REPORTS
IN
INFORMATICS

ISSN 0333-3590

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REPORT NO 251 **October 2003**

Department of Informatics

UNIVERSITY OF BERGEN

Bergen, Norway

This report has URL http://www.ii.uib.no/publikasjoner/texrap/ps/2003-251.ps

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Subexponential parameterized algorithms on graphs of bounded genus and H-minor-free graphs

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Abstract

We introduce a new framework for designing fixed-parameter algorithms with subexponential running time—$2^{O(\sqrt{k})}n^{O(1)}$. Our results apply to a broad family of graph problems, called *bidimensional problems*, which includes many domination and covering problems such as vertex cover, feedback vertex set, minimum maximal matching, dominating set, edge dominating set, clique-transversal set, and many others restricted to bounded genus graphs. Furthermore, it is fairly straightforward to prove that a problem is bidimensional. In particular, our framework includes as special cases all previously known problems to have such subexponential algorithms. Previously, these algorithms applied to planar graphs, single-crossing-minor-free graphs, and map graphs; we extend these results to apply to bounded-genus graphs as well. In a parallel development of combinatorial results, we establish an upper bound on the treewidth (or branchwidth) of a bounded-genus graph that excludes some planar graph H as a minor. This bound depends linearly on the size $|V(H)|$ of the excluded graph H and the genus $g(G)$ of the graph G, and applies and extends the graph-minors work of Robertson & Seymour.

Building on these results, we develop subexponential fixed-parameter algorithms for dominating set, vertex cover, and set cover in any class of graphs excluding a fixed graph H as a minor. In particular, this general category of graphs includes planar graphs, bounded-genus graphs, single-crossing-minor-free graphs, and any class of graphs that is closed under taking minors. Specifically, the running time is $2^{O(\sqrt{k})}n^h$, where h is a constant depending only on H, which is polynomial for $k = O(\log^2 n)$. We introduce a general approach for developing algorithms on H-minor-free graphs, based on structural results about H-minor-free graphs at the heart of Robertson & Seymour's graph-minors work. We believe this approach opens the way to further development for problems on H-minor-free graphs.

1 Introduction

Dominating set is a classic NP-complete graph optimization problem which fits into the broader class of *domination* and *covering* problems on which hundreds of papers

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have been written; see e.g. the survey [22]. A sample application is the problem of finding sites for emergency service facilities such as fire stations. Here we suppose that we can afford to build k fire stations to cover a city, and we require that every building is covered by at least one fire station. This problem is k-dominating set (finding a dominating set of size k) in the graph where edges represent suitable pairings of fire stations with buildings. In this application, we can afford high running time (e.g., several weeks of real time) if the resulting solution builds fewer fire stations (which are extremely expensive). Thus, we prefer exact *fixed-parameter* algorithms (which run fast provided the parameter k is small) over approximation algorithms, even if the approximation were within an additive constant. The theory of fixed-parameter algorithms and parameterized complexity has been thoroughly developed over the past few years; see e.g. [11, 14, 16, 17, 19, 2].

In the last two years, several researchers have obtained exponential speedups in fixed-parameter algorithms for various problems on several classes of graphs. While most previous fixed-parameter algorithms have a running time of $O(2^{O(k)} n^{O(1)})$ or worse, the exponential speedups results in subexponential algorithms with running times of $O(2^{O(\sqrt{k})} n^{O(1)})$. For example, the first fixed-parameter algorithm for k-dominating set in planar graphs [14] has running time $O(11^k |G|)$; subsequently, a sequence of subexponential algorithms and improvements have been obtained, starting with running time $O(4^{6\sqrt{34k}} n)$ [1], then $O(2^{27\sqrt{k}} n)$ [23], and finally $O(2^{15.13\sqrt{k}} k + n^3 + k^4)$ [17]. Other subexponential algorithms for other domination and covering problems on planar graphs also have been obtained [1, 2, 7, 25, 21].

However, all of these algorithms apply only to planar graphs. In another sequence of papers, these results have been generalized to other classes of graphs: map graphs [11], which include planar graphs; $K_{3,3}$-minor-free graphs and K_5-minor-free graphs [13], which include planar graphs; and single-crossing-minor-free graphs [12, 13], which include $K_{3,3}$ or K_5-minor-free graphs. These algorithms [11, 12, 13] apply to dominating set and several other problems related to domination, covering, and logic.

Algorithms for H-minor-free graphs for a fixed graph H have been studied extensively; see e.g. [8, 20, 9, 24, 27]. In particular, it is generally believed that several algorithms for planar graphs can be generalized to H-minor-free graphs for any fixed H [20, 24, 27]. H-minor-free graphs are very general. The deep Graph-Minor Theorem of Robertson & Seymour shows that any graph class that is closed under minors is characterized by excluding a finite set of minors. In particular, any graph class that is closed under minors excludes at least one minor H.

Our results. We introduce a framework for extending algorithms for planar graphs to apply to H-minor-free graphs for any fixed H. In particular, we design subexponential fixed-parameter algorithms for dominating set, vertex cover, and set cover (viewed as one-sided dominating in a bipartite graph) for H-minor-free graphs. Our framework consists of three components, as described below. We believe that many of these components can be applied to other problems and conjectures as well.

First we extend the algorithm for planar graphs to bounded-genus graphs. Roughly speaking, we study the structure of the solution to the problem in $k \times k$ grids, which form a representative substructure in both planar graphs and bounded-genus graphs, and capture the main difficulty of the problem for these graphs. Then using Robertson & Seymour's graph-minor theory, we repeatedly remove handles to reduce the bounded-genus graph down to a planar graph, which is essentially a grid.

Second we extend the algorithm to *almost-embeddable* graphs which can be drawn in a bounded-genus surface except for a bounded number of "local areas of non-planarity", called vortices, and for a bounded number of "apex" vertices, which can have any number of incident edges that are not properly embedded. Because the vortices have bounded pathwidth, their number is bounded, and the number of apexes is bounded, we are able to provide a solution to almost-embeddable graphs using our solution to bounded-genus graphs.

Third we apply a deep theorem of Robertson & Seymour which characterizes H-minor-free graphs as a tree structure of pieces, where each piece is an almost-embeddable graph. Using dynamic programming on such tree structures, analogous to algorithms for graphs of bounded treewidth, we are able to combine the pieces and solve the problem for H-minor-free graphs.

The first step of this procedure, for bounded-genus graphs, applies to a broad class of problems called "bidimensional problems". Roughly speaking, a parameterized problem is *bidimensional* if the parameter is large (linear) in a grid and closed under contractions. Examples of bidimensional problems include vertex cover, feedback vertex set, minimum maximal matching, dominating set, edge dominating set, clique-transversal set, and set cover. We obtain subexponential fixed-parameter algorithms for all of these problems in bounded-genus graphs. As an special case, this generalization settles an open problem about dominating set posed by Ellis, Fan, and Fellows [15]. We also improve substantially on the results of [10]. Along the way, we establish an upper bound on the treewidth (or branchwidth) of a bounded-genus graph that excludes some planar graph H as a minor. This bound depends linearly on the size $|V(H)|$ of the excluded graph H and the genus $g(G)$ of the graph G, and applies and extends the graph-minors work of Robertson & Seymour.

This paper is organized as follows. First, we introduce the terminology used throughout the paper, and formally define tree decompositions, treewidth, and fixed-parameter tractability in Section 2. We construct a general framework for obtaining subexponential parameterized algorithms on graphs of bounded genus in Section 3. First we introduce the concept of bidimensional problem, and then prove that every bidimensional problem has a subexponential parameterized algorithm on graphs of bounded genus. The proof techniques used in this section are very indirect and are based on deep Theorems from Robertson & Seymour's Graph Minors XI and XII. As a byproduct of our results we obtain a generalization of Quickly Excluding Planar Graph Theorem for graphs of bounded genus. In Section 4 we make a step further by developing subexponential algorithms for graphs containing no fixed graph H as a minor. The proof of this result is based on combinatorial bounds from the previous section, a deep structural theorem from Graph Minors XIV (one of the cornerstones of the Graph Minors Theory), and complicated dynamic programming. Finally, in Section 5, we present several extensions of our results and some open problems.

2 Definitions

All the graphs in this paper are undirected without loops or multiple edges. The reader is referred to standard references for appropriate background [5].

The *(disjoint) union* of two disjoint graphs G_1 and G_2, $G_1 \cup G_2$, is the graph G with merged vertex and edge sets: $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is the subgraph of a some contraction of G. We use the notation $H \preceq G$ (resp. $H \preceq_c G$) for H is a minor (a contraction) of G. A graph class $\mathcal{C}$ is a *minor-closed* class if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is H*-minor-free* if $H \notin \mathcal{C}$. For example, a planar graph is a graph excluding both $K_{3,3}$ and K_5 as minors.

The notion of treewidth was introduced by Robertson & Seymour [28] and plays an important role in their fundamental work on graph minors. To define this notion, first we consider the representation of a graph as a tree, which is the basis of our algorithms in this paper. A *tree decomposition* of a graph G, denoted by $TD(G)$, is a pair (χ, T) in which T is a tree and $\chi = \{\chi_i | i \in V(T)\}$ is a family of subsets of $V(G)$ such that: (1) $\bigcup_{i \in V(T)} \chi_i = V(G)$; (2) for each edge $e = \{u, v\} \in E(G)$ there exists an $i \in V(G)$ such that both u and v belong to χ_i; and (3) for all $v \in V(G)$, the set of nodes $\{i \in V(T) | v \in \chi_i\}$ forms a connected subtree of T. To distinguish between vertices of the original graph G and vertices of T in $TD(G)$, we call vertices of T *nodes* and their corresponding χ_i's *bags*. The maximum size of a bag in $TD(G)$ minus one is called the *width* of the tree decomposition. The *treewidth* of a graph G ($\mathbf{tw}(G)$) is the minimum width over all tree decompositions of G. A tree decomposition is called a *path decomposition* if $T = (I, F)$ is a path. The *pathwidth* of a graph G ($\mathbf{tw}(G)$) is the minimum width over all possible path decompositions of G.

A *branch decomposition* of a graph (or a hyper-graph) G is a pair (T, τ), where T is a tree with vertices of degree 1 or 3 and τ is a bijection from the set of leaves of T to $E(G)$. The *order* of an edge e in T is the number of vertices $v \in V(G)$ such that there are leaves t_1, t_2 in T in different components of $T(V(T), E(T) - e)$ with $\tau(t_1)$ and $\tau(t_2)$ both containing v as an endpoint. The *width* of (T, τ) is the maximum order over all edges of T, and the *branchwidth* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G.

Robertson & Seymour [(5.1) in [30]] proved that for any connected graph G where $|E(G)| \geq 3$, $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(G)$. Also, we will need the following result of Robertson, Seymour & Thomas. (Theorems (4.3) in [30] and (6.3) in [35].)

Theorem 2.1 ([35]). *Let $r \geq 1$ be an integer. Every planar graph with no (r, r)-grid as a minor has branch-width $\leq 4r - 3$.*

A *parameter* P is any function mapping graphs to nonnegative integers. The *parameterized problem* associated with P asks, for some fixed k, whether $P(G) \leq k$ for a given graph G.

Let G be a graph and let $v \in V(G)$. Also suppose we have a partition $\mathcal{P}_v = (N_1, N_2)$ of the set of the neighbors of v. Define the *splitting* of G with respect to v and P_v to be the graph obtained from G by (i) removing v and its incident edges, (ii) introducing two new vertices v^1, v^2 and (iii) connecting v^i with the vertices in $N_i, i = 1, 2$. If H is the result of the consecutive application of the above operation on some graph G then we say that H is a *splitting* of G. If additionally in such a splitting process we do not split vertices that are results of previous splittings then we say that H is a *fair splitting* of G.

We say a parameter P is *α-splittable*, if for every graph G and for each vertex $v \in V(G)$ the result of splitting G' with respect to v has $P(G') \leq P(G) + \alpha$. Many natural graph problems are α-splittable for small α. Examples of 1-splittable problems are dominating set, vertex cover, edge dominating set, independent set, clique-transversal set and feedback vertex set among many others.

3 Bidimensional Parameters

In this section, we define a general framework of parameterized problems for which subexponential algorithms with small constants can be obtained. Our framework is sufficiently broad that an algorithmic designer only needs to check two simple properties of any desired parameter to determine the applicability and practicality of our approach.

A *partially triangulated* $(r \times r)$*-grid* is any graph obtained by adding edges between pairs of nonconsecutive vertices on a common face of a planar embedding of a $(r \times r)$-grid.
• A parameter P is called *minor bidimensional with density* δ if (i) contracting or deleting an edge in a graph G cannot increase $P(G)$, and (ii) there exists a function $f, f(x) = o(x)$ such that for the $(r \times r)$-grid R, $P(R) = (\delta r)^2 + f((\delta r)^2)$.
• A parameter P is called *contraction bidimensional with density* δ if (i) contracting an edge in a graph G cannot increase $P(G)$, (ii) there exists a function $f, f(x) = o(x)$ such that for any partially triangulated $(r \times r)$-grid R, $P(R) \geq (\delta r)^2 + f((\delta r)^2)$, and δ is the smallest real number for which this inequality holds.
In either case, P is called *bidimensional*. The *density* δ of P is the minimum of the two possible densities (when both definitions are applicable). We call the sublinear function f *residual function* of P.

Many parameters are bidimensional, mention just a few. Examples of minor bidimensional parameters along with some estimations for their densities are: vertex cover ($\delta = 1/\sqrt{2}$), feedback vertex set (FVS) ($\delta \geq 1/2$) and minimum maximal matching ($\delta \geq 1/\sqrt{8}$). Examples of contraction bidimensional parameters are dominating set ($\delta = 1/3$), edge dominating set ($\delta = 1/\sqrt{14}$) and clique-transversal set ($\delta \geq 1/2\sqrt{2}$).

Notice that density assigns a real number in $(0, 1]$ to any bidimensional parameter. This assignment defines a *total* order on all such parameters.

The class of bidimensional parameterized problems contains all known from the literature planar graph parameters with subexponetial parameterized algorithms. Recently, Cai et al. [6] defined a class *Planar TMIN*$_1$ and proved that for every graph G and parameter $P \in$ *Planar TMIN*$_1$, $\mathbf{tw}(G) = O(\sqrt{P(G)})$. Every problem in *Planar TMIN*$_1$ can be expressed as a special type of dominating set problem on bipartite graphs (we refer to [6] for definitions and further properties of *Planar TMIN*$_1$) and Proposition 3.2 yields immediately the result of Cai et al.

If P is a bidimensional parameter with density δ and residual function f then we define the *normalization factor* of P as $\min\{\beta \mid (\frac{\delta}{\beta} r)^2 \leq (\delta r)^2 + f(\delta r), \text{ for any } r \geq 1 \}$.

3.1 The bounded treewidth approach

Almost all known techniques for obtaining subexponential parameterized algorithms on planar graphs are based on the following "bounded treewidth approach" [1, 17, 23]:

(I1) Prove that the treewidth/branchwidth of G is at most $c\sqrt{P(G)}$ for some constant c;

(I2) Compute the treewidth (or branchwidth) of G;

(I3) If treewidth is $> c\sqrt{P(G)}$ there is no solution to the problem. If treewidth/branchwidth is $\leq c\sqrt{P(G)}$ run standard dynamic programming on graph of bounded treewidth/branchwidth which takes $2^{O(\sqrt{P(G)})} n$ steps.

All previously known ways of obtaining the most important step (I1) are based on rather complicated techniques based on separators. Let us first give some hints why bidimensional parameters are important for the design of subexponential algorithms on planar graphs. The proof of the next lemma is easy and omitted.

Lemma 3.1. *Let P be a contraction (minor) bidimensional parameter with density δ and normalization factor β. Then $P(G) < (\frac{\delta}{\beta} r)^2$ implies that G excludes the $(r \times r)$-grid as a minor (and all partial triangulations of the $(r \times r)$-grid as contractions).*

Theorem 2.1 and Lemma 3.1 imply the following.

Proposition 3.2. *Let P be a bidimensional parameter with density δ and normalization factor β. Then for any planar graph G, $\mathbf{tw}(G) \leq 4\frac{\beta}{\delta}\sqrt{P(G)}$.*

Proposition 3.2 is a first hint on the importance of bidimensionality for easy achieving low bounds like the ones required for step **(I1)** of the bounded treewidth approach. The main result of this section is the following generalization of preposition 3.2. The proof is quite lengthy and technical and can be found in the Appendix.

Theorem 3.3. *Suppose that P is an α-splittable bidimensional parameter (for $\alpha \geq 0$) with density δ and normalization factor β ($\delta \leq 1$ and $\beta \geq 1$). Then for any graph G 2-cell embedded in a surface Σ of Euler genus $\mathbf{eg}(\Sigma)$, $\mathbf{bw}(G) \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{P(G)+1} + 8\alpha(\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1))^2$.*

Theorem 3.3 is a general theorem which applies for any α-splittable bidimensional parameter. For minor bidimensional parameters the bound for branchwidth can be further improved (we omit the proof because it is similar to that of Theorem 3.3).

Theorem 3.4. *Suppose that P is a minor bidimensional parameter with density δ and normalization factor β ($\delta \leq 1$ and $\beta \geq 1$). Then for any graph G 2-cell embedded in a surface Σ of Euler genus $\mathbf{eg}(\Sigma)$, $\mathbf{bw}(G) \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma)+1)\sqrt{P(G)+1}$.*

3.2 Combinatorial Results and Further Improvements

As part of their seminal Graph Minors series, Robertson & Seymour proved the following:

Theorem 3.5 ([29]). *If G excludes a planar graph H as a minor, then the branchwidth of G is at most b_H and the treewidth of G is at most t_H, where b_H and t_H are constants depending only on H.*

The current best estimate of these constants is the exponential upper bound $t_H \leq 20^{2(2|V(H)|+4|E(H)|)^5}$ [35]. However, combining Theorem 2.1 with a lemma of [35] we have that the constants depend only linearly on the size of H.

Theorem 3.6. *If G is a planar graph excluding a planar graph H as a minor, then its branchwidth is at most $b_H \leq 4(2|V(H)| + 4|E(H)|)$.*

Note that the parameter $P(G) = |V(G)|$ is minor bidimensional with δ and β equal to 1. Thus Theorems 3.3 and 3.4 implies the following generalization of Theorem 2.1 for graphs of bounded genus.

Theorem 3.7. *If G is a graph of genus $g(G)$ with branchwidth more than $4r(g(G)+1)$, then G has a $(r \times r)$-grid as a minor.*

In the same way, we are able to quickly exclude any planar graph from bounded-genus graphs. In other words, we generalize Theorem 3.6 as follows:

Theorem 3.8. *If G is a graph of genus $g(G)$ that excludes a planar graph H as a minor, then its branchwidth is at most $b^{\text{genus}}_{H,g(G)} \leq 4(2|V(H)| + 4|E(H)|)(g(G)+1)$.*

3.3 Algorithmic Consequences

As we already discussed, the combinatorial upper bounds for branchwidth/treewidth are used for constructing subexponential parameterized algorithms as follows. Let G be a graph and P be a parameterized problem we need to solve on G. First one constructs a branch/tree decomposition of G that is optimal or 'almost' optimal. A $(\theta, \gamma, \lambda)$-*approximation scheme* for branchwidth/treewidth consists of, for every w, an $O(2^{\gamma w} n^{\lambda})$-time algorithm that, given a graph G, either reports that G has branchwidth/treewidth at least w or produces a branch/tree decomposition of G with width at most θw. For example, the current best schemes are a $(3 + 2/3, 3.698, 3 + \epsilon)$-approximation scheme for treewidth [3] and a $(3, \lg 27, 2)$-approximation scheme for branchwidth [33].

If the branchwidth/treewidth of a graph is "large", then combinatorial upper bounds come into play and we conclude that P has no solution on G. Otherwise we run dynamic programming on graphs of bounded branchwidth/treewidth and compute $P(G)$. Thus we conclude with the main algorithmic result of this section:

Theorem 3.9. *Let P be a bidimensional parameter with density $> \delta$. Suppose there is an algorithm for the associated parameterized problem that runs in $O(2^{aw} n^b)$ time given a tree/branch decomposition of the graph G with width w. Suppose also that we have a $(\theta, \gamma, \lambda)$-approximation scheme for treewidth/branchwidth. Set $\tau = 1$ in the case of branchwidth and $\tau = 1.5$ in the case of treewidth. Then the parameterized problem asking whether $P(G) \leq k$ can be solved in $O(2^{\max\{a\theta,\gamma\}\tau 4 \frac{\beta}{\delta}(g(G)+1)(\sqrt{k+1}+\mu\alpha\frac{\beta}{\delta}(g(G)+1))} n^{\max\{b,\lambda\}})$ time for minor bidimensional parameter $P(G)$ with density δ and normalization factor β, where μ is 0 if P is minor bidimensional and is 2 if P is α-splittable contraction bidimensional.*

The first condition of the theorem holds with small values of a and b for many examples of bidimensional parameters; see [1, 2, 7, 13, 17, 25]. Observe that the correctness of our algorithms is simply based on Theorems 3.3 and 3.4, despite their nonalgorithmic natures, and $(\theta, \gamma, \lambda)$-approximation scheme for branch/tree decomposition. We note that the time bounds we provide do not contain any hidden constants, and the constants are reasonably low for a broad collection of problems covering all the problems for which there already exist $2^{O(\sqrt{k})} n^{O(1)}$-time algorithms.

4 H-minor free graphs

In this section we will generalize the results on graphs of bounded genus to graphs with excluded minors.

4.1 Characterizations of H-minor-free graphs

In this section, we describe the deep theorem of Robertson & Seymour on graphs excluding a fixed graph H as a minor. Intuitively, Robertson-Seymour's theorem says for every graph H, every H-minor-free graph can be expressed as a tree-structure of "pieces", where each piece is a graph which can be drawn in a surface in which H cannot be drawn, except for a bounded number of "apex" vertices and a bounded number of "local areas of non-planarity" called *vortices*. Here the bounds only depend on H.

A graph G is *h-almost embeddable* in S if there exists a vertex set X of size at most h called *apices* such that $G - X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$, where

- G_0 has an embedding in S;
- the graphs G_i, called *vortices*, are pairwise disjoint;
- there are (not necessarily distinct) faces $F_1, \ldots, F_h$ of G_0 in S, and there are pairwise disjoint disks $D_1, \ldots, D_h$ in S, such that for $i = 1, \ldots, h$, $D_i \subset F_i$ and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap D_i$; and
- the graph G_i has a path decomposition $(\mathcal{B}_u)_{u \in U_i}$ of width less than h, such that $u \in \mathcal{B}_u$ for all $u \in U_i$. We note that the sets $\mathcal{B}_u$ are ordered by the ordering of their indices u as points in C_i, where C_i is the boundary cycle of F_i in G_0.

An h-almost embeddable graph is called *apex-free* if the set X of apices is empty.

Suppose G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G'_i $(i = 1, 2)$ be obtained from G_i by deleting some (possibly no) edges from $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \to W_2$. We define a *k-sum*[1] G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G'_1 and G'_2 by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the *join set*. In the rest of this section, when we refer to a vertex v of G in G_1 or G_2, we mean the corresponding vertex of v in G_1 or G_2 (or both).

Now, the result [2] of Robertson & Seymour is as follows.

Theorem 4.1 ([34]). *For every graph H there exists an integer $h \geq 0$ only depending on $|V(H)|$ such that every H-minor-free graph can be obtained by at most h-sums of graphs of size at most h and h-almost-embeddable graphs in some surfaces in which H cannot be embedded.*

In particular, if H is fixed, any surface in which H cannot be embedded has bounded genus. Thus, the summands in the theorem are h-almost-embeddable graphs in bounded-genus surfaces.

This structural theorem plays an important role in obtaining the rest of the results of this paper. From the algorithmic point of view, because Robertson & Seymour [34] have shown that every minor-closed class of graphs has a polynomial-time membership test, one can observe the following theorem used by Grohe [18, Lemma 15]. Also, Seymour [36] claims that we can construct the clique-sum decomposition algorithmically using the proof of the Theorem 4.1.

Theorem 4.2. *For any graph H, there is an algorithm with running time $n^{O(1)}$ that either computes a clique-sum decomposition as in Theorem 4.1 for any given H-minor-free graph G, or outputs that G is not H-minor-free. The exponent in the running time depends on H.*

Let G be an h-almost embeddable on a surface of genus g in a clique-sum decomposition of a graph G^*. Suppose the set of apices in G is X. Assume G has clique-sums with graphs $G_1, \cdots, G_p$ via joinsets $W_1, \cdots, W_p$, where $|W_i| \leq h$, $1 \leq i \leq p$. A clique W_i is called *fully dominated* by a set $S \subseteq V(G)$ if $V(G_i) - X \subseteq N_{G^*}(S)$, otherwise clique W_i is called *partially dominated* by S. A vertex v of G is *fully dominated* by a set S if $N_{G^*[V(G)-X]}(v) \subseteq N_{G^*}(S)$.

We note that in the above definition, the only edges that appear in G, but may not appear in G^* are the edges among vertices of $|W_i|$, $1 \leq i \leq p$.

Theorem 4.3. *Let G be an h-almost embeddable on a surface of genus g in a clique-sum decomposition of a graph G^*. Assume G has clique-sums with graphs $G_1, \ldots, G_p$ via join sets $W_1, \ldots, W_p$, where $|W_i| \leq h$, $1 \leq i \leq p$. Suppose G^* has a dominating set of size at most k. Then there is a subset $S \subseteq V(G)$ of size at most h such that if we remove all fully dominated vertices which are not included in any partially dominated clique W_i from G and obtain graph $\hat{G}$, $\mathbf{tw}(\hat{G}) = O(h^2 g\sqrt{k+h} + g^2) = O(\sqrt{k})$.*

In order to prove Theorem 4.3 we need first some preliminary results. A vertex w is called *r-dominated* by a set S, if the distance from w to a vertex $v \in S$ is at most r. An *r-dominating set* is a set S of vertices such that every vertex of the graph is r-dominated by S. The problem of finding an r-dominating set of size k is

[1]It is worth mentioning that $\oplus$ is not a well-defined operator and it can have a set of possible results.

[2]Theorem 4.1 is very general and has not appeared in print so far. However already several nice applications (see e.g. [4, 18]) are known.

also called the (k, r)-center problem (see Section 5). From the main combinatorial result of [11], r-dominating set is a 1-splittable bidimensional parameter. This and Theorem 3.3 imply the following.

Lemma 4.4. *For any constant r, if a graph G of genus g has an r-dominating set of size at most k, then the treewidth of G is at most $O(g\sqrt{k} + g^2)$.*

Now, we extend this result for apex-free h-almost embeddable graphs (the proof is not hard and is omitted).

Lemma 4.5. *Consider an apex-free h-almost-embeddable graph $G = G_0 \cup G_1 \cup \cdots \cup G_h$. Suppose further that, for each $1 \le i \le h$, $U_i = \{u_i^1, u_i^2, \ldots, u_i^{m_i}\}$ forms a path in G_0. Then $\mathbf{tw}(G) \le (h^2 + 1)(\mathbf{tw}(G_0) + 1) - 1$.*

Lemma 4.6. *For any constant r, an apex-free h-almost-embeddable graph G embedded on a surface of genus g with a set $S \subset V(G)$ of size at most k which r-dominates every vertex of G which is not in a vortex has treewidth at most $O(h^2 g\sqrt{k+h} + g^2) = O(g\sqrt{k})$ (g and h are constants).*

Proof. Consider an apex-free h-almost embeddable graph $G = G_0 \cup G_1 \cup \cdots \cup G_h$ in a surface Σ of genus g. Suppose $U_i = \{u_i^1, u_i^2, \ldots, u_i^{m_i}\}$. Let G_0' be the graph obtained from G_0 by adding new vertices $c_1, c_2, \cdots, c_h$ and edges (c_i, u_i^j) and (u_i^j, u_i^{j+1}) (where $j+1$ is treated modulo m_i) for all $1 \le i \le h$ and $1 \le j \le m_i$. Notice that by adding these edges, vertices U_i, $1 \le i \le h$, form a path in G_0. If G has the aforementioned r-dominating set of size k, then G_0' has an r-dominating set of size at most $k+h$: just delete all vertices in the r-dominating set that are in $G_i - G_0$, $1 \le i \le h$, and add instead all new vertices $c_1, c_2, \cdots, c_h$ to the r-dominating set. Notice that G_0' is embeddable on Σ, since G_0 is embeddable. Thus, according to Lemma B.1 it has treewidth at most $O(g\sqrt{k+h}+g^2)$. By Lemma B.2, the treewidth of $G' = G_0' \cup G_1 \cup \cdots \cup G_h$ is $O((h^2+1)(g\sqrt{k+h}+1) + g^2 - 1)$. U_i forming a path in G_0. Because G is a subgraph of G', the lemma follows. □

Proof of Theorem 4.3 Suppose X is the set of apices in G, so that $G - X$ is an apex-free h-almost embeddable graph. Let D be a dominating set of size k of G^* and let $S = X \cap D$. We claim that S is our desired set. The rest of the proof is as follows: we construct a set $\hat{D}$ of size at most k for $\hat{G} - X$ which 2-dominates every vertex v of $\hat{G} - X$ which is not included in any vortex. Then since $\hat{G} - X$ is an apex-free h-almost-embeddable on a surface of genus g with a 2-dominating-type set of size at most k desired by Lemma B.3, it has treewidth at most $O(h^2 g\sqrt{k+h} + g^2)$. Then we can add vertices of X to all bags and still have a tree decomposition of width $O(h^2 g\sqrt{k+h} + g^2)$, as desired. We construct $\hat{D}$ from D as follows. First, we set $\hat{D} = D \cap V(G)$. For each $1 \le i \le p$, if $D \cap (V(G_i) - W_i) \ne \emptyset$ and $W_i \not\subseteq X$, we add an arbitrary vertex $w \in W_i - X$ to $\hat{D}$. Here we say a vertex v of D is *mapped* to a vertex w of $\hat{D}$ if $v = w$ or if $v \in D \cap (V(G_i) - W_i)$ and vertex $w \in W_i - X$ is the one that we have added to $\hat{D}$. One can easily observe that since each new vertex in $\hat{D}$ is in fact accounted by a unique vertex in D, $|\hat{D}| \le k$. It only remains to show that D is a 2-dominating set for $\hat{G} - X$. If a vertex $v \in V(\hat{G}) - X$ is not fully-dominated, then there exists a vertex $w \in N_G(v)$ which is not dominated by S and thus not dominated by X (since $S = D \cap X$). It means v is 2-dominated by a vertex u of $\hat{G} - X$ which dominates w (we note that u can be originally a vertex u' in $(V(G_i) - W_i) \cap D$ which is mapped to u in $\hat{D}$). Also, we note that for each clique W_i in which there is a mapped vertex of D, this vertex dominates all vertices of $W_i - X$ in $\hat{G} - X$ and thus we keep the whole clique $W_i - X$ in G. It only remains to show that every vertex of a partially dominated clique W_i is 2-dominated by a vertex of $\hat{G} - X$. We consider two cases: if $W_i \cap S = \emptyset$, since $V(G_i) - W_i \ne \emptyset$, there must exists a (mapped) vertex of $\hat{D}$ in $W_i - X$ and we are done. Now assume $W_i \cap S \neq \emptyset$. If $W_i \subset X$ then $W_i \cap (V(\hat{G}) - X) = \emptyset$ and we are done (since there is no clique in $\hat{G} - X$ at all.) Otherwise, there exists a vertex $W_i - X$. If $(V(G_i) - W_i) \subseteq N_{G^*}(S) \neq \emptyset$, then $V(G_i) \cap D \neq \emptyset$. Thus there exists a mapped vertex $w \in W_i - X$ and we have 1-dominated vertices of $W_i - X$. As mentioned before if $D \cap (W_i - X) \neq \emptyset$, vertices $W_i - X$ are 1-dominated and we are done. The only remaining case is the case in which there exists a vertex $w \in W_i - X$ which is dominated by a vertex $x \in V(G)$ and by assumption $w \notin N_{G^*}(S)$ (we note that in this case, there is no dominating vertex in $V(G_i) - W_i$ for any i for which $w \in W_i$.) It means vertex x is not fully dominated and thus it remains in $\hat{G}$. In addition, vertex x 2-dominates all vertices of $W_i - X$, since W_i is a clique in G and thus all vertices of $W_i - X$ are 2-dominated. This completes the proof of the theorem.

4.2 Main result

Theorem 4.7. *One can test whether an H-minor-free graph G^* has a dominating set of size at most k in time $2^{O(\sqrt{k})}n^{O(1)}$, where the constants in the exponents depend on H.*

Proof. First, using the $n^{O(1)}$-time algorithm of Theorem 4.2, we obtain the clique-sum decomposition of graph G^*. In fact, this clique-sum decomposition can be considered as a generalized tree decomposition of G^*.

More precisely, we consider the clique-sum decomposition as a rooted tree. We try to find a k-dominating set in this graph using a two-level dynamic programming. Suppose a graph G is an h-almost embeddable on a surface of genus g in a clique-sum decomposition of a graph G^*. Assume G has clique-sums with graphs $G_0, \ldots, G_p$ via join sets $W_0, W_1, \ldots, W_p$, where $|W_i| \leq h$, $0 \leq i \leq p$. Also assume that G_0 is considered as the parent of G and $G_1, \ldots, G_p$ are considered as children of G.
Colorings. The subproblems in our first-level dynamic program are defined by a coloring of the vertices in W_i. Each vertex will be assigned one of 3 colors, labelled 0, $\uparrow 1$, and $\downarrow 1$. The meaning of the coloring of a vertex v is as follows. Color 0 represents that vertex v belongs to the chosen dominating set. Colors $\downarrow 1$ and $\uparrow 1$ represent that the vertex v is not in the chosen dominating set. Such a vertex v must have a neighbor w in the dominating set (i.e., colored 0); we say that vertex w *resolves* vertex v. Color $\downarrow 1$ for vertex v represents that the dominating vertex w is in the subtree of the clique-sum decomposition rooted at the current graph G, whereas $\uparrow 1$ represents that the dominating vertex w is elsewhere in the clique-sum decomposition. Intuitively, the vertices colored $\downarrow 1$ have already been resolved, whereas the vertices colored $\uparrow 1$ still need to be assigned to a dominating vertex.
Locally valid colorings. A coloring of the vertices of W_i is called *locally valid* with respect to sets $S_1, S_2 \subseteq V(G)$ if the following properties hold:
- for any two adjacent vertices v and w in W_i, if v is colored 0, w is colored $\downarrow 1$; and
- if $v \in S_1 \cap W_i$, then v is colored 0; and
- if $v \in S_2 \cap W_i$, then v is not colored 0.

Our colorings are similar to that of previous work (e.g., [1]), but we use them in a new dynamic-programming framework that acts over clique-sum decompositions instead of tree decompositions.
Dynamic program subproblems. Our first-level dynamic program has one subproblem for each graph G in the clique-sum decomposition and for each coloring c of the vertices in W_0. Because each join set has at most h vertices, the number of subproblems is $O(n \cdot 3^h)$. We define $D(G, c)$ to be the size of the minimum "semi"-dominating set of the vertices in subtree rooted at G subject to the following restrictions:

1. Vertices colored $\downarrow 1$ are adjacent to at least one vertex in the dominating set. (Vertices colored $\uparrow 1$ are dominated "for free".)
2. Vertices colored 0 are precisely the vertices in the dominating set.
3. Vertices in W_0 are colored according to c.

If we solve every such subproblem, then in particular, we solve the subproblems involving the root node of the clique-sum decomposition in which every vertex is colored 0 or $\downarrow 1$. The final dominating set of size k is given by the best solution to these subproblems.

Induction step. Suppose for each coloring c of W_i, $1 \leq i \leq p$, we know $D(G_i, c)$. If the graph G is of size at most h, then we can try all colorings in $O(3^h \cdot h^2) = O(1)$ time (where the factor of h^2 is for checking validity). Thus, we focus on almost-embeddable graphs G. First, we guess a subset X of size at most h. Then for each subset S of X, we put the vertices of S in the dominating set and forbid vertices of $X - S$ from being in the dominating set. Now we remove from G all fully dominated vertices of $G - X$ that are not included in any partially dominated clique W_i. Call the resulting graph $\hat{G}$. By Theorem 4.3, $\mathbf{tw}(\hat{G}) = O(\sqrt{k})$. We can obtain such a tree decomposition of width $3 + 2/3$ times optimum, in $2^{O(\sqrt{k})} n^{3+\epsilon}$ time by a result of Amir [3]. All vertices absent from this tree decomposition are fully dominated and thus, in any minimum dominating set that includes S, they will not appear except the following case. It is possible that up to $|X - S| = O(h)$ vertices, which are either fully dominated or belong to $V(G_i) - W_i$ where W_i is fully dominated, appear in the dominating set to dominate vertices of $X - S$. Call the set of such vertices S'. We can guess this set S' by choosing at most h vertices among the discarded vertices that have at least one neighbor in $X - S$, and then add S' to the dominating set. On the other hand, for any partially dominated clique W_i, we know that all of its vertices are present in the tree decomposition; because they form a clique, there is a bag α_i in any tree decomposition that contains all vertices of W_i. We find α_i in our tree decomposition and map W_i and G_i to this bag. We also assume W_0 is contained in all bags, because its size is at most h. Now, for each coloring c of W_0, we run the dynamic program of Alber et al. [1] on the tree decomposition, with the restriction that the colorings of the bags are locally valid with respect to $S_1 := S \cup S'$ and $S_2 := X - S$, and are consistent with the coloring c of W_0. For each bag α_i to which we mapped G_i, we add to the cost of the bag the value $D(G_i, c')$ for the current coloring c' of W_i. Using this dynamic program, we can obtain $D(G, c)$ for each coloring c of W_0.

Running time. The running time for each coloring c of W_0 and each choice of S is $2^{O(\sqrt{k})} n$ according to [1]. We have 3^h choices for c, $O(n^{h+1})$ choices for X, $O(2^h)$ choices for S, and $O(n^{h+1})$ choices for S'. Thus the running time for this inductive step is $6^h n^{2h+2} 2^{O(\sqrt{k})}$. There are $O(n)$ graphs in the clique-sum decomposition of G. Therefore, the total running time of the algorithm is $O(6^h n^{2h+3} 2^{O(\sqrt{k})}) + n^{O(1)}$ (the latter term for creating the clique-sum decomposition), which is $2^{O(\sqrt{k})} n^{O(1)}$ as desired. □

5 Conclusions and Future Work

Theorem 4.7 can be used to obtain subexponential algorithms not only for dominating set problems.

For example, for vertex cover one can use the following reduction. For a graph G let G' be the graph obtained from G by adding a path of length two between any pair of adjacent vertices. The following lemma is obvious.

Lemma 5.1. *For any K_h-minor free graph G, $h \geq 4$, and integer $k \geq 1$*
- *G' is K_h-minor free,*

• *G has vertex cover of size $\leq k$ if and only if G' has a dominating set of size $\leq k$.*

Combining Lemma 5.1 with Theorem 4.7 we conclude that parameterized vertex cover can be solved in subexponential time on graphs with an excluded minor.

Another example is the set cover problem. Given a collection $C = (C_1, C_2, \ldots, C_m)$ of subsets of a finite set $S = (s_1, s_2, \ldots, s_n)$, a set cover is a subcollection $C' \subseteq C$ such that $\cup_{C_i \in C'} = S$. Minimum set cover (SC) problem is to find a cover of minimum size. For a SC problem (C, S) its graph G_S is a bipartite graph with bipartition (C, S). Vertices s_i and C_j are adjacent in G_S if and only if $s_i \in C_j$. Theorem 4.7 can be used to prove that SC with G_S H-minor free for some fixed graph H, can be solved in subexponential time. In fact, for a given graph G_S we construct an auxiliary graph A_S by adding new vertices v, u, w and making adjacent v to $\{u, w, C_1, C_2, \ldots, C_m\}$. Then

• (C, S) has a set cover of size $\leq k$ if and only if A_S has a dominating set of size $\leq k + 1$.

• If G_S is K_h-minor free then A_S is K_{h+1}-minor free.

We believe that we can generalize Theorem 4.7 in order to obtain a fixed-parameter algorithm with exponential speed-up for the (k, r)*-center problem* on H-minor-free graphs. The (k, r)-center problem is a generalization of the dominating set problem in which one asks whether an input graph G has $\leq k$ vertices (called centers) such that every vertex of G is within distance $\leq r$ from some center. Demaine et al. [11] consider this problem for planar graph and map graphs and present a generalization of dynamic programming mentioned in the proof of Theorem 4.7 to solve the (k, r)-center problem for graphs of bounded treewidth/branchwidth. Using this dynamic programming and a generalization of Lemma 4.3, one can obtain the desired result for H-minor-free graphs. Similar technique can be used for solving the dominating set problem in constant powers of H-minor-free graphs, the most general class of graphs so far for which one can obtain the exponential speed-up.

However it is an open and tempting question if our technique can be generalized to solve in subexponential time on graphs with excluded minors every problem solved in subexponential time on bounded genus graphs.

We also suspect that there is a strong connection between bidimensional parameters and the existence of linear-size kernels for the corresponding parameterized problems in bounded-genus graphs.

The final question is if the upper bounds Theorems 3.3 and 3.4 can be extended to larger graph classes. The first step in this direction was obtained by the authors for minor-closed graph families: A graph family $\mathcal{F}$ has domination-treewidth property if there is some function $f(d)$ such for that every graph $G \in \mathcal{F}$ with dominating set of size $\leq k$, $\mathbf{tw}(G) \leq f(k)$. It was shown that a minor-closed graph family has domination-treewidth property if and only if this is bounded local treewidth family. We conjecture that for any bidimensional parameter P and minor-closed graph family $\mathcal{F}$, $\mathbf{tw}(G) = O(\sqrt{P(G)})$ for every $G \in \mathcal{F}$ if and only if $\mathcal{F}$ is of bounded local treewidth.

Acknowledgments. The authors are indebated to Paul D. Seymour for many discussions that led to combinatorial results of this paper and for providing a portal into the Graph Minor Theory. We also thank Naomi Nishimura and Prabhakar Ragde for encouragement and helpful discussions.

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A Proof of Theorem 3.3

We need first some basic definitions and results.

A *surface* Σ is a compact 2-manifold, without boundary. A *line* in Σ is subset homeomorphic to $[0,1]$. An O-arc is a subset of Σ homeomorphic to a circle. Let G be a graph 2-cell embedded in Σ. To simplify notations we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider G as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$. We call by *region* of G any connected component of $\Sigma - E(G) - V(G)$. (Every region is an open set.) We use the notation $V(G), E(G)$, and $R(G)$ for the set of the vertices, edges and regions of G.

If $\Delta \subseteq \Sigma$, then $\overline{\Delta}$ denotes the *closure* of Δ, and the boundary of Δ is $\mathbf{bd}(\Delta) = \overline{\Delta} \cap \overline{\Sigma - \Delta}$. An edge e (a vertex v) is incident with a region r if $e \subseteq \mathbf{bd}(r)$ ($v \subseteq \mathbf{bd}(r)$).

A subset of Σ meeting the drawing only in vertices of G is called *G-normal*. If an O-arc is G-normal then we call it *noose*. The length of a noose is the number of its vertices. $\Delta \subseteq \Sigma$ is an open disc if it is homeomorphic to $\{(x,y) : x^2 + y^2 < 1\}$. We say that a disc D is *bounded* by a noose N if $N = \mathbf{bd}(D)$. A graph G 2-cell embedded in a connected surface Σ is *θ-representative* if every noose of length $< \theta$ is contractable (null-homotopic in Σ).

A *separation* of a graph G is a pair (A,B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and its order is $|V(A \cap B)|$. Tangles were introduced by Robertson & Seymour in [30]. A *tangle of order* $\theta \geq 1$ is a set $\mathcal{T}$ of separations of G, each of order θ, such that

(i) for every separation (A,B) of G of order $< \theta$, $\mathcal{T}$ contains one of (A,B), (B,A)

(ii) if $(A_1,B_1),(A_2,B_2),(A_3,B_3) \in \mathcal{T}$ then $A_1 \cup A_2 \cup A_3 \neq G$.

(iii) if $(A,B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Let G be a graph embedded in a connected surface Σ. A tangle $\mathcal{T}$ of order θ is *respectful* if for every noose N in Σ with $|N \cap V(G)| < \theta$, there is a closed disc $\Delta \subseteq \Sigma$ with $\mathbf{bd}(\Delta) = N$ such that separation $(G \cap \Delta, G \cap \overline{\Sigma - \Delta}) \in \mathcal{T}$.

Theorem A.1 ((4.1) in [31]). *Let Σ be a connected surface, not a sphere, let $\theta \geq 1$, and let G be a θ-representative graph 2-cell embedded in Σ. Then there is a unique respectful tangle in G of order θ.*

Theorem A.1 will be useful for bounding the representativity of graphs excluding some graphs as a minor/contraction. This is done by the following lemma.

Lemma A.2. *Let G be a graph 2-cell embedded in a non-planar surface Σ of representativity at least θ. Then G contains as a contraction a partially triangulated $(\theta/4 \times \theta/4)$-grid.*

Proof sketch of Lemma A.2. By Theorem A.1, G has a respectful tangle of order θ. Let $A(R_G)$ be the set of vertices, edges, and regions (collectively, *atoms*) in the radial[3] graph R_G. According to Section 9 of [31] (see also [32]), the existence of a respectful tangle makes it possible to define a metric d on $A(R_G)$ as follows:

- If $a = b$, then $d(a, b) = 0$.
- If $a \neq b$, and a and b are interior to a contractible closed walk of radial graph of length $< 2\theta$, then $d(a, b)$ is half the minimum length of such a walk (here by *interior* we mean the direction in which the walk can be contracted).
- Otherwise, $d(a, b) = \theta$.

Assume for simplicity that θ is even. Let c be any vertex in G. For $0 \leq i < \theta/2$, define Z_{2i} to be the union of all atoms of distance at most $2i$ from c. (Notice that, in radial graphs, all closed walks have even length.) By Theorem 8.10 of [31], Z_{2i} is a nonempty simply connected set, for all i. (A patch of a surface is *simply connected* if it has no noncontractible closed curves.) Thus, the boundary ∂Z_{2i} of each Z_{2i} is a closed walk in the radial graph.

We claim that the closed walks ∂Z_{2i} and ∂Z_{2i+2} are vertex-disjoint. Consider any atom a on ∂Z_{2i} and an adjacent atom b outside Z_{2i}. The distance between a and b is 2 because there is a length-2 closed walk connecting them, doubling the edge (a, b). By Theorem 9.1 of [31], the metric satisfies the triangle inequality, and hence $d(s, b) \leq d(s, a) + 2 = 2i + 2$. In fact, this bound must hold with equality, because $b \notin Z_{2i}$. Therefore, every atom a on ∂Z_{2i} is surrounded on the exterior of Z_{2i} by atoms at distance exactly $2i + 2$ from c, so ∂Z_{2i} is strictly enclosed by ∂Z_{2i+2}.

Consider the "annulus" $\mathcal{A} = Z_{2\theta-2} - Z_{\theta}$. We claim that there are at least $\theta/2$ vertex-disjoint paths in the radial graph connecting vertices in ∂Z_{θ} to vertices in $\partial Z_{2\theta-2}$. By Menger's Theorem, the contrary implies the existence of a cut in $\mathcal{A}$ of size $< \theta/2$ separating the two sets, implying the existence of a cycle of length $< \theta$, but such a cycle must be contained in Z_{θ}.

Now we form a $(\theta/2 \times \theta/2)$-grid in the radial graph. The rows in the grid are formed by taking cycles enclosing c that are subsets of the closed walks ∂Z_{2i} for $i = \theta, \theta + 2, \theta + 4, \ldots, 2\theta - 2$. The column lines in the grid are formed by the $\theta/2$ vertex-disjoint paths found above. Thus we obtain a subdivision of the $(\theta/2 \times \theta/2)$-grid as a subgraph of the radial graph.

Finally, we transform this grid into a $(\theta/4 \times \theta/4)$-grid in the original graph G. Each grid edge in the radial graph corresponds in the original graph to a sequence of faces surrounding the edge. We replace this grid edge by the upper half of each face. In this way, each row in the radial graph maps in the original graph to a curve above this row. Two adjacent mapped rows may touch but cannot properly cross, so rows of distance 2 or more cannot overlap. Thus, by discarding the odd-numbered rows, and similarly for the columns, we obtain a subdivision of the $(\theta/4 \times \theta/4)$-grid in the original graph. Because each Z_{2i} was simply connected, the grid is embedded in a planar patch on Σ, so if we apply contractions without deletions, we obtain a partially triangulated grid. □

[3]Informally, the radial graph of a 2-cell embedded in Σ graph G is the bipartite graph R_G obtained by selecting a point in every region r of G and connecting it to every vertex of G incident to that region. However, a region maybe "incident more than once" with the same vertex, so one needs a more formal definition. A *radial drawing* R_G is a radial graph of a 2-cell embedded in Σ.

The Euler genus $\mathbf{eg}(\Sigma)$ of a nonorientable surface Σ is equal to the nonorientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The Euler genus $\mathbf{eg}(\Sigma)$ of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ.

The following lemma is very useful in proofs by induction on the genus. The first part of the lemma follows from Lemma 4.2.4 (corresponding to nonseparating cycle) and the second part follows from Proposition 4.2.1 (corresponding to surface separating cycle) in [26].

Lemma A.3. *Let G be a connected graph 2-cell embedded in a non-planar surface Σ, and let N be a noncontractible noose on G. Then there is a fair splitting G' of G affecting the set $S = (v_1, \ldots, v_\rho)$ of the vertices of G met by N such that one of the following holds*
- *G' can be 2-cell embedded in a surface with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$.*
- *each connected component G_i of G' can be 2-cell embedded in a surface with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$ and is a contraction of some graph G_i^* obtained from G after $\leq \rho$ splittings.*

Proof of Theorem 3.3. We use induction on the Euler genus of Σ.

In case $\mathbf{eg}(\Sigma) = 0$, Lemma 3.1 implies that if $P(G) < (\frac{\delta}{\beta} r)^2$, then G excludes the $(r \times r)$-grid as a minor. Indeed, this is obvious in case P is minor bidimensional. If P is contraction bidimensional, then it is enough to observe that if the planar graph G can be transformed to H via a sequence of edge contractions or removals, then by applying only the contractions in this sequence we get a partial triangulation of H. Using now Theorem 2.1 we get that if $P(G) < (\frac{\delta}{\beta} r)^2$, then $\mathbf{bw}(G) \leq 4r - 6$. If we set $r = \lfloor \frac{\beta}{\delta} \sqrt{P(G)} \rfloor + 1$, we have that $\mathbf{bw}(G) \leq 4 \lfloor \frac{\beta}{\delta} \sqrt{P(G)} \rfloor - 2$. As $\alpha, \beta, \delta \geq 0$, the induction base is done.

Suppose now that $\mathbf{eg}(\Sigma) \geq 1$ and that induction hypothesis holds for any graph 2-cell embedded in a sphere with Euler genus less than $\mathbf{eg}(\Sigma)$. Let G be a graph embedded in Σ. We set $k = P(G)$ and we claim that the representativity of G is $\leq 4 \lfloor \frac{\beta}{\delta} \sqrt{k+1} \rfloor$. Lemma 3.1 implies that if $k < (\frac{\delta}{\beta} r)^2$, then G excludes any triangulation of the $(r \times r)$-grid as a contraction. By the contrapositive of Lemma A.2, this implies that the representativity of G is $< 4r$. If we set $r = \lfloor \frac{\delta}{\beta} \sqrt{k+1} \rfloor + 1$, we have that the representativity of G is $\leq 4 \lfloor \frac{\beta}{\delta} \sqrt{k+1} \rfloor$. Let N be a minimum size non-contractible noose N on Σ meeting ρ vertices of G where $\rho \leq 4 \lfloor \frac{\beta}{\delta} \sqrt{k+1} \rfloor$. By Lemma A.3, there is a fair splitting along the vertices met by N such that one of the conditions (1) or (2) holds. Let G' be the resulting graph and let Σ' be a sphere such that $\mathbf{eg}(\Sigma') \leq \mathbf{eg}(\Sigma) - 1$ and every component of G' is 2-cell embedable in Σ'. We claim that in each of the cases (1), (2), $\mathbf{bw}(G') \leq 4 \frac{\beta}{\delta} \mathbf{eg}(\Sigma) \sqrt{k + \alpha\rho + 1} + 8\alpha(\frac{\beta}{\delta})^2 (\mathbf{eg}(\Sigma))^2$.

Case (1): We apply the induction hypothesis on G' and get that $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma') + 1)\sqrt{P(G') + 1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma') + 1)^2$. As G' is obtained from G after $\leq \rho$ splittings and P is an α-splittable parameter, we have $P(G') \leq k + \alpha\rho$. Taking in mind that $eg(\Sigma') \leq \mathbf{eg}(\Sigma) - 1$, we obtain $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k + \alpha\rho + 1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$.

Case (2): We apply the induction hypothesis on each of the connected components of G. Let G_i be such a component. We get that $\mathbf{bw}(G_i) \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma') + 1)\sqrt{P(G_i) + 1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma') + 1)^2$. As G_i is a contraction of some graph G_i^* obtained from G after $\leq \rho$ splittings and P is an α-splittable parameter, we get that $P(G_i) \leq P(G_i^*) \leq k + \alpha\rho$. Again since $\mathbf{eg}(\Sigma') \leq \mathbf{eg}(\Sigma) - 1$, we have $\mathbf{bw}(G_i) \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k + \alpha\rho + 1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$. Notice that $\mathbf{bw}(G') = \max_i(\mathbf{bw}(G_i))$ which in turn implies that $\mathbf{bw}(G') \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k + \alpha\rho + 1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2$. As G' is the result of ρ consecutive vertex splittings on G and the splitting operation cannot increase the branchwidth more than one we get that $\mathbf{bw}(G) \leq \mathbf{bw}(G') + \rho$. Therefore, $\mathbf{bw}(G) \leq 4\frac{\beta}{\delta}\mathbf{eg}(\Sigma)\sqrt{k + \alpha\rho + 1} + 8\alpha(\frac{\beta}{\delta})^2(\mathbf{eg}(\Sigma))^2 + \rho \leq 4\frac{\beta}{\delta}(\mathbf{eg}(\Sigma) + 1)\sqrt{k+1} + 8\alpha(\frac{\beta}{\delta}(\mathbf{eg}(\Sigma) + 1))^2$. □

B Proof of Theorem 4.3

In order to prove Theorem 4.3 we need first some preliminary results. A vertex w is called *r-dominated* by a set S, if the distance from w to a vertex $v \in S$ is at most r. An *r-dominating set* is a set S of vertices such that every vertex of the graph is r-dominated by S. The problem of finding an r-dominating set of size k is also called the (k, r)-center problem (see Section 5). ¿From the main combinatorial result of [11], r-dominating set is a 1-splittable bidimensional parameter. This and Theorem 3.3 imply the following.

Lemma B.1. *For any constant r, if a graph G of genus g has an r-dominating set of size at most k, then the treewidth of G is at most $O(g\sqrt{k}+g^2)$.*

Now, we extend this result for apex-free h-almost embeddable graphs (the proof is not hard and is omitted).

Lemma B.2. *Consider an apex-free h-almost-embeddable graph $G = G_0 \cup G_1 \cup \cdots \cup G_h$. Suppose further that, for each $1 \le i \le h$, $U_i = \{u_i^1, u_i^2, \ldots, u_i^{m_i}\}$ forms a path in G_0. Then $\mathbf{tw}(G) \le (h^2+1)(\mathbf{tw}(G_0)+1)-1$.*

Lemma B.3. *For any constant r, an apex-free h-almost-embeddable graph G embedded on a surface of genus g with a set $S \subset V(G)$ of size at most k which r-dominates every vertex of G which is not in a vortex has treewidth at most $O(h^2 g\sqrt{k+h}+g^2) = O(g\sqrt{k})$ (g and h are constants).*

Proof. Consider an apex-free h-almost embeddable graph $G = G_0 \cup G_1 \cup \cdots \cup G_h$ in a surface Σ of genus g. Suppose $U_i = \{u_i^1, u_i^2, \ldots, u_i^{m_i}\}$. Let G_0' be the graph obtained from G_0 by adding new vertices $c_1, c_2, \cdots, c_h$ and edges (c_i, u_i^j) and (u_i^j, u_i^{j+1}) (where $j+1$ is treated modulo m_i) for all $1 \le i \le h$ and $1 \le j \le m_i$. Notice that by adding these edges, vertices U_i, $1 \le i \le h$, form a path in G_0. If G has the aforementioned r-dominating set of size k, then G_0' has an r-dominating set of size at most $k+h$: just delete all vertices in the r-dominating set that are in $G_i - G_0$, $1 \le i \le h$, and add instead all new vertices $c_1, c_2, \cdots, c_h$ to the r-dominating set. Notice that G_0' is embeddable on Σ, since G_0 is embeddable. Thus, according to Lemma B.1 it has treewidth at most $O(g\sqrt{k+h}+g^2)$. By Lemma B.2, the treewidth of $G' = G_0' \cup G_1 \cup \cdots \cup G_h$ is $O((h^2+1)(g\sqrt{k+h}+1)+g^2-1)$. U_i forming a path in G_0. Because G is a subgraph of G', the lemma follows. □

Proof of Theorem 4.3. Suppose X is the set of apices in G, so that $G - X$ is an apex-free h-almost embeddable graph. Let D be a dominating set of size k of G^* and let $S = X \cap D$. We claim that S is our desired set. The rest of the proof is as follows: we construct a set $\hat{D}$ of size at most k for $\hat{G} - X$ which 2-dominates every vertex v of $\hat{G} - X$ which is not included in any vortex. Then since $\hat{G} - X$ is an apex-free h-almost-embeddable on a surface of genus g with a 2-dominating-type set of size at most k desired by Lemma B.3, it has treewidth at most $O(h^2 g\sqrt{k+h}+g^2)$. Then we can add vertices of X to all bags and still have a tree decomposition of width $O(h^2 g\sqrt{k+h}+g^2)$, as desired. We construct $\hat{D}$ from D as follows. First, we set $\hat{D} = D \cap V(G)$. For each $1 \le i \le p$, if $D \cap (V(G_i) - W_i) \neq \emptyset$ and $W_i \not\subseteq X$, we add an arbitrary vertex $w \in W_i - X$ to $\hat{D}$. Here we say a vertex v of D is *mapped* to a vertex w of $\hat{D}$ if $v = w$ or if $v \in D \cap (V(G_i) - W_i)$ and vertex $w \in W_i - X$ is the one that we have added to $\hat{D}$. One can easily observe that since each new vertex in $\hat{D}$ is in fact accounted by a unique vertex in D, $|\hat{D}| \le k$. It only remains to show that D is a 2-dominating set for $\hat{G} - X$. If a vertex $v \in V(\hat{G}) - X$ is not fully-dominated, then there exists a vertex $w \in N_G(v)$ which is not dominated by S and thus not dominated by X (since $S = D \cap X$). It means v is 2-dominated by a vertex u of $\hat{G} - X$ which dominates w (we note that u can be originally a vertex u' in $(V(G_i) - W_i) \cap D$ which is mapped to u in $\hat{D}$). Also, we note that for each clique W_i in which there is a mapped vertex of D, this vertex dominates all vertices of $W_i - X$ in $\hat{G} - X$ and thus we keep the whole clique $W_i - X$ in G. It only remains to show that every vertex of a partially dominated clique W_i is 2-dominated by a vertex of $\hat{G} - X$. We consider two cases: if $W_i \cap S = \emptyset$, since $V(G_i) - W_i \neq \emptyset$, there must exists a (mapped) vertex of $\hat{D}$ in $W_i - X$ and we are done. Now assume $W_i \cap S \neq \emptyset$. If $W_i \subset X$ then $W_i \cap (V(\hat{G}) - X) = \emptyset$ and we are done (since there is no clique in $\hat{G} - X$ at all.) Otherwise, there exists a vertex $W_i - X$. If $(V(G_i) - W_i) \subseteq N_{G^*}(S) \neq \emptyset$, then $V(G_i) \cap D \neq \emptyset$. Thus there exists a mapped vertex $w \in W_i - X$ and we have 1-dominated vertices of $W_i - X$. As mentioned before if $D \cap (W_i - X) \neq \emptyset$, vertices $W_i - X$ are 1-dominated and we are done. The only remaining case is the case in which there exists a vertex $w \in W_i - X$ which is dominated by a vertex $x \in V(G)$ and by assumption $w \notin N_{G^*}(S)$ (we note that in this case, there is no dominating vertex in $V(G_i) - W_i$ for any i for which $w \in W_i$.) It means vertex x is not fully dominated and thus it remains in $\hat{G}$. In addition, vertex x 2-dominates all vertices of

$W_i - X$, since W_i is a clique in G and thus all vertices of $W_i - X$ are 2-dominated. This completes the proof of the theorem. □