L_1 - L_2, c \ge 3 \quad \text{or} \quad v > 3, \\ \text{and} \quad \varepsilon(13) - \min_{p_2} \varepsilon(p_2) = 0 \quad (v = 0) \tag{70}$$

For the remaining cases, since $d = 0$, $i_2 = L_1 - L_2$, we have

$$m(p_2) = \frac{1}{3}(i_1 + \delta) - \varepsilon(p_2) - \lfloor \frac{10}{3} c \rfloor - \zeta(p_2) = 0.$$

Hence

$$i_1 = 3\Big(\lfloor \frac{10}{3} c \rfloor + \varepsilon(p_2) + \zeta(p_2)\Big) - \delta, \tag{71}$$

where $\varepsilon(p_2) = \varepsilon(\delta, c', i_2 = L_1, p_2)$, $\zeta(p_2) = \zeta(v = L_2, p_2)$. On the other hand, for $i_2 = L_1$ we have

$$m(p) = \frac{1}{2}(i_1 - i_0 + \delta_1) - 4c - \varepsilon_1(p) - \varepsilon(p) = 0, \quad p \in \{14, 17\},$$

where $\varepsilon(p) \geq 0$. Hence

$$\frac{1}{2}(i_1 - i_0 + \delta_1) - \varepsilon_1(p) - 4c = \varepsilon(p) \geq 0, \quad p \in \{14, 17\} \tag{72}$$

Therefore, for the remaining cases, if we take c, v then by (71), Table 1 we get i_1; from $i_0 \leq i_1 \leq 3i_0 - 4$ we get i_0. It is easy to check that if $c > 2$ then (72) is not satisfied. Hence we only need to consider $c \leq 2$. For $i_2 = L_1$ we have $u = \frac{1}{2}(i_1 - i_0) - 4c$. In order to satisfy (72) we only need to consider smaller i_0. From the values of u, $v = L_2$, c, $\varepsilon(u,p)$, $\zeta(v,p)$ we can get the upper bounds U of $\big(\zeta(v,p_1) - \zeta(0,p_1)\big) + \big((\varepsilon(u,p_1) - \varepsilon(0,p_1)\big)$ for all p_1.
The following table show all above values for all remaining cases.

c	$v = L_2$	$\varepsilon(p_2) + \zeta(p_2)$	δ	i_1	i_0	u	U 39	U 48	U 51	U other p_1
2	2	2	0	24	8	0	1	1	1	1
	2	2	1	14	6	0	1	1	1	1
1	3	2	0	15	7	0	1	1	1	1
	4	3	2		7	1	1	2	1	1
				16	8	0		1		
	1	1	0	3	3	0	1	1	1	1
	2	2	2	4	3	1	1	2	1	1
					4	0		1		
0					3	2	2	2		
	3	2	1	5	4	1	1	2	1	1
					5	0	1	1		
					4	2			2	
	4	2	0	6	5	1	1	1	2	1
					6	0			1	

If $p_1 \neq 39, 48, 51$, then $U = 1$, by (63) we have

$$\begin{aligned} m(p_1 \mid d = 0, i_2 = L_1 - L_2) &\geq m(p_1 \mid d = v = u = 0) - U \\ &\geq m(13 \mid d = v = u = 0) - U \\ &= m(13 \mid d = 0, i_2 = L_1 - L_2) + 1 - U \\ &\geq 0 \end{aligned} \tag{73}$$

except for $v = 2$, $c = 0$ i.e. $i_1 = 4$, $i_0 = 3, 4$; and $v = 1$, $c = 0$ (Subcase IIIb).
For $i_1 = 4$, $i_0 = 3, 4$ we have

$$m(13 \mid d = v = u = 0) = m(13 \mid d = 0, i_2 = L_1 - L_2).$$

If $p_1 = 39, 48, 51$, then $U = 2$, but $\varepsilon(13) \geq \varepsilon(p_1) + 1$ for $d = v = u = 0$. Hence $m(p_1) \geq m(13) + 1$ for $d = v = u = 0$. Similar to (73) we get $m(p_1) \geq 0$, for $d = 0$, $i_2 = L_1 - L_2$.

If $i_1 = 4$, $i_0 = 4 - \tau$ where $\tau = 0, 1$, then we have the following table for $i_2 = L_1 - L_2 = 7 - \tau$:

p	$m(p)$
11 3 51 32,37	$4-\tau$ $3-\tau$ $2-\tau$ $1-\tau$
1, 5, 30, 33, 39, 48	2
4, 12, 15, 35, 38, 41, 42, 47, 50, 53	1
otherwise	0

Subcase IIId

For the following constructions we assume the conditions of Case III, and, in addition that $i_2 < L_1 - L_2$. In particular, as mentioned above, this implies that

$$i_2 \le i_0 + i_1 + \delta - 2, \tag{74}$$

where $\delta \equiv -i_1 \pmod 3$, $\delta \in \{0,1,2\}$. Our constructions will be valid under this condition (which may be weaker than $i_2 < L_1 - L_2$).

Let $\delta_2 \equiv -(i_0 + i_1 + 2) \pmod 9$, $0 \le \delta_2 \le 8$.

Define c and u by

$$i_0 + i_1 + \delta - 2 - i_2 = 18c + u, \quad \text{where} \quad 0 \le u \le 17.$$

We first define $\varepsilon_2(p) = \varepsilon_2(\delta, \delta_2, p)$ and $\varepsilon(0,p) = \varepsilon(\delta, \delta_2, 0, p)$ by the following tables.

$\varepsilon_2(p) = \varepsilon_2(\delta, \delta_2, p)$									
δ_2	30	39	48	33	42	51	27	36	45
0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0
4	1	1	1	1	0	0	0	0	0
5	1	1	1	1	1	0	0	0	0
6	1	1	1	1	1	1	0	0	0
7	1	1	1	1	1	1	1	0	0
8	1	1	1	1	1	1	1	1	0

$\varepsilon_2(p) = \varepsilon_2(\delta, \delta_2, p)$			
δ	29,38,47,35,44,53,32,41,50	28,37,46	34,43,52
0	1	1	1
1	1	1	0
2	1	0	0

$\varepsilon_2(p) = 0$ otherwise.

$\varepsilon(0,p) = \varepsilon(\delta_2, 0, p)$								
δ_2	47,35	44,53	32,41	50,28	37,46	34,43	52,31	40,49
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	1	1
3	0	0	0	0	0	1	1	1
4	0	0	0	0	1	1	1	1
5	0	0	0	1	1	1	1	1
6	0	0	1	1	1	1	1	1
7	0	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1

$\varepsilon(p) = 0$ otherwise.

Let

$$I \triangleq (\widehat{3,9,28} \cup \widehat{3,9,29}) \setminus \overline{3,9}$$

Define f, g, f_1, g_1 by

$$\begin{aligned} \tilde{m} &= 3f + g, \text{ where } g \in \{0,1,2\}, \\ \frac{1}{3}(i_1 + \delta) - 1 &= 6f_1 + g_1, \text{ where } g_1 \in \{0,1,2,3,4,5\}. \end{aligned}$$

Here

$$\tilde{m} = \frac{1}{9}(i_0 + i_1 + 2 + \delta_2) - \max\{\varepsilon_2(p) + \varepsilon(0,p) \mid p \in I\}.$$

For $p \in \{1,4,5,9,12,15\}$ we define $\varepsilon_1(p)$. We have $\varepsilon_1(1) = \delta - 1$ and $\varepsilon_1(4) = \varepsilon_1(5) = 0$ in all cases. For $p \in \{9,12,15\}$, $\varepsilon_1(p) = 0$, except possibly when $f = f_1 = c$, for this case we return to the definition in Subsubcase IIId4 below.

We consider four subsubcases of subcase IIId, depending on which is larger of c, f and f_1. The definition of $\varepsilon(u,p)$ for $u > 0$ will depend on the subsubcase. However, the definition of $m(p)$ will be the same for all four subsubcase, and we give this first.

$m(p)$	for
i_0	$p = 11$
$i_0 - 1$	$p = 3$
$\frac{1}{3}(i_1 + \delta) - \varepsilon_1(p)$	$p \in \{1,4,5\}$
0	$p \in \{10,13,14,16,17\}$
$\frac{1}{3}(i_1 + \delta) - 1 - \varepsilon_1(p) - \varepsilon(u,p) - 6c$	$p \in \{9,12,15\}$
$\frac{1}{9}(i_0 + i_1 + 2 + \delta_2) - \varepsilon_2(p)$	$p \in \{27,30,33,36,39,42$ $45,48,51\}$
$\frac{1}{9}(i_0 + i_1 + 2 + \delta_2) - \varepsilon_2(p) - \varepsilon(u,p) - 3c$	otherwise

Similar to Subcase IIIc, $\varepsilon(\omega,p) = \varepsilon(\omega - 1, p)$ except in 4 points. In the following, we will repeatedly refer to some lines when describing the four points. Therefore we find it convenient to list those lines in a table now. The lines are denoted l_i, $1 \le i \le 18$.

i	line l_i				i	line l_i			
1	15	49	28	43	2	15	40	46	34
3	15	31	37	52	4	15	50	29	44
5	15	41	47	35	6	15	32	38	53
7	12	49	34	37	8	12	40	52	28
9	12	31	43	46	10	12	50	35	38
11	12	41	53	29	12	12	32	44	47
13	9	49	40	31	14	9	52	43	34
15	9	46	37	28	16	9	50	41	32
17	9	53	44	35	18	9	47	38	29

Table 2

Subsubcase IIId1: $c < \min\{f, f_1\}$

For $1 \le \omega \le u$, we let $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p) + 1$ for the four points on the line l_ω and $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p)$ otherwise.

Subsubcase IIId2: $f > f_1 = c$

For $1 \le \omega \le u$, we let $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p) + 1$ for the four points on the line given by the following table and $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p)$ otherwise.

ω	1	2	3	4	5	6	7	8
line	l_1	l_7	l_{13}	l_2	l_8	l_{14}	l_3	l_9
ω	9	10	11	12	13	14	15	
line	l_{15}	l_4	l_{10}	l_{16}	l_5	l_{11}	l_{17}	

Subsubcase IIId3: $c = f < f_1$

For $1 \le \omega \le 6g$, we let $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p) + 1$ for $p \in l_\omega$ and $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p)$ otherwise.

For $6g + 1 \le \omega \le u$, we let $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p) + 1$ for the four points on the line given by the following table and $\varepsilon(\omega, p) = \varepsilon(\omega - 1, p)$ otherwise.

ω	6g+1	6g+2	6g+3	6g+4	6g+5
$\delta = 0$	l_{13}	l_{18}	l_{17}	l_{16}	l_{15}
$\delta = 1$	l_{14}	l_{13}	l_{18}	l_{17}	l_{16}
$\delta = 2$	l_{15}	l_{14}	l_{13}	l_{18}	l_{17}

Subsubcase IIId4: $c = f = f_1$

Let

$$\iota_0 = i_0 - 9c \quad \text{and} \quad \iota_1 = i_1 - 18c.$$

In this subsubcase there are many combinations of small values of (ι_0, ι_1) and a number of them have to be treated separately.

Let $N(\delta, \delta_2)$ be defined by the following table.

	$N(\delta, \delta_2)$								
δ_2 :	0	1	2	3	4	5	6	7	8
$\delta = 0$	1	0	5	5	4	3	3	2	1
$\delta = 1$	2	1	0	0	5	4	4	3	2
$\delta = 2$	3	2	1	1	0	5	5	4	3

We have $\frac{1}{3}(\iota_1 + \delta) - 1 = g_1 \le 5$, and so

$$\iota_1 \le 18 - \delta.$$

The treatment will depend on whether $\frac{1}{3}(\iota_1 + \delta) - 1 = g_1 > \max(3g - 1, N(\delta, \delta_2) - 1)$ or not, that is, if

$$\iota_1 > 3 \max(3g, N(\delta, \delta_2)) - \delta. \tag{75}$$

The situation when (75) is satisfied.

In this case, $g = 0$ or $g = 1$. If $g = 1$, then for $\omega \leq 6$, $\varepsilon(\omega, p)$ is defined by

$$\varepsilon(\omega,p) - \varepsilon(\omega-1,p) = 1 \text{ if and only if } p \in \begin{cases} l_\omega, & \text{for } 1 \leq \omega \leq 3, \\ l_{\omega+6}, & \text{for } 4 \leq \omega \leq 6. \end{cases}$$

For both $g = 0$ and $g = 1$, when $\omega \geq 6g + 1$, $\varepsilon(\omega,p) - \varepsilon(\omega-1,p)$ is defined in the same way as for Subsubcase IIId3, except in the following special cases: (i_0, i_1, i_2) = (5,10,7), (6,10,7), (6,10,8), (7,10,8), (7,10,9). For these cases we have $\varepsilon(\omega,p) - \varepsilon(\omega-1,p) = 1$ for the following sets of points:

ω	7	8	9
set	l_{15}	$\{12\} \cup l_{14} \setminus \{9\}$	$\{15\} \cup l_{13} \setminus \{9\}$

The situation when (75) is not satisfied.

For $p \in \{9, 12, 15\}$, we have

$$\varepsilon_1(9) = 2, \quad \varepsilon_1(12) = -1, \quad \varepsilon_1(15) = -1$$

in the following cases:

- $\iota_1 \in \{7, 10, 16, 17, 18\}$,
- $\iota_1 = 8$ and $\iota_0 \geq 4$,
- $\iota_1 = 9$ and $\iota_0 \geq 2$.

In all other cases $\varepsilon_1(9) = \varepsilon_1(12) = \varepsilon_1(15) = 0$.

Further $\varepsilon(\omega,p) - \varepsilon(\omega-1,p) = 1$ for p in the sets given by the following table. We use the notation $a \ldots b$ to denote the sequence $a, a+1, a+2, \ldots, b$ and the further notation $l_i\langle p \rangle = \{p\} \cup l_i \setminus \{9\}$.

ι_1	ι_0	ω	set	ω	set	ω	set
18	$6 \ldots 14$	$1 \ldots 13$	l_ω	14	l_{18}	15	l_{17}
17	$5 \ldots 7$	$1 \ldots 12$	l_ω	13	l_{14}		
17	$8 \ldots 13$	$1 \ldots 14$	l_ω	15	l_{18}		
16	$5 \ldots 7$	$1 \ldots 12$	l_ω	13	l_{15}		
16	8	$1 \ldots 12$	l_ω	13	l_{14}	14	l_{15}
16	$9 \ldots 13$	$1 \ldots 15$	l_ω				
15	$-2, -1$	$1 \ldots 3$	$l_{\omega+3}$	4	l_{13}	5	l_{15}
15	$7 \ldots 17$	$1 \ldots 3$	l_ω	$4 \ldots 6$	$l_{\omega+6}$	$7 \ldots 9$	$l_{\omega+9}$
		10	l_{13}	11	$l_{15}\langle 12 \rangle$	12	$l_{14}\langle 15 \rangle$
14	-2	$1 \ldots 3$	$l_{\omega+3}$	4	l_{13}	5	l_{14}
14	$7 \ldots 16$	$1 \ldots 3$	l_ω	$4 \ldots 6$	$l_{\omega+6}$	$7 \ldots 9$	$l_{\omega+9}$
		10	l_{14}	11	$l_{13}\langle 12 \rangle$	12	$l_{15}\langle 15 \rangle$
13	$-3, -2$	$1 \ldots 3$	l_ω	4	l_{18}	5	l_{17}
13	8	$1 \ldots 3$	l_ω	$4 \ldots 9$	$l_{\omega+6}$	10	l_{18}
		11	$l_{17}\langle 12 \rangle$	12	$l_{16}\langle 15 \rangle$		
13	$6 \ldots 16$, $\neq 8$	$1 \ldots 3$	l_ω	$4 \ldots 9$	$l_{\omega+6}$	10	l_{18}
		11	$l_{17}\langle 12 \rangle$	12	$l_{15}\langle 15 \rangle$		

ι_1	ι_0	ω	set	ω	set	ω	set
12	$0\ldots2$	$1\ldots3$	$l_{\omega+3}$	4	l_{13}	5	l_{15}
12	$9\ldots20$	$1\ldots3$	l_{ω}	$4\ldots6$	$l_{\omega+6}$	$7\ldots9$	$l_{\omega+9}$
11	-1	$1\ldots2$	$l_{\omega+12}$	3	l_{17}	4	$l_{18}\langle12\rangle$
11	$0,1$	$1\ldots2$ 5	$l_{\omega+4}$ l_4	3	l_{13}	4	l_{14}
11	$8\ldots19$	$1\ldots3$ 8	l_{ω} l_{14}	$4\ldots6$ 9	$l_{\omega+6}$ l_{17}	7	l_{13}
10	$-1\ldots1$	$1\ldots3$	l_{ω}	4	l_{18}	5	$l_{17}\langle12\rangle$
10	$7\ldots19$	$1\ldots3$ 8	l_{ω} $l_{14}\langle12\rangle$	$4\ldots6$ 9	$l_{\omega+6}$ $l_{13}\langle15\rangle$	7	l_{15}
9	$0,1$	1	l_{18}	2	$l_{13}\langle12\rangle$	3	l_{17}
9	2	1	l_5	2	l_6	3	$l_{13}\langle12\rangle$
9	$3,4$	$1\ldots3$	$l_{\omega+3}$	4	$l_{13}\langle12\rangle$	5	$l_{15}\langle12\rangle$
9	5	$1\ldots3$	$l_{\omega+3}$	4	l_7	5	l_9
9	$6\ldots23$	$1\ldots3$	l_{ω}	$4\ldots6$	$l_{\omega+6}$		
8	$1\ldots3$	$1\ldots2$	$l_{\omega+12}$	3	$l_{18}\langle12\rangle$	4	$l_{17}\langle12\rangle$
8	4	$1\ldots3$	$l_{\omega+3}$	4	$l_{14}\langle12\rangle$	5	$l_{13}\langle12\rangle$
8	$5\ldots22$	$1\ldots3$	l_{ω}	$4\ldots6$	$l_{\omega+6}$		
7	$0\ldots3$	$1\ldots3$	l_{ω}	4	$l_{18}\langle12\rangle$	5	$l_{17}\langle12\rangle$
7	4	$1\ldots3$	l_{ω}	4	l_{11}	5	l_{12}
7	$5\ldots22$	$1\ldots3$	l_{ω}	$4\ldots6$	$l_{\omega+6}$		
6	3	1	l_{18}	2	$l_{13}\langle15\rangle$		
6	$4\ldots9$	1	l_{18}	2	$l_{17}\langle12\rangle$	3	$l_{13}\langle15\rangle$
6	$10\ldots26$	1	l_{13}	2	l_{12}	3	l_3
5	$2,3$	1	l_{13}	2	$l_{14}\langle12\rangle$		
5	$4\ldots9$	1	l_{18}	2	$l_{14}\langle12\rangle$	3	$l_{13}\langle15\rangle$
5	$10\ldots25$	1	l_{14}	2	l_{12}	3	l_3
4	$2\ldots8$	1	l_{15}	2	$l_{14}\langle12\rangle$	3	$l_{13}\langle15\rangle$
4	$9\ldots25$	1	l_{15}	2	l_{12}	3	l_3

For $2 \le \iota_1 \le 3$ we have $u = 0$.

The corresponding DS is ENDS $(i_0, i_1, i_2, 3i_2 - 4)$

Subsubcase IIId5

For $(i_0, i_1, i_2) = (4, 4, 8)$, the value assignment is given by the following table:

p	$m(p)$
11	4
3	3
$4, 5, 45$	2
$1, 9, 12, 15, 27, 28, 30, 31, 33..40, 42, 43, 46, 48, 49, 51, 52$	1
otherwise	0

Proof (Subcase IIId). We sketch the proof for $c = 0$. For $c > 0$ the proof is similar.

For $u = 0$ the proof is similar to Construction 3.1, the details of the proof are omitted.

Consider $u > 0$. First we show that

$$0 \le m(p) < i_0, \quad \text{for} \quad p \in V_3 \setminus \{11\}.$$

Since

$$m(i_0, i_1, i_2, p) \le m(i_0, i_1, i_2 + 1, p)$$

and $m(i_0, i_1, i_2, p) < i_0$, for $p \in V_3 \setminus \{11\}$ and $u = 0$, so $m(p) < i_0$, for $p \in V_3 \setminus \{11\}$, and we only have to consider the smallest value of i_2 to prove $m(p) \ge 0$, namely

$$i_2 = \max\Big(i_0 + 1, \Big\lceil \frac{1}{3}(i_0 + i_1 + 5)\Big\rceil\Big).$$

In Subsubcase IIId1, from the definition of $\varepsilon(p)$ and

$$\begin{array}{l} \min\{m(p) \mid p \in I\} \ge 3f + g \ge 3, \\ m(9) = m(12) = m(15) = 6f_1 + g_1 \ge 6. \end{array}$$

we get $m(p) \ge 0$.

In Subsubcase IIId2, if $u = 3g_1$ then $m(9) = m(12) = m(15) = 0$ and $i_2 = i_0 + 1$. In this case $\tilde{m} = 3f + g \ge 3$ and so $m(p) \ge 0$ for $p \in I$. Hence $i_3 \ge i_0 + i_1 + 1$. From $i_3 = 3i_2 - 4$ we get $i_2 \ge \lceil \frac{1}{3}(i_0 + i_1 + 5)\rceil$, $i_2 = \max(i_0 + 1, \frac{1}{3}(i_0 + i_1 + 5))$.

In Subsubcase IIId3, for $u_1 = N(\delta, \delta_2)$ we have $m(p) = 0$ for $p \in I$ except that $m(p) = 1$ in the following cases:

δ_2	0	1	2	3	4	5	6	7	8
$m(p) = 1$	N_0	31	34, 43	N_0	28	41, 32	N_0	35	38, 29

Let

$$\delta_1 \triangleq |\{p \mid m(p) = 1, p \in I, u_1 = N(\delta, \delta_2)\}|.$$

We have $\delta_1 \equiv \delta_2 \pmod 3$. From $6f_1 + g_1 \ge 6$ and the definition of $\varepsilon(p)$ we have $m(9) = m(12) = m(15) \ge 0$. Hence by $i_3 = i_0 + i_1 + 1 + \delta_1$ and $i_3 = 3i_2 - 4$ we get

$$i_2 = \frac{1}{3}(i_0 + i_1 + 5 + \delta_1) = \Big\lceil \frac{1}{3}(i_0 + i_1 + 5)\Big\rceil \ge i_0 + 1.$$

In Subsubcase IIId4, if (75) is satisfied, then the situation is similar to Subsubcase IIId3. If (75) is not satisfied, then we give one exampe as an illustration:

Let $i_1 = 12$. We have $\delta = 0$. If $3 \le i_0 \le 8$, then $g = 1$ and (75) is satisfied. For the other i_0, (75) is not satisfied, and we have the following table

i_0	2	9	10	11	12
δ_2	2	4	3	2	1
$i_0 + 1$	3	10	11	12	13
$\lceil \frac{1}{3}(i_0 + i_1 + 5)\rceil$	7	9	9	10	10
$n(i_0, i_1)$	5	9	9	9	9

where

$$n(i_0, i_1) \triangleq 3\Big(\lceil \frac{i_1}{3}\rceil - 1\Big) - \max\Big(\lceil \frac{1}{3}(i_0 + i_1 + 5)\rceil - i_0 - 1, 0\Big).$$

If $9 \le i_0 \le 12$, $u = n(i_0, i_1)$, then $m(9) = m(12) = m(15) = 0$, $m(p) \ge 0$ for $p \in I$. If $i_0 = 2$, $u = 5$, then $m(p) = 0$ for $p \in I$ except $m(34) = m(43) = 1$, and $m(12) = 3$, $m(9) = 1$, $m(15) = 0$. Hence

$$i_2 = \Big\lceil \frac{1}{3}(i_0 + i_1 + 5)\Big\rceil \ge i_0 + 1.$$

For the lines, it is sufficient to consider $\omega = 0$.

For the planes, if the 4 points which satisfy $\varepsilon(\omega, p) \neq \varepsilon(\omega - 1, p)$ are in a line l, then since each plane has a joint point with l, so the corresponding DS is the ENDS $(i_0, i_1, i_2, 3i_2 - 4)$. If these 4 points are not in a line and are in the set $\{p_2\} \cup l_j \setminus \{p_1\}$, then we only need show that

$$m(P) < m(\widehat{3, 9, 27})$$

for the 9 planes P containing p_1, except the 4 planes containing $\overline{3, 9}$. We give one exampe as an illustration.

Example, $i_0 = 4$, $i_1 = 5$, $u = 3$, $\varepsilon(3, p) - \varepsilon(2, p) = 1$ iff $p \in \{15\} \cup l_{13} \setminus \{9\}$.

We have $i_2 = 5$, $c = 0$, $\delta = 1$, $\delta_2 = 7$, and $m(p)$ is given by the following table:

$m(p)$	p
4	11
3	3
2	1, 4, 5, 45, 36
1	30, 39, 48, 33, 42, 51, 27, 35
0	otherwise

The values of the planes containing the point 9 but not the line $\overline{3, 9}$ are given by the following table.

P	$\widehat{9, 4, 27}$	$\widehat{9, 4, 30}$	$\widehat{9, 4, 33}$	$\widehat{9, 5, 27}$	$\widehat{9, 5, 30}$	$\widehat{9, 5, 33}$
$m(p)$	8	5	5	7	6	5
P	$\widehat{9, 1, 27}$	$\widehat{9, 1, 30}$	$\widehat{9, 1, 33}$			
$m(p)$	11	9	10			

Hence $m(P) \leq 13 = m(\widehat{3, 9, 27}) - 1$ in all cases.

Case IV

The conditions of Case IV and Theorem 1 i) and ii) easily imply the following relations.

$$i_1 \leq i_0 - 1 \tag{76}$$

$$i_2 \leq \min(3i_0 - 1, i_0 + 3i_1 - 5). \tag{77}$$

Hence by $i_1 \geq \frac{1}{3}(i_0 + 4), i_0 \geq 2$ we have $i_0 \geq 4, i_1 \geq 3$.

Let

$$i_3 = \min(12i_1 - i_2 - 13, \quad 3i_1 - 4), \tag{78}$$

and

$$\delta \equiv -i_0 \pmod 3, \ \delta \in \{0, 1, 2\}.$$

Case IVa

For this case we assume the additional conditions

$$i_2 \leq 3i_1 - 3 \quad \text{and} \quad i_2 < 2i_0.$$

Let

$$\begin{aligned} &3i_1 - 3 - i_2 = 3c + u, && u \in \{0, 1, 2\}, \\ &\delta_1 \equiv -(i_1 - 1 - c) \pmod 2, && \delta_1 \in \{0, 1\}, \\ &\delta_2 \equiv -(i_0 + 1) \pmod 9, && 0 \leq \delta_2 \leq 8. \end{aligned}$$

Define $\varepsilon(p)$ by the following tables.

$\varepsilon(0,p)=\varepsilon(\delta_2,0,p)$									
δ_2	30	39	48	33	42	51	27	36	45
0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0	0	0
3	1	1	1	1	1	1	0	0	0
4	1	1	1	1	1	1	1	1	0
5	2	1	1	1	1	1	1	1	1
6	2	2	2	1	1	1	1	1	1
7	2	2	2	2	2	1	1	1	1
8	2	2	2	2	2	2	2	1	1

$\varepsilon(0,p)=\varepsilon(\delta_2,0,p)$								
δ_2	31,40	35,49	34,43	29,52	28,37	46,32	41,50	44,53
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0	1
4	1	1	0	0	1	0	0	1
5	1	1	0	0	1	1	0	1
6	1	1	0	0	1	1	1	1
7	1	1	1	0	1	1	1	1
8	1	1	1	1	1	1	1	1

$\varepsilon(p)=\varepsilon(\delta,\omega,p)$				
		1,9	4,12	5,15
$\delta=0$		1	-1	-1
$\delta=1$	$\omega=0$	0	0	0
$\delta=1$	$\omega>0$	1	0	-1
$\delta=2$		1	0	0

We have $\varepsilon(\omega,p)=\varepsilon(\omega-1,p)$ except in the following cases:

	$\varepsilon(\omega,p)-\varepsilon(\omega-1,p)=-1$ iff $p=$			
$\omega=1,\ \delta_1=0$	9	53	44	35
$\omega=1,\ \delta_1=1$	9	50	41	32
$\omega=2$	9	49	40	31

For $c>0$, $\varepsilon_2(p)$ is given by the following table:

δ_1	δ_2	31,40,49	34,43,52,35,44,53	28,37,46	otherwise
0	$\neq 3,4$	0	0	1	0
0	3,4	1	0	0	0
1		1	1	1	0

If $c=0$, then the same values apply with the following modifications:

$\varepsilon_2(p)$ are changed by $+1$ or -1 for the following p								
condition	31	28	37	38	40	41	43	44
$\delta_2\in\{1,5\}$, $\delta_1=0$					$+1$	-1		
$\delta_2\in\{4,8\}$			$+1$	-1				
$\delta_2\in\{2,7\}$, $(i_0,i_1)\neq(6,4)$							$+1$	-1
$(i_0,i_1)=(6,5)$	$+1$	-1					$+1$	-1

Note that if $\delta_2 \in \{2,7\}$ and $(i_0, i_1) = (6,5)$, then $\varepsilon_2(43)$ and $\varepsilon_2(44)$ are modified twice.

p	$m(p)$
11	i_0
3	$i_1 - 1$
10, 13, 16, 14, 17	0
1, 4, 5,	$\frac{1}{3}(i_0+\delta) - \varepsilon(u,p)$
27, 30, 33, 36, 39, 42 45, 48, 51	$\frac{2}{9}(i_0+1+\delta_2) - \varepsilon(u,p)$
9, 12, 15	$i_1 - c - 1 - \frac{1}{3}(i_0+\delta) + \varepsilon(u,p)$
otherwise	$\frac{1}{2}(i_1 - 1 - c + \delta_1) - \varepsilon_2(p) - \frac{1}{9}(i_0+\delta_2+1)$ $+\varepsilon(u,p)$

The corresponding DS is ENDS $(i_0, i_1, i_2, 3i_2 - 4)$.

Case IVb

For this case we assume the additional conditions

$$2i_0 \le i_2 \le 3i_1 - 3.$$

Let

$$3i_1 - 3 - i_2 = 3c + u, \text{ where } u \in \{0,1,2\},\ c \ge 0.$$

Define $\varepsilon(p)$ by following tables.

$\varepsilon(p) = \varepsilon(\delta, 0, p)$									
δ	16,27,31 36,40 45,49	30 39 48	32,35 41,44 53	12,29 38,47	9 50	10 34 43	13 52	15	1
0,1	δ	0	0	$-\delta$	$-\delta$	1	1	-1	$\delta - 1$
2	1	1	-1	-1	-2	1	2	-2	1

ω	$\varepsilon(\omega,p) - \varepsilon(\omega-1,p) = 1$ iff p is in following line
1	l_1
2	l_{15}

Define $m(p)$ by following table.

p	$m(p)$
11	i_0
3	$i_1 - 1$
14, 17	0
9, 12, 15, 32, 41, 50 35, 44, 53, 29, 38, 47	$i_1 - 1 - c - \frac{2}{3}(i_0+\delta) - \varepsilon(u,p)$
otherwise	$\frac{1}{3}(i_0+\delta) - \varepsilon(u,p)$

The corresponding DS is ENDS $(i_0, i_1, i_2, 3i_1 - 4)$.

Case IVc

For this case we assume the additional conditions

$$i_2 \geq 3i_1 - 2, \quad i_1 < \frac{2}{3}(i_0 + 2) \quad \text{(i.e.} \quad i_2 \leq i_0 + 3i_1 - 5),$$

and $(i_0, i_1, i_2) \neq (4, 3, 8)$.

Define c, u by

$$i_0 + 3i_1 - 5 - i_2 = 9c + u, \quad 0 \leq u \leq 8.$$

For some p there corresponds a point p_1 given as follows.

p	9	12	15	$3\pi + 1$ or $3\pi + 2$ where $9 \leq \pi \leq 17$
p_1	11	14	17	3π

For $p \in \{3\pi + 1, 3\pi + 2 \mid 9 \leq \pi \leq 17\}$ let

$$\delta_1(p) \equiv -(i_1 - 1 - (\frac{1}{3}(i_0 + \delta) - c - \varepsilon_1(p_1) - \varepsilon(p_1))) \pmod 2, \quad \delta_1(p) \in \{0, 1\}.$$

Define $\varepsilon_1(p)$ by the following table (and $\varepsilon_1(p) = 0$ for p not in the table).

$\varepsilon_1(p) = \varepsilon_1(\delta, p)$			
δ	27, 36, 45	33,42,51	11
0	0	0	−1
1	1	0	0
2	1	1	1

Next, define $\varepsilon(\omega, p)$ by following table (and $\varepsilon(\omega, p) = 0$ for p not in the table).

$\varepsilon(p) = \varepsilon(u, p)$								
u	30	39	48	33	42	51	27	36
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0
4	1	1	1	1	0	0	0	0
5	1	1	1	1	1	0	0	0
6	1	1	1	1	1	1	0	0
7	1	1	1	1	1	1	1	0
8	1	1	1	1	1	1	1	1

If $i_2 < 3i_1 - 2$, then $\varepsilon_2(p)$ is defined as follows.

$$\begin{aligned} \varepsilon_2(p) &= 1 \text{ if } \delta_1(p) = 1 \text{ and } p \notin \{28, 32, 35, 37, 41, 44, 46, 50, 53\}, \\ \varepsilon_2(p) &= 0 \text{ otherwise.} \end{aligned}$$

If $i_2 = 3i_1 - 2$, then some values are modified as follows.

	31	32	37	38	49	51	
$u=2$	$\delta_1(30)=\delta_1(39)=1$						
$u=5$	$\delta_1(33)=\delta_1(42)=1$	-1	$+1$				
$u=8$	$\delta_1(27)=\delta_1(36)=0$						
$u=1$	$\delta_1(30)=1$						
$u=4$	$\delta_1(33)=1$						
$u=7$	$\delta_1(27)=0$			-1	$+1$		
$u=2$	$\delta_1(30)=\delta_1(39)=0$						
$u=5$	$\delta_1(33)=\delta_1(42)=0$						
$u=8$	$\delta_1(27)=\delta_1(36)=1$						
$u=1$	$\delta_1(30)=0$						
$u=4$	$\delta_1(33)=0$					-1	$+1$
$u=7$	$\delta_1(27)=1$						

Now $m(p)$ is defined as follows.

p	$m(p)$
1	i_0
3	i_1-1
4, 5, 10, 13, 16	0
11, 14, 17	$\frac{1}{3}(i_0+\delta)-\varepsilon_1(p)$
27, 30, 33 36, 39, 42 45, 48, 51	$\frac{1}{3}(i_0+\delta)-c-\varepsilon_1(p)-\varepsilon(u,p)$
9, 12, 15	$i_1-(\frac{1}{3}(i_0+\delta)-\varepsilon_1(p_1))-1$
otherwise	$\frac{1}{2}(i_1-1-(\frac{1}{3}(i_0+\delta)-c-\varepsilon_1(p_1)-\varepsilon(u,p_1))+\delta_1)$ $-\varepsilon_2(p)$

The corresponding DS is ENDS $(i_0,i_1,i_2,12i_1-i_2-13)$.

Case IVd

For $(i_0,i_1,i_2)=(4,3,8)$, the value assignment is given by the following table:

p	$m(p)$
1	4
3,14,17,30,39,51	2
9,11,27,28,33,35,36,37,42,44,45,46,48,50	1
otherwise	0

The corresponding DS is ENDS (4,3,8,15).

Case IVe

For this case we assume the additional conditions

$$i_2 \geq 3i_1-2, \quad i_1 \geq \frac{2}{3}(i_0+2) \quad (\text{ i.e.} i_2 \leq 3i_0-1)$$

Let $u \equiv -i_2 \pmod 9$ for $0 \leq u \leq 8$. For $p_1(p)$, $\varepsilon(p)$ and $\varepsilon_2(p)$ we use the definitions as in Subcase IVc. Further, $p_2(p)$ is colinear with 1, p and $p_1(p)$.

Let

$$\delta_1(p) \equiv -(i_1-1-(\frac{1}{9}(i_2+u)-\varepsilon(p_1(p))) \pmod 2, \quad \delta_1 \in \{0,1\}$$

Define $\varepsilon_1(p)$ and $m(p)$ by following tables.

$\varepsilon_1(p) = \varepsilon_1(p,\delta)$				
δ	15	12	9, 14, 17	11
0	0	0	0	-1
1	1	0	0	0
2	1	1	0	1

p	$m(p)$
1	i_0
3	$i_1 - 1$
4, 5	0
9, 12, 15, 11, 14, 17	$\frac{1}{3}(i_0+\delta) - \varepsilon_1(u,p)$
10, 13, 16	$i_1 - m(p_1(p)) - m(p_2(p)) - 1$
27, 30, 33, 36, 39, 42 45, 48, 51	$\frac{1}{9}(i_2+u) - \varepsilon(u,p)$
otherwise	$\frac{1}{2}(i_1 - 1 - m(p_1(p)) + \delta_1(p)) - \varepsilon_2(p)$

The corresponding DS is ENDS $(i_0, i_1, i_2, 12i_1 - i_2 - 13)$.

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