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Michał Walicki and Uwe Wolter

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Department of Informatics

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Abstract

Multialgebras are relational structures where selection of one argument as the "result" yields strong algebraic properties missing in the case of relations. However, such properties can be obtained only by choosing an appropriate definition of homomorphism and this question has been neglected or left implicit in most of the literature on power structures. We summarize our earlier results on the possible notions of compositional homomorphisms of multialgebras and investigate in detail one of them, the outer-tight homomorphisms which yield rich structural properties not offered by other alternatives. A series of classical algebraic properties is demonstrated for the resulting category and the notion of associated congruence – bireachability, which is dual to bisimulation equivalence – is presented. The category is cocomplete. Final objects have quite interesting nature but, unfortunately, are not guaranteed to exist. To guarantee the existence of final objects, we have to extend the category by admitting algebras over proper classes, in the same way as it has to be done for coalgebras involving power-set functor. We give an exact characterisation of the large objects as colimits of small algebras or, equivalently, as algebras were each element is reachable from at most a set of other elements. Finally, we give a construction of products (which has not been given in the earlier version of the report). A particular case gives construction of products for coalgebras over power-set functor. The results (for the category of small algebras) extend to this category which is thus complete and cocomplete. The category of small algebras may lack final objects but possesses other limits and all colimits. Examples and remarks illustrate relations to total and partial algebras, coalgebras, automata theory and topology.

This report extends and completes the results from – and, repeating also the previous results, replaces – the earlier report no 292.

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Contents

1 Background

Multialgebras are algebras where operations can return not only single values but also sets thereof. Multialgebras, or variants of power-set structures, have been given some attention in the mathematical community, e.g., [35, 36, 13, 38, 6, 29], with [5] presenting a comprehensive overview. The seminal work here was [24, 25] which introduced algebras of complexes for representation of relational structures and demonstrated general representability of Boolean algebras with operators by such algebras. As Kripke-frames are naturally represented by such algebras, their relevance for modal logic has also been acknowledged, e.g., [4], if not widely recognized. (The works of McKinsey and Tarski, [30, 31, 32], provided the semantics for S4 logic directly in terms of Boolean algebras with closure operator.) Likewise, automata can be modeled as multialgebras where the power-set operation allows for a natural inclusion of nondeterminism. In the tradition of algebraic specifications, multialgebras have been used as an extension of algebraic semantics precisely for the purpose of modelling nondeterminism, e.g., [19, 20, 22, 43, 45]. In this context, it is important to distinguish between arbitrary sets and one-element sets (nondeterministic operations vs. usual functions), as well as to pay attention to the distinction between sets being second-order or first-order objects – the former corresponds to multialgebras (application of operations to sets is obtained by pointwise extension and hence is monotone) and the latter to power-set algebras (where operation applied to a larger set may yield a smaller result) – the distinction was investigated and used in [44, 46]. Some variants of multialgebras disallow empty result-sets, e.g., [43, 13], but most do not. Then, applying the standard isomorphism

$$A_1 \times \ldots \times A_n \to \mathcal{P}(A) \;\simeq\; \mathcal{P}(A_1 \times \ldots \times A_n \times A), \tag{1.1}$$

one obtains another representation of relational structures, although with more algebraic properties, as will be observed below. This is the variant of multialgebras we will be using.

Following [17] (definition 3.1.2), a one-sorted multialgebraic operation α over a set A can be seen as a dialgebra $\langle A, \alpha\rangle$, namely, a function $\alpha : F(A) \to \mathcal{P}(A)$, where the functor $F : \mathsf{SET} \to \mathsf{SET}$ gives the source of the operation and $\mathcal{P} : \mathsf{SET} \to \mathsf{SET}$ is the covariant existential-image power-set functor, i.e., sending a function $f : A \to B$ onto $\mathcal{P}(f)(X) = \{f(x) \mid x \in X\}$, for $X \subseteq A$. Although we will not use this model of multialgebras, we may occasionally refer to it. [40] presents a series of basic facts about dialgebras (called "bialgebras") which can be instantiated to either algebraic or colagebraic version depending on the choice of the functors. In general, instead of $\mathcal{P}$ one can use any endofunctor $G : \mathsf{SET} \to \mathsf{SET}$ and a morphism $\langle A, \alpha\rangle \to \langle B, \beta\rangle$ in the category SET^F_G is a function $f : A \to B$, such that $F(f); \alpha = \beta; G(f)$. The variations in the definitions of homomorphisms to be encountered below could be then seen as variations over this notion of morphism (requiring, in addition, lax transformations). Less abstractly, we can use the isomorphism (1.1), and view a multialgebra as a relational structure where, for each relation, one argument is designated as its "result" and used for composing the relation with others. This composition is obtained by pointwise extension.

Definition 1.2 *For a signature $\Sigma = \langle \mathcal{S}, \mathcal{F}\rangle$, a Σ-multialgebra M is given by:*

- *a (family of) carrier set(s) $|M| = \{s^M \mid s \in \mathcal{S}\}$,*
- *a function $f^M : s_1^M \times \ldots \times s_n^M \to \mathcal{P}(s^M)$ for each $f : s_1 \times \ldots \times s_n \to s \in \mathcal{F}$, with composition defined through additive extension to sets, i.e. $f^M(X_1, \ldots, X_n) = \bigcup_{x_i \in X_i} f^M(x_1, \ldots, x_n)$.*

We will not distinguish in the notation between an algebra A and its carrier. Expressions involving set operations, e.g., $x \in A, X \subseteq A$, suggest that the carrier of A is meant. The only structures addressed in the paper are multialgebras, so "multialgebra" and "algebra" will be used interchangeably. We assume a given signature with f/R ranging over all operation/relation symbols.

Selection of the "result" argument corresponds, in a sense, to turning our considerations to binary relations with the additional operation of tupling the arguments. Composition of relations $R_1 : X_{11} \ldots X_{1n} \to X_1, \ldots, R_k : X_{k1} \ldots X_{kn} \to X_k$ and $R : X_1 \ldots X_k \to X$, corresponds to application of R to the tupling $\langle R_1 \ldots R_k\rangle$. When using relational notation, we write composition in diagrammatic order, $R; \phi$, resp. $\phi; R$, assuming implicitly ϕ to be binary (homomorphism or, strictly speaking, a tuple $\langle \phi_1, \ldots, \phi_{n+1}\rangle$ of unary functions, for each relevant argument/sort i.) The composition is, as just explained, an abbreviation for the multialgebraic one, i.e.:

$$\begin{array}{rlcl} & \langle\langle a_1...a_n\rangle, b\rangle \in R;\phi & \iff & \exists a : \langle\langle a_1...a_n\rangle, a\rangle \in R \wedge \langle a,b\rangle \in \phi_{n+1} \\ resp. & \langle\langle a_1...a_n\rangle, b\rangle \in \phi;R & \iff & \exists b_1...b_n : \langle a_i,b_i\rangle \in \phi_i \wedge \langle\langle b_1...b_n\rangle, b\rangle \in R \end{array} \tag{1.3}$$

Having made these precautions, we will write things as if all relations were binary, (most) algebras were one-sorted and homomorpisms simple functions (and not their families), but all considerations apply to the general case. (Occasionally, we may write argument sequences explicitly.)

Selection of the "result" among the relational arguments leads to more algebraic structure reflected by homomorphisms. (In particular, derived operators of a multialgebra are analogous to those of classical algebra, so that for a given signature Σ, the term structure T_Σ is itself a Σ-algebra, and preservation/reflection of Σ operations leads to the corresponding behaviour of the derived operators. For relational structures (without specified composition argument), on the other hand, derived operators are just boolean operators which are only very weakly related to the actual signature and need not be preserved by homomorphisms preserving the basic relations. [10], V.3, p.203, considers this the reason for the subordinate role of homorphisms in the study of relational structures.) This, however, does not simplify the study of the resulting structure – the number of possible definitions of homomorphisms, congruences, etc. hardly diminishes.

As the first step towards a simplification of the rather complicated picture, we have earlier in [42] classified compositional homomorphisms of (relational structures modeled as) multialgebras. In order to motivate our choice of the outer-tight homomorphisms, we recall now these results and in 1.2 review finite (co)completeness of the respective categories. Section 2 studies the basic algebraic notions (congruences, subalgebras) in the category of multialgebras with outer-tight homomorphisms. Section 3 extends this category allowing algebras with proper classes as carriers and shows its cocompleteness and existence of equalizers and final objects. Section 4 demonstrates then the existence of products, thus completing the proof of completeness of the category. Except for the existence of final objects, all other constructions can be performed also (for small diagrams) in the category of small algebras. Section 5 contains a brief summary and suggestions for further development. The appendix 6 summarizes the main assumptions used in the treatement of classes.

1.1 Compositional homomorphisms of multialgebras

Besides some preservation properties, the first minimal requirement for a definition of homomorphism seems to be compositionality: composition of two homomorphisms should yield a homomorphism. In fact, various proposed definitions (used to obtain specific results) violate this requirement. We therefore start by inquiring into the possible compositional definitions.

Definition 1.4 *A definition $\Delta[\_]$ of a function $\phi : A \to B$ being a homomorphism of the multialgebraic structures $A \to B$ has the form:*

$$\Delta[\phi] \iff l_1[\phi]; R^A; r_1[\phi] \quad \bowtie \quad l_2[\phi]; R^B; r_2[\phi]$$

where $l[\_]$'s and $r[\_]$'s are relational expressions (using only relational composition and converse), and $\bowtie \in \{=, \subseteq, \supseteq\}$.

One can certainly consider other formats but most proposed definitions of homomorphisms conform to this one as do, in particular, all compositional definitions which we have encountered.[1]

Definition 1.5 *A definition is* compositional *iff for all $\phi : A \to B$, $\psi : B \to C$, we have $\Delta[\phi]$ & $\Delta[\psi] \Rightarrow \Delta[\phi;\psi]$, i.e.:*

$$\begin{array}{lclcl} l_1[\phi]; R^A; r_1[\phi] & \bowtie & l_2[\phi]; R^B; r_2[\phi] \ \ \& & & \\ l_1[\psi]; R^B; r_1[\psi] & \bowtie & l_2[\psi]; R^C; r_2[\psi] & & \\ & \Rightarrow & l_1[\phi;\psi]; R^A; r_1[\phi;\psi] & \bowtie & l_2[\phi;\psi]; R^C; r_2[\phi;\psi] \end{array}$$

[1] Of course, one can consider homomorphisms which are themselves relations, but such a generalisation goes beyond the scope of the present investigation.

The number of syntactic expressions of the kind $l[\phi]$ is infinite, however, since homomorphisms are functions we have the simple fact:

Fact 1.6 *a)* $\phi^-;\phi;\phi^- = \phi^-$ *b)* $\phi;\phi^-;\phi = \phi$ *c)* $\phi^-;\phi = id_{\phi[A]}$

Thus the length of each of the expression $l[\phi]$, resp. $r[\phi]$ (measured by the number of occurring ϕ's or ϕ^-'s) can be limited to 2.

On the other hand, both sides of definition 1.4 must yield relational expressions of the same type, i.e., of one of the four types $A \times A, A \times B, \ldots$, which will be abbreviated as $AA, AB, \ldots$

For each choice of $\bowtie$, this leaves us with four possibilities for each type. For instance, for AB we have the following four possibilities:

$\top_{AB} : \phi;\phi^-;R^A;\phi \bowtie \phi;R^B;\phi^-;\phi$ $\qquad \perp_{AB} : R^A;\phi \bowtie \phi;R^B$

$E_{AB} : \phi;\phi^-;R^A;\phi \bowtie \phi;R^B$ $\qquad W_{AB} : R^A;\phi \bowtie \phi;R^B;\phi^-;\phi$

The symbols denoting the respective possibilities are chosen for the following reason. Relational composition preserves each of the relations $\bowtie$, i.e., given a particular choice of $\bowtie$ and any relations C, D (of appropriate type), we have: $R_1 \bowtie R_2 \Rightarrow C;R_1 \bowtie C;R_2$ and $R_1 \bowtie R_2 \Rightarrow R_1;D \bowtie R_2;D$. Starting with $\perp_{AB}$ and pre-composing (on the "*E*ast") both sides of $\bowtie$ with $\phi;\phi^-;(\_)$, we obtain E_{AB}; post-composing (on the "*W*est") both sides of $\bowtie$ with $(\_);\phi^-;\phi$, we obtain W_{AB}. Dual compositions lead from there to $\top_{AB}$. Thus we have that $\perp_{AB} \Rightarrow E_{AB}, W_{AB} \Rightarrow \top_{AB}$ and the corresponding lattices are obtained for the other three types starting, respectively, with

$$\perp_{AA} : R^A \bowtie \phi;R^B;\phi^- \qquad \perp_{BB} : \phi^-;R^A;\phi \bowtie R^B \qquad \perp_{BA} : \phi^-;R^A \bowtie R^B;\phi^-$$

Figure 1.7 shows the four lattices for each type (the choice of $\bowtie$ is uniform for all four).

$$\begin{tikzcd}
& \top_{AB} \arrow[rrrr, <->, dotted] & & & & \top_{AA} & \\
E_{AB} \arrow[ur, "(_);\phi^-;\phi"] \arrow[ddd, <->, dotted, "\phi;(_)", "\phi^-;(_)"'] & & W_{AB} \arrow[ul, "\phi;\phi^-;(_)"'] \arrow[rr, <->, dotted, "(_);\phi", "(_);\phi^-"'] & & W_{AA} \arrow[ur, "\phi;\phi^-;(_)"] & & E_{AA} \arrow[ul, "(_);\phi;\phi^-"'] \arrow[ddd, <->, dotted, "\phi;(_)"', "\phi^-;(_)"] \\
& R^A;\phi \bowtie \phi;R^B \arrow[ul, "\phi;\phi^-;(_)"] \arrow[ur, "(_);\phi^-;\phi"'] \arrow[uu, dotted] \arrow[d, dotted] & & & & R^A \bowtie \phi;R^B;\phi^- \arrow[ul, "(_);\phi;\phi^-"] \arrow[ur, "\phi;\phi^-;(_)"'] \arrow[uu, dotted] \arrow[d, dotted] & \\
& \top_{BB} \arrow[rrrr, <->, dotted] & & & & \top_{BA} & \\
E_{BB} \arrow[ur, "(_);\phi^-;\phi"] & & W_{BB} \arrow[ul, "\phi^-;\phi;(_)"'] \arrow[rr, <->, dotted, "(_);\phi", "(_);\phi^-"'] & & W_{BA} \arrow[ur, "\phi^-;\phi;(_)"] & & E_{BA} \arrow[ul, "(_);\phi;\phi^-"'] \\
& \phi^-;R^A;\phi \bowtie R^B \arrow[ul, "\phi^-;\phi;(_)"] \arrow[ur, "(_);\phi^-;\phi"'] & & & & \phi^-;R^A \bowtie R^B;\phi^- \arrow[ul, "(_);\phi;\phi^-"] \arrow[ur, "\phi^-;\phi;(_)"'] &
\end{tikzcd}$$

Figure 1.7: Lattices for each relation type (for each choice of $\bowtie$).

The additional equivalences (indicated with dotted arrows) are easily verified using the fact that composition preserves each of $\bowtie$ and Fact 1.6. Also all the top definitions are equivalent which follows by simple calculation.

These observations simplify the picture a bit, leading, for each choice of $\bowtie$, to the order of 9 possible definitions shown in figure 1.8.

When the mappings between the structures are, as in our case, functions and not arbitrary relations, several elements of the ordering from 1.8 collapse.

Fact 1.9 *All definitions (of the form 1.4) involving* $\subseteq$ *are equivalent.*

We are thus left with one definition involving $\subseteq$ and 18 other definitions obtained from two instances (with $=$, resp. $\supseteq$ for $\bowtie$) of the orderings in figure 1.8. The following, main theorem shows that only the bottom elements of these orderings yield compositional definitions.

Theorem 1.10 *A definition is compositional iff it is equivalent to one of:*

1) $R^A;\phi \bowtie \phi;R^B$ $\qquad$ 2) $\phi^-;R^A;\phi \triangleright R^B$ $\qquad$ 3) $\phi^-;R^A \triangleright R^B;\phi^-$ $\qquad$ 4) $R^A \triangleright \phi;R^B;\phi^-$

where $\bowtie \in \{=, \subseteq, \supseteq\}$ *and* $\triangleright \in \{=, \supseteq\}$.

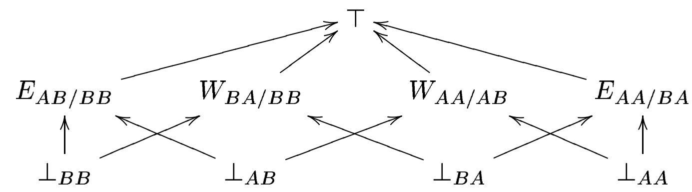


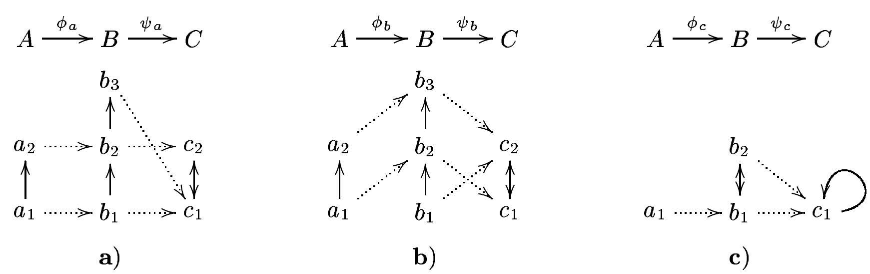
Figure 1.8: Possible definitions (for a given choice of $\bowtie$).

PROOF: For the "if" part, one easily checks that 1)–4) do yield compositional definitions. In fact, this part of the theorem holds for *any* transitive set-relation $\bowtie$. For instance, for 1) we verify:

$$
\begin{array}{rrclcrcl}
 & \phi^-;R^A & \bowtie & R^B;\phi^- & \& & \psi^-;R^B & \bowtie & R^C;\psi^- \\
\Rightarrow & \psi^-;\phi^-;R^A & \bowtie & \psi^-;R^B;\phi^- & \& & \psi^-;R^B;\phi^- & \bowtie & R^C;\psi^-;\phi^- \\
 & & & & \Rightarrow & (\phi;\psi)^-;R^A & \bowtie & R^C;(\phi;\psi)^-
\end{array}
$$

The "only if" part is shown providing counter-examples for the remaining possibilities. Although there are 10 cases left, they are easily shown by the following three counter-examples. In all cases, the given homomorphisms ϕ, ψ satisfy the respective definition with $=$ for $\triangleright$ (hence, also for $\supseteq$), while their composition does not satisfy the respective definition with $\supseteq$ for $\triangleright$. Thus we obtain immediately counter-examples for both $\triangleright \in \{=, \supseteq\}$.

Vertical arrows represent the relation (R) in respective multialgebras; the dotted arrows illustrate the images under the respective homomorphisms:



a) for $W_{BB} : \phi^-;R^A;\phi \ \triangleright\ R^B;\phi^-;\phi$. We have: $\phi_a^-;R^A;\phi_a = R^B;\phi_a^-;\phi_a$ and $\psi_a^-;R^B;\psi_a = R^C;\psi_a^-;\psi_a$. However, for the composition $\rho_a = \phi_a;\psi_a$, we have $\langle c_2, c_1\rangle \in R^C;\rho_a^-;\rho_a$ but $\langle c_2, c_1\rangle \notin \rho_a^-;R^A;\rho_a$, i.e., $\rho_a^-;R^A;\rho_a \not\supseteq R^C;\rho_a^-;\rho_a$.

b) for $E_{BB} : \phi^-;R^A;\phi \triangleright \phi^-;\phi;R^B$ is quite analogous. $\phi_b^-;R^A;\phi_b = R^B;\phi_b^-;\phi_b$ and $\psi_b^-;R^B;\psi_b = R^C;\psi_b^-;\psi_b$, but $\rho_b^-;R^A;\rho_b \not\supseteq \rho_b^-;\rho_b;R^C$ with $\langle c_2, c_1\rangle$ as a witness to this negation.

Both these examples can also be used as counter-examples for compositionality of $\top$, represented by $\top_{BB}$. For instance, in the first case, we have $R^B;\phi_a^-;\phi_a = \phi_a^-;\phi_a;R^B;\phi_a^-;\phi_a$ and the corresponding equality holds for ψ_a and R^C – so exactly the same argument yields a counter-example also for this case.

c) $W_{AA/AB}$ and $E_{AA/BA}$: ϕ_c and ψ_c are obviously W_{AB}: $R^A;\phi_c = \phi_c;R^B;\phi_c^-;\phi_c$ and $R^B;\psi_c = \psi_c;R^C;\psi_c^-;\psi_c$. But their composition gives: $\emptyset = R^A;\rho_c \not\supseteq \rho_c;R^C;\rho_c^-;\rho_c = \langle c_1, c_1\rangle$. This gives also counter-example for $E_{BA} : \phi_c^-;R^A \triangleright \phi_c^-;\phi_c;R^B;\phi_c^-$. □

Table 1.10 summarises the naming conventions for the compositional cases. The name consists of two parts, the first (inner/left/...) indicating one of the four main cases in the theorem and the second (closed/tight/weak) the choice of the set relation.

[11] studied in detail the four cases of weak morphisms as models of simulations between data types. However, as we observed in lemma 1.9, these four cases coincide when the morphisms are, as in our case, functions and not arbitrary relations, as in [11].

	$R^A;\phi \bowtie \phi;R^B$	$\phi^-;R^A;\phi \triangleright R^B$	$\phi^-;R^A \triangleright R^B;\phi^-$	$R^A \triangleright \phi;R^B;\phi^-$
	inner	left	outer	right
closed	$\mathsf{MAlg}_{IC}(\Sigma)$: $R^A;\phi \supseteq \phi;R^B$	$\mathsf{MAlg}_{LC}(\Sigma)$: $\phi^-;R^A;\phi \supseteq R^B$	$\mathsf{MAlg}_{OC}(\Sigma)$: $\phi^-;R^A \supseteq R^B;\phi^-$	$\mathsf{MAlg}_{RC}(\Sigma)$: $R^A \supseteq \phi;R^B;\phi^-$
tight	$\mathsf{MAlg}_{IT}(\Sigma)$: $R^A;\phi = \phi;R^B$	$\mathsf{MAlg}_{LT}(\Sigma)$: $\phi^-;R^A;\phi = R^B$	$\mathsf{MAlg}_{OT}(\Sigma)$: $\phi^-;R^A = R^B;\phi^-$	$\mathsf{MAlg}_{RT}(\Sigma)$: $R^A = \phi;R^B;\phi^-$
weak	$\mathsf{MAlg}_{W}(\Sigma) : R^A;\phi \subseteq \phi;R^B$			

Table 1.10: Compositional homomorphisms

1.2 (Finite) completeness and cocompleteness

Earlier study of finite (co)completeness of resulting categories, [42], is summarized in table 1.11.

	initial	co-prod.	co-equal.	final	prod.	equal.
$\mathsf{MAlg}_{W}(\Sigma)$	+	+	+	+	+	+
$\mathsf{MAlg}_{IC}(\Sigma)$	−	−	−	+	−	−
$\mathsf{MAlg}_{IT}(\Sigma)$	−	−	+	−	−	−
$\mathsf{MAlg}_{LC}(\Sigma)$	−	−	+	+	−	−
$\mathsf{MAlg}_{LT}(\Sigma)$	−	−	+	−	−	−
$\mathsf{MAlg}_{OC}(\Sigma)$	+	+	−	+	−	+
$\mathsf{MAlg}_{OT}(\Sigma)$	+	+	+	+/−	?	+
$\mathsf{MAlg}_{RC}(\Sigma)$	+	+	+	+	+	+
$\mathsf{MAlg}_{RT}(\Sigma)$	+	−	−	−	−	+

Table 1.11: Finite limits and colimits in the categories of multialgebras

The present paper addresses the category of outer-tight homomorphisms (the double row) and, in particular, provides the full answers to the places marked +/− and ?. First, however, a few words about the possible alternatives.

Remark 1.12 *Viewing (binary) relations as coalgebras for the existential image power-set functor ($\mathcal{P}(f)(X) = \bigcup_{x\in X} f(x)$), yields the homomorphism condition $R^A;\phi = \phi;R^B$, that is, the inner-tight homomorphisms. As we see from the table, the category $\mathsf{MAlg}_{IT}(\Sigma)$ has rather few (co)limits. This, of course, looks suspicious, since we know from [37] that this category of coalgebras over sets will be, at least, cocomplete. The difference is, however, due to the fact that although the homomorphism conditions look the same, the respective representations of relations are not:*

The absence of final objects is here due to the fact that the table addresses only categories based on sets. The non-existence of colimits is due to the algebraic character of operations, in particular, constants which correspond to predicates. (Restriction to signatures containing only *binary relations would yield the same category as coalgebras mentioned in the first line of this remark.) For instance, for a signature with a single sort and constant $c :\to S$, the category $\mathsf{MAlg}_{IT}(\Sigma)$ has no intial multialgebra I. A multialgebraic constant is $c^I \subseteq S^I$, which corresponds to the arrow $c^I : \mathbf{1} \to \mathcal{P}(S^I)$, where $\mathbf{1}$ is a one-element set. Consequently, for any (in particular, empty) c^I there is no IT-homomorphism $\phi : I \to A$ making $\phi(c^I) = c^A$ when $|c^I| < |c^A|$. The desired equality $c^I;\phi = \phi;c^A$, for $I = \emptyset$, is achieved by viewing constants as coalgebraic arrows, namely, $c^I : S^I \to \mathbf{2}$ (with $\mathbf{2}$ being, for instance, $\{\bot, \top\}$). The two* *diagrams illustrate the difference.*

$$\begin{array}{ccc}
\emptyset & \xrightarrow{\phi} & A \\
\\
\mathbf{2} & \xrightarrow{id_{\mathbf{2}}} & \mathbf{2} \\
{\scriptstyle c^I}\uparrow & & \uparrow{\scriptstyle c^A} \\
\emptyset & \xrightarrow{\phi} & A
\end{array}
\qquad\qquad
\begin{array}{ccc}
\emptyset & \xrightarrow{\phi} & A \\
\\
\mathcal{P}(\emptyset) & \xrightarrow{\mathcal{P}(\phi)} & \mathcal{P}(A) \\
{\scriptstyle c^I}\uparrow & & \uparrow{\scriptstyle c^A} \\
\mathbf{1} & \xrightarrow{id_{\mathbf{1}}} & \mathbf{1}
\end{array}$$

$$c^I; id_{\mathbf{2}} = \emptyset = \phi; c^A \qquad\qquad c^I; \mathcal{P}(\phi) = \emptyset \neq c^A = id_{\mathbf{1}}; c^A$$

The meaning of the condition is different in the two cases in that for coalgebras it requires equality of two functions while for multialgebras of two sets. As an example, take the carrier $X = \{1,2\}$ *and one constant* c*. Let, in a multilagebra* M*,* $c^M = \{1,2\}$*, while in a coalgebra* C*,* $c(1) = c(2) = \top$*. Let* $X' = \{1,2,3\}$ *and* $c^{M'} = \{1,2,3\}$ *while in a coalgebra* C'*,* $c'(1) = c'(2) = c'(3) = \top$*. Although both* M *and* C*, resp.,* M' *and* C' *represent the same predicates, the inclusion* $i : X \to X'$ *is a coalgebraic homomorphism, since indeed* $c; id_{\mathbf{2}} = i; c'$*, but it is not a multialgebraic IT-homomorphism since* $i(c^M) = i(\{1,2\}) = \{1,2\} \neq \{1,2,3\} = c^{M'}$*.*

This might be taken as a suggestion that the multialgebraic representation of relations is not the most sucessful one. However, using coalgebras as models of relations is by no means straightforward. For the first, one has to decide whether to use the functor $\mathcal{P}(X^n)$ or $\mathbf{2}^{(X^n)}$ – the difference in homomorphisms will be similar to that suggested in the above remark (between equality of sets and of functions). In either case one has to decide which power-set functor to use. Any choice involves sacrificing the pleasant and well understood behavior of polynomial functors. Additional complications arise if one wants to model many-sorted relations. (Although these are hardly theoretically demanding, they *are* complications, at least of the same order as in the case of many-sorted algebras.) Multialgebraic model, on the other hand, is in agreement with the traditional notion of relation/predicate as a subset. It deals with many-argument, as well as many-sorted, relations in the uniform and elementary way. In addition, one should also remark that multialgebras were introduced not merely as representations of relational structures but of Boolean algebras with operators and, on the other hand, as a generalisation of algebraic semantics to handle nondeterminism and partiality (most common institutions can be naturally embedded into the institution of multialgebras, with weak homomorphisms as morphisms in the model categories, [27]). The investigation of homomorphisms arises from this background and was motivated primarily by the search for the interesting canonical objects (initial or final) for algebraic specifications with nondeterminism.

Now, weak homomorphisms are those which are most commonly used. Unfortunately, this is an extremely weak notion which is also reflected in its name. Although the initial objects exist, they are of little interest having all predicates and relations empty. Lifiting existence of initial objects to the axiomatic classes depends, of course, on the language one wants to use, and this is by no means a clarified issue. Most approaches suggest, at least, the use of inclusions, but this again leads only to empty relations in the initial objects. Furthermore, even simplest formulae are not preserved. E.g., having two constants a, b interpreted in A as $\{1\}$, resp., $\{1,2\}$ makes $A \models a \subseteq b$. But the inclusion, which is a weak homomorphism, into B with $a^B = \{1,3\}$ and $b^B = \{1,2\}$ does not preserve this formula. Counterexamples can be easily found also when we restrict attention to preservation under homomorphic images. (Similar remarks apply to the other (co)complete category $\mathsf{MAlg}_{RC}(\Sigma)$.) One way would be to design a specific syntax ensuring adequate restrictions of the model classes, as was done, for instance, with membership alegbras, [34]. But this amounts to a specialisation of the problem motivated by particular applications which we are not addressing here.

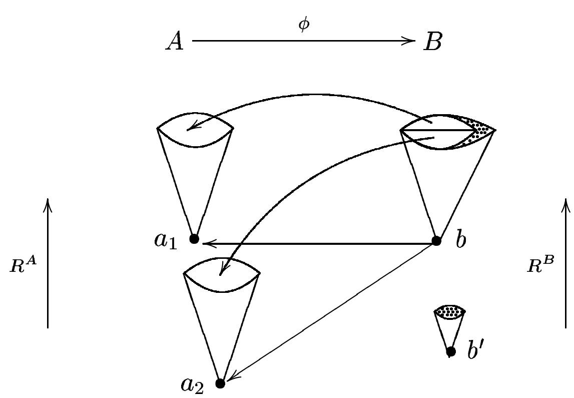
The outer-tight homomorphisms seem to possess many desirable properties which are absent in the case of weak homomorphisms and vainly sought in other cases. (The condition $\phi^-; R^A = R^B; \phi^-$ is suggested as the definition of homomorphism between Boolean polyalgebras (yet another name for multialgebras) in [23], p.262 and p.264, def. 2.3.3. It is, however, not investigated there and seems to arise in order to preserve the Boolean structure which is not part of the definition of our multialgebras.) The objective of this paper is to substantiate the positive aspect of this claim: firstly, by showing the existence of several universal constructions in the category $\mathsf{MAlg}_{OT}(\Sigma)$ for an arbitrary signature Σ and, secondly, by observing how these constructions give rise to tight algebraic relationships missing in the case of weak homomorphism. The following section 2 summarizes some basic facts concerning the category $\mathsf{MAlg}_{OT}(\Sigma)$, discusses OT-congruences, subalgebras and illustrates the character of final objects. However, as the +/− in the table 1.11 indicates, such objects can be constructed only in special cases and, generally, do not exist due to the simple cardinality reasons. (The problem here is exactly the same as with coalgebras involving power-set functor.) Subsection 2.4 shows two special cases when final objects exist – the one is obtained by restrictions on the signature and the other on the objects admitted to the category of multialgebras. The existence of final objects is shown in section 3 for the extended category $\mathsf{MAlg}^*_{OT}(\Sigma)$ where algebras may have carriers being proper classes. (The proof is analogous to that used for showing the corresponding fact for the categories of coalgebras for "set-based" functors in [2].) We show cocompleteness of this category (which result transfers easily to $\mathsf{MAlg}_{OT}(\Sigma)$). In section 4 we give a construction of products in $\mathsf{MAlg}^*_{OT}(\Sigma)$ and show that the corresponding construction can be performed in $\mathsf{MAlg}_{OT}(\Sigma)$. Thus, for any signature Σ, we obtain complete and cocomplete category of large algebras $\mathsf{MAlg}^*_{OT}(\Sigma)$, and a cocomplete category of small algebras $\mathsf{MAlg}_{OT}(\Sigma)$, with products and equalizers, but in general without final objects.

2 The category Outer-Tight, $\mathsf{MAlg}_{OT}(\Sigma)$

The outer-tight homomorphism, OT-homomorphism, $\phi : A \to B$, is a function from the carrier of A to that of B, satisfying the condition that for every relation $R \in \Sigma$:

$\phi^-; R^A = R^B; \phi^-$ i.e., in the functional notation: $R^A(\phi^-(b)) = \phi^-(R^B(b))$

which for constants specializes to $c^A = \phi^-(c^B)$.



1) $b \in \phi[A] \Rightarrow$

$\bigcup_{a_i \in \phi^-(b)} R^A(a_i) = \phi^-(R^B(b))$

2) $b' \notin \phi[A] \Rightarrow R^B(b') \cap \phi[A] = \emptyset$
or, equivalently
$x \in \phi[A] \wedge x \in R^B(b) \Rightarrow b \in \phi[A]$

Figure 2.1: OT-homomorphisms

The converse of the defining equation yields an equivalent definition: $(R^A)^-; \phi = \phi; (R^B)^-$, or functionally, $\phi((R^A)^-(a)) = (R^B)^-(\phi(a))$. This requirement of preservation and reflection of pre-images of the operations gives a simpler picture:

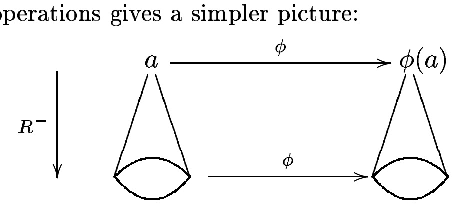


Figure 2.2: OT-homomorphisms

The OT-requirement is strictly stronger than that of the weak homomorphism which requires merely preservation of images, i.e., $R^A; \phi \subseteq \phi; R^B$. In fact, $\phi^-; R^A = R^B; \phi^- \Rightarrow \phi; \phi^-; R^A; \phi = \phi; R^B; \phi^-; \phi$, and since $id_A \subseteq \phi; \phi^-$ and $\phi^-; \phi \subseteq id_B$, this equality yields $R^A; \phi \subseteq \phi; R^B$. Thus, every OT-homomorphism is also weak.

Remark 2.3 *As OT implies weakness and, in the special case when the involved multialgebras are classical (with all operations being total, deterministic functions), weakness implies classical homomorphism condition so, in this special case, the OT-homomorphisms become classical homomorphisms, i.e., $\phi^-; R^A = R^B; \phi^- \Rightarrow R^A; \phi = \phi; R^B$. (For any a, $R^B(\phi(a))$ is then a unique value and so is $R^A(a)$; hence the inclusion $R^A; \phi \subseteq \phi; R^B$ becomes the equality $\phi(R^A(a)) = R^B(\phi(a))$ of single values.)*

However, not every classical homomorphism $\psi : A \to B$ can be obtained as such a special case of an OT-homomorphism. E.g., for a signature with one operation $R : s \to t$ and the two algebras as shown below, the mapping ψ is a classical homomorphism satisfying $\forall a : \psi(R^A(a)) = R^B(\psi(a))$:

$$\begin{tikzcd}
 & a_2 \arrow[rr, "\psi"] & & b_2 \\
t: & a_1 \arrow[r] & b_1 & \\
s: & a \arrow[u, "R^A"] \arrow[r] & b \arrow[u, "R^B"'] & b' \arrow[uu]
\end{tikzcd}$$

However, ψ is not OT since $R^A(\psi^-(b')) = \emptyset \neq \{a_2\} = \psi^-(R^B(b'))$. In general, for classical algebras, we only have the implication $R^A; \psi = \psi; R^B \Rightarrow \psi^-; R^A = \psi^-; \psi; R^B; \psi^- \subseteq R^B; \psi^-$ and the above example shows that the inclusion can be proper. Thus, if we restrict the category $\mathsf{MAlg}_{OT}(\Sigma)$ to classical algebras only, we will obtain a wide – but not full – subcategory of the category $\mathsf{Alg}(\Sigma)$ of classical algebras and homomorphisms.

Remark 2.4 *Partial algebras can be seen as deterministic multialgebras where operations return either 1-element sets or the empty set. OT-homomorphisms have here close associates, namely, the full homomorphisms. A mapping $\phi : A \to B$ is a full homomorphism iff*

1) $\phi(f^A(a)) \subseteq f^B(\phi(a))$

2) $\phi(a) \in (f^B)^- \wedge f^B(\phi(a)) \in \phi[A] \Longrightarrow \exists a' \in (f^A)^- : \phi(a) = \phi(a')$

where membership in the inverse image of an operation, $a \in f^-$, is the same as membership in its definition domain, $a \in dom(f)$. OT implies fullness: the first condition is just the requirement of weak homomorphism, while the second follows since for OT homomorphism, the mere fact of $f^B(b) \in \phi[A]$ implies that $b \in \phi[A]$ and, moreover, that $f^A(\phi^-(b)) = \phi^-(f^B(b)) \neq \emptyset$, i.e., $\exists a' \in (f^A)^- : \phi(a') = b$.

Full surjective homomorphisms are quite central since they are exactly the quotient homomorphisms. Also full injective homomorphism are central since they provide the concept of a relative subalgebra: $A' \subseteq A$ is a relative subalgebra of A if the inclusion is a full homomorphism. However, full homomorphisms without any additional (e.g., surjectivity) requirement are not compositional, as the following example from [7], 2.4.5, illustrates:

$$\begin{tikzcd}
A \arrow[r, "\phi"] & B \arrow[r, "\psi"] & C & \\
a_2 \arrow[r] & b_2 \arrow[r] & c_2 & \\
a_1 \arrow[r] & b_1 \arrow[r] & c_1 \arrow[u] & {} \\
 & b_4 \arrow[uur] & & {} \arrow[u, "f"'] \\
 & b_3 \arrow[u] \arrow[uur] & &
\end{tikzcd}$$

Both ϕ and ψ are full, but $\phi; \psi$ is not. ψ is, of course, weak, but it is neither OT nor even OC, since $f^B(\psi^-(c_1)) = \{b_4\} \not\supseteq \{b_2, b_4\} = \psi^-(f^C(c_1))$.

One considers a stronger notion (implying fullness) of closed homomorphisms, which are compositional. A mapping is a closed homomorphism iff it satisfies 1) and

3) $\phi(a) \in (f^B)^- \Longrightarrow a \in (f^A)^-$.

This notion appears rather strong, as it requires all ϕ-preimages of a $b \in (f^B)^-$ to be in the domain of f^A. Thus, for instance, ϕ in the left diagram is OT (and hence full) but not closed:

$$\begin{tikzcd}
 & A \arrow[r, "\phi"] & B \\
 & a_1 \arrow[r] & b_1 \\
a \arrow[rr, bend right] & a' \arrow[u] \arrow[r] & b \arrow[u]
\end{tikzcd}
\qquad \uparrow f \qquad
\begin{tikzcd}
A \arrow[r, "\psi"] & B & \\
a_1 \arrow[r] & b_1 & \\
a \arrow[u] \arrow[r] & b \arrow[u] & b' \arrow[ul]
\end{tikzcd}$$

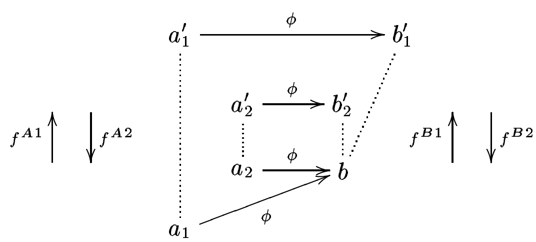
On the other hand, closedness does not imply OT, as shown by ψ in the right diagram.

As the OT-condition is expressible in terms of the inverse images of operations (cf. figure 2.2), that is, in terms of their definition domains, it might be a possible candidate for consideration in connection with partial algebras. Such considerations, however, fall outside the scope of this report.

We are dealing exclusively with the OT-homomorphisms, and so we will not qualify the name – saying "homomorphism" we will mean an OT-homomorphism unless qualified otherwise.

Remark 2.5 *One relational structure can be, in general, represented by various multialgebras depending on the choice of the signature for the multialgebraic operations. This is reflected in the "essentially the same" mapping between structures qualifying or not qualifying as OT-homomorphism.*

The OT-homomorphism condition is, namely, sensitive to the chosen representation of a relation, i.e., it is not invariant under permutation of relational arguments. For instance, two relations $R^A = \{\langle a_1, a_1' \rangle, \langle a_2, a_2' \rangle\}$ and $R^B = \{\langle b, b_1' \rangle, \langle b, b_2' \rangle\}$, can be represented as the multifunctions, $f^{A1}(a_1) = a_1', f^{A1}(a_2) = a_2'$ or $f^{A2}(a_1') = a_1, f^{A2}(a_2') = a_2$ and, respectively, $f^{B1}(b) = \{b_1', b_2'\}$ or $f^{B2}(b_1') = b = f^{B2}(b_2')$.



Now, the mapping $\phi(a_1) = \phi(a_2) = b$ and $\phi(a_i') = b_i'$ is OT homomorphism between f^{A1} and f^{B1} but not between f^{A2} and f^{B2}. (The example concerns, of course, not just the converse of a binary relation but the general situation, where the choice of the relational argument to function as the result of the multioperation can determine whether a given mapping is or is not an OT homomorphism.) Thus, although in the trivial sense of the isomorphism (1.1), multialgebras are only *representations of relational structures, so when homomorphisms are taken into consideration, the algebraic character of this representation becomes quite significant, as will become evident from the rest of this paper.*

As a possible example of OT-homomorphism consider the following.

Example 2.6 *Consider a multialgebra M over the signature Θ with one sort and one unary operation $x \mapsto \overline{x}$. By the definition of multialgebra, we obtain that:*

1) $\overline{\emptyset} = \emptyset$
2) $\overline{X} = \bigcup_{x \in X} \overline{x}$, *for each subset $X \subseteq M$, in particular, 2.b)* $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$

Let us restrict the class of Θ-multialgebras to those where the operation $\overline{x}$ satisfies two closure conditions, for every element $x \in M$:

3) $x \subseteq \overline{x}$ *(and hence, $X \subseteq \overline{X}$, for all subsets $X \subseteq M$),*
4) $\overline{\overline{x}} = \overline{x}$ *(and hence, $\overline{\overline{X}} = \overline{X}$ for all $X \subseteq M$).*

In short, such a multialgebra is a topological space. The condition 2 is more general than that required for a topological closure operator, namely, 2.b). Consequently, the Θ-multialgebras will make closed not only finite but also arbitrary unions of closed sets. If, for instance, $\overline{x} = x$ for all $x \in M$, then M is a $\mathbf{T}_1$ space (where, by 2., $\overline{X} = X$ for every subset $X \subseteq M$, i.e., the topology with all subsets of M being clopen.)

The OT-homomorphism condition for $\phi : A \to B$ becomes now: $\forall y \in B : \overline{\phi^-(y)} = \phi^-(\overline{y})$ which yields, for every $Y \subseteq B$:

$$\phi^-(\overline{Y}) \stackrel{2}{=} \phi^-(\bigcup_{y \in Y} \overline{y}) = \bigcup_{y \in Y} \phi^-(\overline{y}) \stackrel{OT}{=} \bigcup_{y \in Y} \overline{\phi^-(y)} = \overline{\phi^-(Y)}$$

which implies, in particular, $\phi^-(\overline{Y}) \supseteq \overline{\phi^-(Y)}$, i.e., continuity of ϕ. If, in addition, we restrict ϕ to be injective, the above equality amounts to the requirement of ϕ being a homeomorphism between the spaces A and B.

The paranthetical "hence" phrase (in point 3-4) follows due to the simple but useful fact:

Fact 2.7 *For any terms $t(x), s(x) \in \mathcal{T}(\Sigma, Y)$ and Σ-multialgebra M :*

$$M \models \forall x \in M : s(x) = t(x) \iff M \models \forall X \subseteq M : s(X) = t(X).$$

Proof: $\Rightarrow$ follows directly from additivity of operations: $s^M(X) = \bigcup_{x \in X} s^M(x) = \bigcup_{x \in X} t^M(x) = t^M(X)$, while $\Leftarrow$ since $x \in M$ is but a special case of $X \subseteq M$, for $X = \{x\}$. □

Remark 2.8 *Alternatively to the above example, we can endow any multialgebra M over arbitray Σ with a topology by taking (for each sort s^M) as the subbasis, all the sets of the form $f^M(\overline{x})$, for $f \in \Sigma$ and $\overline{x} \in M$, together with s^M and $\emptyset$. (Opens will be the interpretations of all ground terms, as well as, all "reachable" sets, i.e., of the form $t^M(\overline{x})$, for some term t and all possible assignments to $\overline{x}$. The latter follows by induction on the depth of the term t: $f^M(x)$ is open and so is $g^M(y)$ for each $y \in f^M(x)$, hence also is $\bigcup_{y \in f^M(x)} g^M(y) = g(f(x))^M$.) Viewing opens as observations, [39], this amounts to viewing an operation f applied to an x as an f-observation, and the topology classifies all possible finitely verifiable observations.*

The OT-homomorphism condition implies then continuity, since $f^A(\phi^-(x)) \stackrel{OT}{=} \phi^-(f^B(x))$ makes, for any open $t^B(x)$, its ϕ-preimage open in A, as a union of opens $\bigcup_{a \in \phi^-(x)} t^A(a)$. (Trivially, also, $\phi^-(X \cap Y) = \phi^-(X) \cap \phi^-(Y)$ and $\phi^-(\bigcup_i X_i) = \bigcup_i \phi^-(X_i)$, so that, e.g., $\phi^-(f^B(x) \cap g^B(y)) = \phi^-(f^B(x)) \cap \phi^-(g^B(y)) \stackrel{OT}{=} f^A(\phi^-(x)) \cap g^A(\phi^-(y))$.)

In fact, the OT condition is stronger than mere continuity and falls between it and homeomorphism since, as observed in the example above, it is equivalent to homeomorphism provided that the mapping is injective. This topological aspect will not concern us much, but it will be encountered occasionally.

The following illustrates another way of arriving at OT-homomorphisms, establishing a tight connection between the category $\mathsf{MAlg}_{OT}(\Sigma)$ and the category of coalgebras for the corresponding Σ-functor.

Remark 2.9 *Consider, as an example, a functor $\Sigma : \mathsf{SET} \to \mathsf{SET}$, given by $\Sigma(X) = X \times X$. A Σ-coalgebra is then a function $\alpha : A \to \Sigma(A)$, i.e., $\alpha : A \to A \times A$. The converse $\alpha^- : A \times A \to A$ is not, however, a function in case α is not injective: in general, it is a multifunction, i.e., $\alpha^- : A \times A \to \mathcal{P}(A)$. Thus, a multialgebra can be seen as a converse of a coalgebra for an arbitrary polynomial functor Σ (i.e., a coproduct of products). There are, of course, multialgebras which can not be obtained in this way, namely, the ones whith $f : S \to \mathcal{P}(S)$ such that for two elements $s_1 \neq s_2 \in S : f(s_1) \cap f(s_2) \neq \emptyset$ (i.e., when the converse $f^- : \mathcal{P}(S) \to \mathcal{P}(S)$ is not determined by any function $S \to S$.) Thus, coalgebras (over polynomial functors) can be represented by multialgebras but not vice versa.*

A Σ-coalgebra homomorphism $\phi : \langle A, \alpha \rangle \to \langle B, \beta \rangle$ is a function $\phi : A \to B$ such that the diagram to the left commutes:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & A \times A \arrow[d, "\phi \times \phi\ = \Sigma(\phi)"] \\ B \arrow[r, "\beta"] & B \times B \end{tikzcd} \qquad \begin{tikzcd} A & A \times A \arrow[l, "\alpha^-"'] \\ B \arrow[u, "\phi^-"] & B \times B \arrow[l, "\beta^-"'] \arrow[u, "\phi^- \times \phi^-"'] \end{tikzcd}$$

i.e., such that

$$\alpha;\Sigma(\phi) \quad = \quad \phi;\beta. \tag{2.10}$$

By taking the converse of both sides of this equation, we obtain $(\alpha;\Sigma(\phi))^- = (\phi;\beta)^-$*, i.e.,*

$$\Sigma(\phi)^-;\alpha^- \quad = \quad \beta^-;\phi^- \tag{2.11}$$

which is the OT *condition on* ϕ *for the multialgebras as shown in the diagram to the right. Thus, we not only obtain a pointwise representation of coalgebras, but also their morphisms are represented as the* OT*-morphisms between the corresponding multialgebras.*

Notice, however, that while the condition (2.10) involves only arrows in the category SET*, each of the arrows* $\alpha^-, \beta^-, \phi^-$ *in the equation resulting in (2.11) can be a multifunction returning, for each argument, a subset of the target. The equality is thus not the equality of arrows in* SET*, but of the result sets for each choice of the argument.*

Remark 2.12 *As a special variation of the above remark, we can view coalgebras for the (direct image) power-set functor as multialgebras for the signature* $f : S \to S$*.*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & \mathcal{P}(A) \arrow[d, "\mathcal{P}(\phi)"] \\ B \arrow[r, "\beta"] & \mathcal{P}(B) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{P}(A) & A \arrow[l, "\alpha^-"'] \\ \mathcal{P}(B) \arrow[u, "\phi^-"] & B \arrow[l, "\beta^-"'] \arrow[u, "\phi^-"'] \end{tikzcd}$$

$$\alpha;\mathcal{P}(\phi) = \phi;\beta \qquad\qquad \phi^-;\alpha^- = \beta^-;\mathcal{P}(\phi^-)$$

We represent any coalgebra $\alpha : A \to \mathcal{P}(A)$ *by a multialgebra* $\alpha^-(x) = \{y \in A \mid x \in \alpha(y)\}$*. Since* $(\alpha^-)^- = \alpha$ *we obtain the bijection between coalgebras and multialgebras. This bijection on objects extends to the isomorphism of the respective categories, since coalgebraic homomorphism condition is then equivalent to the* OT*-condition.*

We can thus see multialgebras as both generalisation of algebras to handle partiality and nondeterminism and, on the other hand, as a possible representation of a large class of coalgebras. This representation is nevertheless to be studied from the algebraic perspective. It is mentioned primarily to emphasize again (in addition to the just mentioned topological aspect), that mutlialgebras capture also the notion of observability, which appears in a somehow dual form to the coalgebraic one. This duality will become well visible in considerations of bireachability, subsection 2.3 and later, as a converse of the coalgebraic bisimilarity.

2.1 Some preliminaries

Proposition 2.13 *An* OT*-homomorphism* ϕ *is*

1) *injective iff it is mono;*
2) *surjective iff it is epi; generally, a collection* $\{\phi_i : A_i \to B \mid i \in I\}$ *is jointly surjective iff it is epi-sink;*
3) *bijective iff it is iso.*

Proof: 1. $\Rightarrow$) Assume injectivity of ϕ and let (*) $\psi_1;\phi = \psi_2;\phi$ for two given homomorphisms $\psi_1, \psi_2 : X \to A$. All arrows can be seen as morphisms in SET, where injectivity of ϕ is equivalent to it being a monomorphism. But then (*) implies that $\psi_1 = \psi_2$ as SET-morphism, which implies their equality as OT-homomorphisms.

$\Leftarrow$) In section 2.3 we show that if ϕ is an OT-homomorphism then its kernel, $ker(\phi) = \phi;\phi^-$, is an OT-congruence (fact 2.38) which can be endowed with the algebraic structure (definition 2.50) such that the projections are homomorphisms (fact 2.51). Then, in the diagram $\begin{tikzcd} ker(\phi) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "\phi"] & B \end{tikzcd}$ we obtain $\pi_1;\phi = \pi_2;\phi$ and, assuming ϕ to be mono, $\pi_1 = \pi_2$. But this means that $ker(\phi) = id_A$, i.e., that ϕ is injective. Below, we spell out this proof in details without refering to the results to be introduced later on.

Assuming ϕ is not injective. Then there is at least one element $b \in B$ and a set of two or more elements $A_1 \subseteq A$ such that $a \in A_1 \Leftrightarrow \phi(a) = b$. Let a_i range over all elements in A_1. Since ϕ is OT: $\phi^-(f^B(b)) = \bigcup_{a_i \in A_1} f^A(a_i)$. We define an algebra X on the set

$X = \{\langle x,y\rangle \mid x,y \in A \wedge \phi(x) = \phi(y)\}$ (in particular, $\forall x \in A : \langle x,x\rangle \in X$), by letting, for all constants, functions and arguments $\langle x_1,y_1\rangle \ldots \langle x_n,y_n\rangle \in X$:

$$\begin{aligned} c^X &= \{\langle x,y\rangle \in c^A \times c^A \mid \phi(x) = \phi(y)\} \\ f^X(\langle x_1,y_1\rangle \ldots \langle x_n,y_n\rangle) &= \{\langle x,y\rangle \in f^A(x_1 \ldots x_n) \times f^A(y_1 \ldots y_n) \mid \phi(x) = \phi(y)\} \end{aligned} \tag{2.14}$$

Let $\psi_1, \psi_2 : X \to A$ be projections. By (2.14) and the fact that $\forall x \in A : \langle x,x\rangle \in X$, we have:

$$\begin{aligned} \psi_1(c^X) &= c^A \\ \psi_1(f^X(\langle x_1,y_1\rangle \ldots \langle x_n,y_n\rangle)) &= f^A(x_1 \ldots x_n) \end{aligned}$$

and the corresponding equations hold for ψ_2. To prove that ψ_i are OT we have to show:

$$\psi_i^-(f^A(a_1 \ldots a_n)) = f^X(\psi_i^-(a_1) \ldots \psi_i^-(a_n))$$

for arbitrary $a_1 \ldots a_n \in A$. We show it for $i = 1$ as the proof for ψ_2 is entirely analogous. By definition of ψ_1 we obtain:

$$\psi_1^-(a) = \{\langle a,y\rangle \mid y \in [a]_\phi\} \tag{2.15}$$

where $[a]_\phi = \{a' \in A \mid \phi(a') = \phi(a)\}$. Furthermore, since ϕ is OT:

$$\forall \langle x,y\rangle \in X : \phi(x) = \phi(y) \wedge x \in f^A(a_1 \ldots a_n) \;\Rightarrow y \in f^A([a_1]_\phi \ldots [a_n]_\phi)$$

which means

$$\psi_1^-(f^A(a_1 \ldots a_n)) = \{\langle x,y\rangle \in f^A(a_1 \ldots a_n) \times f^A([a_1]_\phi \ldots [a_n]_\phi) \mid \phi(x) = \phi(y)\}$$

On the other hand

$$\begin{aligned} f^X(\psi_1^-(a_1) \ldots \psi_1^-(a_n)) &\overset{(2.15)}{=} f^X(\{\langle a_1,y_1\rangle \ldots \langle a_n,y_n\rangle \mid y_i \in [a_i]_\phi, 1 \le i \le n\}) \\ &\overset{(2.14)}{=} \{\langle x,y'\rangle \in f^A(a_1 \ldots a_n) \times f^A([a_1]_\phi \ldots [a_n]_\phi) \mid \phi(x) = \phi(y')\} \end{aligned}$$

Hence $\psi_1^-(f^A(a_1 \ldots a_n)) = f^X(\psi_1^-(a_1) \ldots \psi_1^-(a_n))$ and thus ψ_1 is OT.
By assumption we have at least two $a_1, a_2 \in A_1$, i.e., $\phi(a_1) = \phi(a_2) \wedge a_1 \neq a_2$. This means that $\langle a_1,a_2\rangle \in X$, and since $\psi_1(\langle a_1,a_2\rangle) = a_1$ while $\psi_2(\langle a_1,a_2\rangle) = a_2$, $\psi_1 \neq \psi_2$. But $\psi_1;\phi = \psi_2;\phi$, and thus ϕ is not mono.

2. We show the general statement for epi-sinks, from which the result for epis follows as a special case.
$\Rightarrow$) Assume joint surjectivity of ϕ_i and (*) $\phi_i;\psi_1 = \phi_i;\psi_2$ for all i and some $\psi_1, \psi_2 : B \to X$. Then $\forall b \in B\ \exists i\ \exists a \in A_i : b = \phi_i(a)$ and so $\psi_1(b) = \psi_1(\phi_i(a)) \overset{(*)}{=} \psi_2(\phi_i(a)) = \psi_2(b)$.
$\Leftarrow$) Assume that ϕ_i are not jointly surjective. Writing $\phi[A] = \bigcup_i \phi_i[A_i]$, we then have that $B_1 = B \setminus \phi[A]$ is non-empty. Since every ϕ_i is OT so for any $b_1 \ldots b_n \in B$:

$$f(b_1 \ldots b_n) \cap \phi_i[A_i] \neq \emptyset \Rightarrow b_1 \ldots b_n \in \phi_i[A_i]$$

and, furthermore

$$\{b_1, \ldots, b_n\} \cap B_1 \neq \emptyset \Rightarrow f^B(b_1 \ldots b_n) \subseteq B_1 \tag{2.16}$$

We let $B_2 \simeq B_1$ be disjoint from B, and denote the bijections

$$\iota_{21} : (B_2 \cup \phi[A]) \longleftrightarrow (B_1 \cup \phi[A]) : \iota_{12}. \tag{2.17}$$

which are identities on the elements in $\phi[A]$.
We define an algebra structure on the set $X = B_1 \cup \phi[A] \cup B_2$ as follows:

$$\begin{aligned} c^X &= c^B \cup \iota_{12}(c^B \cap B_1) \\ f^X(x_1 \ldots x_n) &= \begin{cases} f^B(x_1 \ldots x_n) \cup \iota_{12}(f^B(x_1 \ldots x_n)) & \text{iff } x_1 \ldots x_n \in \phi[A] \\ f^B(x_1 \ldots x_n) & \text{iff } x_1 \ldots x_n \in B_1 \cup \phi[A] = B \\ \iota_{12}(f^B(\iota_{21}(x_1) \ldots \iota_{21}(x_n))) & \text{iff } x_1 \ldots x_n \in B_2 \cup \phi[A] \\ \emptyset & \text{otherwise} \end{cases} \end{aligned} \tag{2.18}$$

The four disjuncts of the above definition are to be understood exclusively, i.e., the second case applies only when the first does not, etc.

We define two mappings $\psi_1, \psi_2 : B \to X$, as follows: $\psi_1(b) = b$ for all $b \in B$, while $\psi_2(b) = b$ for all $b \in \phi[A]$ and $\psi_2(b) = \iota_{12}(b)$ for all $b \in B_1$.

To prove that ψ_1 and ψ_2 are OT we observe first that both are injective, and so:

$$\psi_1^-(x) = \begin{cases} x & \text{if } x \in \phi[A] \\ x & \text{if } x \in B_1 \\ \emptyset & \text{otherwise; } x \in B_2 \end{cases} \qquad \psi_2^-(x) = \begin{cases} x & \text{if } x \in \phi[A] \\ \iota_{21}(x) & \text{if } x \in B_2 \\ \emptyset & \text{otherwise; } x \in B_1 \end{cases} \tag{2.19}$$

We consider four (disjoint) cases, corresponding to those in (2.18):

1) If $x_1 \dots x_n \in \phi[A]$:

$$\begin{aligned} \psi_1^-(f^X(x_1 \dots x_n)) &\overset{(2.18)}{=} \psi_1^-(f^B(x_1 \dots x_n) \cup \iota_{12}(f^B(x_1 \dots x_n))) \\ &\overset{(2.19)}{=} \psi_1^-(f^B(x_1 \dots x_n)) \\ &\overset{(2.19)}{=} f^B(\psi_1^-(x_1) \dots \psi_1^-(x_n)) \end{aligned}$$

2) If $x_1 \dots x_n \in B$:

$$\begin{aligned} \psi_1^-(f^X(x_1 \dots x_n)) &\overset{(2.18)}{=} \psi_1^-(f^B(x_1 \dots x_n)) \\ &\overset{(2.19)}{=} f^B(\psi_1^-(x_1) \dots \psi_1^-(x_n)) \end{aligned}$$

3) If $x_1 \dots x_n \in B_2 \cup \phi[A]$, with at least one $x_i \in B_2$:

$$\begin{aligned} \psi_1^-(f^X(x_1 \dots x_n)) &\overset{(2.18)}{=} \psi_1^-(\iota_{12}(f^B(\iota_{21}(x_1) \dots \iota_{21}(x_n)))) \\ &\overset{(2.17)}{=} \psi_1^-(\iota_{12}(f^B(b_1 \dots b_n))), b_1 \dots b_n \in B, b_i \in B_1 \\ &\overset{(2.16)}{=} \psi_1^-(\iota_{12}(B_1')), \quad B_1' = \{b \in f^B(b_1 \dots b_n)\} \subseteq B_1 \\ &\overset{(2.17)}{=} \psi_1^-(B_2'), \quad B_2' = \{b \in \iota_{12}(B_1')\} \subseteq B_2 \\ &\overset{(2.19)}{=} \emptyset \\ f^B(\psi_1^-(x_1) \dots \psi_1^-(x_n)) &\overset{(2.19)}{=} \emptyset, \text{ since for at least one } x_i \in B_2 : \psi_1^-(x_i) = \emptyset \end{aligned}$$

4) Otherwise (there are at least two elements x_i and x_j such that $x_i \in B_1$ and $x_j \in B_2$):

$$\begin{aligned} \psi_1^-(f^X(x_1 \dots x_n)) &\overset{(2.18)}{=} \psi_1^-(\emptyset) = \emptyset \\ f^B(\psi_1^-(x_1) \dots \psi_1^-(x_n)) &\overset{(2.19)}{=} \emptyset, \text{ since for } x_i \in B_2, x_j \in B_1 : \psi_1^-(x_j) = \emptyset = \psi_2^-(x_i) \end{aligned}$$

Thus, for all $x_1 \dots x_n \in X : \psi_1^-(f^X(x_1 \dots x_n)) = f^B(\psi_1^-(x_1) \dots \psi_1^-(x_n))$, i.e., ψ_1 is OT. We defined the algebraic structure on $B_2 \simeq B_1$ and the proof for ψ_2 is entirely analogous.

Now, $\psi_1(b) \neq \psi_2(b)$ for any $b \in B_1$, while for all $b \in \phi[A] : \psi_1(b) = \psi_2(b)$. Hence for all $i : \phi_i; \psi_1 = \phi_i; \psi_2$, while $\psi_1 \neq \psi_2$, i.e., $\{\phi_i \mid i \in I\}$ is not en epi-sink.

3. If ϕ is not bijective, there can be no inverse. If it is, then ϕ^- is easily verified to be OT. $\square$

Remark 2.20 *The general fact about dialgebras (e.g., proposition 18 in [40]) is that for a function $f : A \to B$ which is a dialgebra morphism in SET^F_G:*

1) if F preserves weak pushouts, then f is epi iff it is surjective, and

2) if G preserves weak pullbacks, then f is mono iff it is injective.

In our case, both these conditions are satisfied, since F is the polynomial functor (coproduct of products) while $G = \mathcal{P}$ which does preserve weak pullbacks. However, the proposition cannot be applied since our OT-homomorphisms are not the same as the morphisms in $\mathsf{SET}^F_{\mathcal{P}}$.

2.2 Subalgebras

We say that A' is a subalgebra of A, $A' \sqsubseteq A$, if A' is an algebra with $A' \subseteq A$ and such that the inclusion is a homomorphism. (The following considerations would not be significantly affected, if we adopted the categorical definition, according to which subobject is an equivalence class of monomorphisms.) If $A, A' \in \mathsf{MAlg}_{OT}(\Sigma)$ and $A' \subset A$, this does not mean that the inclusion is an OT-homomorphism, i.e., it may still happen that A' is not a subalgebra of A, $A' \not\sqsubseteq A$. E.g., $A' : \begin{array}{c} b \\ \uparrow \\ a_1 \end{array}$ is not a subalgebra of $A : \begin{array}{ccc} & & b \\ & \nearrow & \uparrow \\ a_1 & & a_2 \end{array}$. If b is in the carrier of a subalgebra, then so must be all its pre-images: all the elements of the argument-sorts, from which b is reachable by some operations (cf. condition 2 in figure 2.1.) Hence, the only subalgebra of A containing b is A itself. This closure condition is – by requiring the presence of all elements from which a present element is reachable – converse of the classical one which requires closure under the results of the operations. Notice the equivalence of the two following closure conditions, for a subset $A \subseteq B$, with $\overline{A} = B \setminus A$:

$$\begin{array}{l} 1) \;\; x \in A \Longrightarrow f(x) \subseteq A \text{ and} \\ 2) \;\; y \in \overline{A} \;\wedge\; y \in f(x) \Longrightarrow x \in \overline{A} \end{array} \tag{2.21}$$

(The equivalence for the classical/deterministic algebras is obtained by taking $f(x) \in A$ and $y = f(x)$, respectively.) That is, closure of a subset A under images (of operations) is equivalent to closure of its complement under pre-images. What happens in our case of OT-homomorphisms, is that the latter is taken as the closure condition on the subset A and not on its complement (cf. condition 2. in Figure 2.1). It reflects the similarly converse character of the OT-congruences to be studied shortly.

Inclusion is not necessarily a homomorphism, but it is when restricted to subalgebras of the same algebra.

Fact 2.22 *Inclusions between subalgebras of the same algebra are OT homomorphisms. I.e., if $A_1 \sqsubseteq A$ and $A_2 \sqsubseteq A$ and $A_2 \subseteq A_1$, then also $A_2 \sqsubseteq A_1$.*

PROOF: We have two inclusion homomorphisms $\iota_k : A_k \to A$, $k = 1, 2$, and inclusion $i : A_2 \to A_1$ which we want to show is a homomorphism. We thus have: 1) $\iota_1^-; R^{A_1} = R^A; \iota_1^-$, 2) $\iota_2^-; R^{A_2} = R^A; \iota_2^-$ and 3) $i; \iota_1 = \iota_2$. 2,3) $\Rightarrow \iota_1^-; i^-; R^{A_2} = R^A; \iota_1^-; i^- \overset{1)}{\Rightarrow} \iota_1^-; i^-; R^{A_2} = \iota_1^-; R^{A_1}; i^- \Rightarrow \iota_1; \iota_1^-; i^-; R^{A_2} = \iota_1; \iota_1^-; R^{A_1}; i^-$. Since ι_1 is inclusion, we have that $\iota_1; \iota_1^- = id_{A_1}$ and so we obtain $i^-; R^{A_2} = R^{A_1}; i^-$. □

Taking into account the equivalence (2.21), we also have the following characterisation of subalgebras:

Fact 2.23 *Given $A, A' \in \mathsf{MAlg}_{OT}(\Sigma)$ with $A' \subseteq A$, the following conditions are equivalent:*

1) $A' \sqsubseteq A$, i.e., inclusion $i : A' \to A$ is OT

2) A' is closed under pre-images of A-operations (i.e., $a' \in A' \wedge a' \in f^A(a) \Longrightarrow a \in A'$)

3) $A \setminus A'$ is closed under images of operations

Remark 2.24 *Consider for the moment only classical, i.e., deterministic algebras. As observed in remark 2.3, OT-homomorphism becomes then a special case of IT, i.e., the classical homomorphism. (Similarly, LT and RT will be special cases of IT, and the observation below applies also to these alternatives.) One can then define a more specific concept of an OT-subalgebra by requiring that the inclusion is not only a homomorphism but an OT-homomorphism. In view of the above fact, such a subalgebra $A' \sqsubseteq A$ would be closed under pre-images (inclusion being OT) but also under images (OT being IT). Thus also the complement $A'' = A \setminus A'$ would be closed under images and pre-images, i.e., would be a subalgebra of A. In other words, an OT-subalegbra would amount to partioning the algebra A into two disjoint subalgebras.*

Given a collection of subalgebras, $A_k \sqsubseteq A$, their intersection C is obtained as $C = \bigcap_{k \in K} A_k$, with $f^C(a) = f^A(a) \cap C$ for all $a \in C$. The drawing below gives one example with two subalegebras $A_1, A_2 \sqsubseteq A$, and their intersection C :

$$
\begin{array}{cccc}
\begin{tikzcd} b_1 & & b_2 \\ A & a \arrow[ul] \arrow[ur] & \end{tikzcd}
&
\begin{tikzcd} b_1 & A_1 \\ & a \arrow[ul] \end{tikzcd}
&
\begin{tikzcd} A_2 & b_2 \\ a \arrow[ur] & \end{tikzcd}
&
\begin{array}{c} C \\ \\ a \end{array}
\end{array}
$$

Notice that the reverse situation, with taking only intersection of the results, does not work in the same way, as suggested by the example above of $A' \not\sqsubseteq A$.

We do have the counterpart of the classical result that intersection of subalgebras yields a subalgebra.

Fact 2.25 *Given a collection $\{A_k \mid k \in K, A_k \sqsubseteq A\}$, then also $\bigcap_{k \in K} A_k = C \sqsubseteq A$.*

PROOF: For each $k \in K$ we have the inclusion homomorphism $i_k : A_k \hookrightarrow A$ and also the inclusion $c_k : C \subseteq A_k$. If at least for one such k, c_k is a homomorphism, the claim follows. We will show it for an arbitrary (and hence every) k.
Since we consider only inclusions, for every k, l we have that $c_k; i_k = c_l; i_l$ and hence also

$$i_k^- ; c_k^- = i_l^- ; c_l^- . \tag{2.26}$$

Moreover, just like for an $X \subseteq A : i_k^-(X) = X \cap A_k$, so for $Y \subseteq A_k$:

$$c_k^-(Y) = Y \bigcap_{l \neq k} A_l. \tag{2.27}$$

Let $k \in K$ be arbitrary, and consider two cases for the expression $R^C(c_k^-(a))$, where $a \in A_k$.
1) $c_k^-(a) = \emptyset$ (for at least one argument a, which we simplify in notation by ignoring other arguments), and thus also $R^C(c_k^-(a)) = \emptyset$ but, in particular,

$$
\begin{array}{rcll}
a \in A_k \ \&\ a \notin C & \Rightarrow & \exists A_l : a \notin A_l, \ i.e., \ i_l^-(a) = \emptyset & (2.27) \\
& \Rightarrow & R^{A_l}(i_l^-(a)) = \emptyset & R(\emptyset) = \emptyset \\
& \Rightarrow & i_l^-(R^A(a)) = \emptyset & i_l \ \mathit{is}\ OT \\
& \Rightarrow & c_l^-(i_l^-(R^A(a))) = \emptyset & c_l^-(\emptyset) = \emptyset \\
& \Rightarrow & c_k^-(i_k^-(R^A(a))) = \emptyset & (2.26) \\
& \Rightarrow & c_k^-(R^{A_k}(i_k^-(a))) = \emptyset & \mathit{since}\ i_k \ \mathit{is}\ OT \\
& \Rightarrow & c_k^-(R^{A_k}(a)) = \emptyset & \mathit{since}\ i_k(a) = a
\end{array}
$$

Thus, if $c_k^-(a) = \emptyset$ then the condition $R^C(c_k^-(a)) = c_k^-(R^{A_k}(a))$ is satisfied.
2) The second case assumes $c_k^-(a) \neq \emptyset$. Then $c_k^-(a) = a \in C$.
a) $R^C(c_k^-(a)) = R^C(a) \stackrel{def.\ of\ C}{=} R^A(a) \cap \bigcap_{l \in K} A_l$.
b) $c_k^-(R^{A_k}(a)) \stackrel{(2.27)}{=} R^{A_k}(a) \cap \bigcap_{k \neq l \in K} A_l$.
c) $R^{A_k}(a) = R^{A_k}(i_k^-(a)) = i_k^-(R^A(a)) = R^A(a) \cap A_k$ and substituting this into b) gives equality with a). $\square$

Hence, given an algebra A, the collection of its subalgebras, $\downarrow(A)$, with the subalgebra relation, $\langle \downarrow(A), \sqsubseteq \rangle$ is a lower semilattice with the greatest element A, and so:

Fact 2.28 *For an algebra A, $\langle \downarrow(A), \sqsubseteq \rangle$ is a complete lattice with meets given by intersection.*

In view of the equivalences from fact 2.23, we obtain thus also a "complementary" lattice of subsets of A closed under images, since every $A' \sqsubseteq A$ determines such a closed subset $A \setminus A'$ and vice versa.

The above verifies also the following fact – according to which the diagram of subalgebras is directed – which, however, we also prove separately providing the explicit construction.

Fact 2.29 *For every set $X \subseteq A$, there is a smallest subalgebra $A_X \sqsubseteq A$ with $X \subseteq A_X$.*

PROOF: The construction extends the given set X to obtain a subalgebra. X is sorted, and the construction extends in each step each sort (if at all):

1) $X_0 = X$
2) For all $x \in A$, if $f^A(x) \cap X_i \neq \emptyset$ then include into X_{i+1} also all such x.

3) $X_\omega = \bigcup_{i\in\omega} X_i$

We define Σ structure A_X on X_ω by letting, for all $x \in X_\omega$ and every operation f from the signature: $f^{A_X}(x) = f^A(x) \cap X_\omega$. This makes A_X obviously closed under all operations.

The inclusion $\iota : X_\omega \hookrightarrow A$ is OT. We have $\iota^-(Y) = Y \cap X_\omega$, and have to check that $f^{A_X}(\iota^-(a)) = \iota^-(f^A(a)) = f^A(a) \cap X_\omega$. Now if $f^A(a) \cap X_\omega \neq \emptyset$ then, by 2., $a \in X_\omega$ and we have $f^{A_X}(\iota^-(a)) = f^A(a) \cap X_\omega$, i.e., the required equality holds.

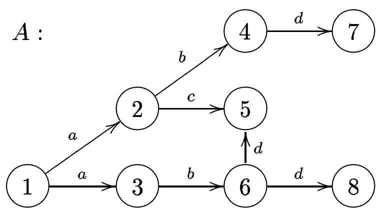
If, on the other hand, $f^A(a) \cap X_\omega = \emptyset$, then either $a \notin X_\omega$ and so $f^{A_X}(a) = \emptyset$, or else $a \in X_\omega$ and then $f^{A_X}(a) = f^A(a) \cap X_\omega = \emptyset$. So the equality holds also in this case.

A_X is in fact smallest subalgebra of A containing X. For removing any element from its carrier, would require removing it either from X or else from among elements added in step 2). In the former case, the result would not contain X, while in the latter would not be a subalgebra of A (inclusion would not be an OT-homomorphism). □

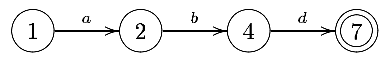
Thus, if $A_1, A_2 \sqsubseteq A$, then there is also (a smallest) $A_3 \sqsubseteq A$, with $A_1 \cup A_2 \subseteq A_3$.

Example 2.30 *Given an alphabet, all its symbols can be viewed as operations acting on the single sort of states. A given set of states and definition of these functions determine then a possibly nondeterministic automaton. For instance, the automaton (multialgebra) A has 8 elements in the sort of states and, e.g., $a^A(1) = \{2,3\}$ while $b^A(1) = \emptyset$, $c^A(2) = \{5\}$ and $d^A(6) = \{5,8\}$. The subalgebras generated by the state 3, resp. 7 are shown to the right:*

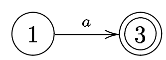
$A :$



$A_{\{7\}} :$



$A_{\{3\}} :$



The subalgebra generated by $X \subseteq A$ is thus, in this example, the maximal set of states A_X all reaching X (with the Σ-structure inherited from A), i.e., such that $s \in A_X$ iff there exists a path (derived operator) p for which $p^A(s) \cap X \neq \emptyset$. If we think of multialgebra as a (possibly action-labeled) OR *search (or game, like minimax) graph, the subalgebra generated by X will thus pick up the paths/strategies leading to the goals in X.*

We also have a dual construction of a largest subalgebra $A^X \sqsubseteq A$ with $A^X \subseteq X$.

Fact 2.31 *For every set $X \subseteq A$, there exists a largest subalgebra $A^X \sqsubseteq A$ with $A^X \subseteq X$.*

PROOF: The construction is, in a sense, dual to that from the previous fact and it removes now, from the given set X, elements to obtain a subalgebra.

1) $X_0 = X$
2) If $\exists x \in A \setminus X_i : f^A(x) \cap X_i \neq \emptyset$ then remove these result elements from X_{i+1}, i.e., $X_{i+1} = X_i \setminus \bigcup_{f\in\Sigma} f^A(A \setminus X_i)$
3) $X_\omega = \bigcap_{i\in\omega} X_i$

More explicitly, in point 2) we remove from X_i the elements which are reachable from outside of X_i, i.e., $X_{i+1} = X_i \setminus \bigcup_{x\in A\setminus X_i \wedge f^A(x)\cap X_i \neq \emptyset} f^A(x)$ as f ranges over all operation symbols in Σ.

We define Σ structure A^X on X_ω by letting, for all $x \in X_\omega$ and every operation f from the signature: $f^{A^X}(x) = f^A(x) \cap X_\omega$. This makes A^X obviously closed under all operations.

The inclusion $\iota : X_\omega \hookrightarrow A$ is OT. We have $\iota^-(Y) = Y \cap X_\omega$, and have to check that $f^{A^X}(\iota^-(a)) = \iota^-(f^A(a)) = f^A(a) \cap X_\omega$. Now if $f^A(a) \cap X_\omega \neq \emptyset$ then, by 2., $a \in X_\omega$ and we have $f^{A^X}(\iota^-(a)) = f^A(a) \cap X_\omega$, i.e., the required equality holds.

If, on the other hand, $f^A(a) \cap X_\omega = \emptyset$, then either $a \notin X_\omega$ and so $f^{A^X}(a) = \emptyset$, or else $a \in X_\omega$ and then $f^{A^X}(a) = f^A(a) \cap X_\omega = \emptyset$. So the equality holds also in this case.

A^X is in fact the largest subalgebra of A contained in X. For adding any element from $A \setminus A^X$, would require adding it either to X or else among elements removed in step 2). In the former case, the result would not be contained in X, while in the latter would not be a subalgebra of A (inclusion would not be an OT-homomorphism). □

For instance, for A from example 2.30, $A^{\{7\}} = A^{\{3\}} = \emptyset$. We can easily see that $A^X = \emptyset$ if the set X is not downward closed, i.e., whenever $\forall x \in X \exists y \in A \setminus X : x \in f^A(y)$.

Utilising fact 2.23, we can reformulate the constructions of A_X and A^X. Let $Cl(\overline{X})$ be the supremum (in the respective lattice, mentioned after fact 2.28) of all subsets of A closed under A-operations and not intersecting X, i.e.,

$$Cl(\overline{X}) = \bigsqcup_i X_i : X_i \cap X = \emptyset \wedge (x \in X_i \wedge y \in f^A(x) \Rightarrow y \in X_i).$$

On the other hand, let $cl(X)$ be the infimum of all $X_i \subseteq A$ which are closed under A-operations and whose complement is contained in X ($A \setminus X_i \subseteq X$ or, equivalently, $X_i \cup X = A$), i.e.,

$$cl(X) = \bigsqcap_i X_i : A \setminus X_i \subseteq X \wedge (x \in X_i \wedge y \in f^A(x) \Rightarrow y \in X_i).$$

We then have alternative formulations of the two facts:

2.29. $A_X = A \setminus Cl(\overline{X})$.

2.31. $A^X = A \setminus cl(X)$.

Finally, we have the expected relations between homomorphic images and subalgebras.

Lemma 2.32 *Let $\phi : A \to B$ be a homomorphism:*

1) *The image $\phi[A] \sqsubseteq B$ is a subalgebra of B.*
2) *For any $B' \sqsubseteq B : \phi^-[B'] \sqsubseteq A$.*
3) *For any $A' \sqsubseteq A : \phi[A'] \sqsubseteq B$.*

Proof:

1) By fact 2.23, it is enough to show that $\phi[A]$ is closed under pre-images of B-operations. If $b' \in \phi[A]$ and $b' \in f^B(b)$ then, since ϕ is OT, $\emptyset \neq \phi^-(b') \subseteq f^A(\phi^-(b))$. But this implies that $f^A(\phi^-(b)) \neq \emptyset$, i.e., $b \in \phi[A]$.
2) By fact 2.23, it is enough to show that $\phi^-[B']$ is closed under pre-images of A-operations. If $a' \in \phi^-[B']$ then $\phi(a') \in B'$, and if $a' \in f^A(a)$ then also $\phi(a') \in \phi(f^A(a)) \subseteq f^B(\phi(a))$, since OT are also weak. By assumption, B' is closed under pre-images of B-operations, so the last inclusion impilies $\phi(a) \in B'$, i.e., $a \in \phi^-[B']$.
3) Follows directly from 1, since the restriction of ϕ to A' (pre-composition with the inclusion $A' \subseteq A$) is a homomorphism. □

Point 1 gives immediately epi-mono factorisation of homomorphisms: any $\phi : A \to B$ can be factored as $\phi = e; m$ where $e : A \to \phi[A]$ is epi and $m : \phi[A] \to B$ is mono. We will address this factorisation in connection to congruences below (lemma 2.40).

2.3 OT-congruences

In order for the quotient construction performed on a carrier of a (classical) Σ-algebra to yield a (quotient) Σ-algebra, the equivalence must be a Σ-congruence. However, for any (classical) algebra A and any equivalence $\sim$ on its carrier, the quotient $A/_\sim$, with operations collecting the possibly non-congruent results (i.e., defined by $R^{A/\sim}([a]) = \{[n] \mid n \in R^A(a'),\ a' \in [a]\}$), is a multialgebra, and the construction works in the same way if we start with a multialgebra, and not only classical algebra, A. Defining the mapping $q : A \to A/_\sim$ by $q(a) = [a]$, the operations are obtained as $R^{A/\sim} = q^-; R^A; q$. In general, this mapping is only a weak homomorphism, just like the kernel of a weak homomorphism is, in general, only an equivalence. (This correspondence is perhaps the clearest expression of the weakness of this homomorphism notion.) OT-homomorphisms come along with a much stronger notion of congruence.

Definition 2.33 *An equivalence $\sim$ on A is an OT-congruence iff: $\sim; R^A; \sim\ =\ \sim; R^A$.*

More explicitly, the inclusion $\subseteq$ says that

$$\forall a'', a', b, b' : a'' \sim a' R^A b' \sim b \Rightarrow \exists a : a'' \sim a R^A b, \tag{2.34}$$

which, when $\sim$ is equivalence, is equivalent to

$$\forall a', b, b' : a' R^A b' \sim b \Rightarrow \exists a : a' \sim a R^A b. \tag{2.35}$$

((2.35) is a special case of (2.34) whenever $\sim$ is reflexive, while transitivity (and symmetry) of $\sim$ yields the opposite implication.) Any equivalence satisfying this last condition is OT, since the opposite inclusion $\sim; R^A; \sim\ \supseteq\ \sim; R^A$ holds trivially for any reflexive $\sim$.

This characterisation of OT-congruence can be viewed as a converse (bi)simulation.[2] (Bi)simulation requires propagation of $\sim$ forward, while OT-congruence backward. Let us call a relation satisfying two symmetric conditions (for each $R \in \Sigma$):

$$\begin{array}{lcl} \forall b, b'\ \forall a' : b \sim b' \wedge a' R^A b' & \Rightarrow & \exists a : a \sim a' \ \wedge\ a R^A b \\ \&\ \ \forall b, b'\ \forall a\ : b \sim b' \wedge\ a R^A b & \Rightarrow & \exists a' : a \sim a' \wedge a' R^A b' \end{array} \tag{2.36}$$

"bireachability" – OT-congruence is then an equivalence which is also bireachability or, simply, equivalence satisfying (2.35) (since symmetry makes (2.35) imply (2.36)).

$$\begin{array}{ccc} \text{(bi)simulation} & \qquad & \text{bireachability} \\ \begin{tikzcd} b \arrow[r, dotted, no head, "\sim"] & b' \\ a \arrow[u, "R"] \arrow[r, no head, "\sim"] & a' \arrow[u, dotted, "R"'] \end{tikzcd} & & \begin{tikzcd} b \arrow[r, no head, "\sim"] & b' \\ a \arrow[u, "R"] \arrow[r, dotted, no head, "\sim"] & a' \arrow[u, dotted, "R"'] \end{tikzcd} \\ \sim; R \subseteq R; \sim & & R; \sim\ \subseteq\ \sim; R \end{array} \tag{2.37}$$

We can describe bireachability/OT-congruence in the following terms dual to the classical congruence. Classical congruence requires propagation of the relation: if two elements are related, $a_1 \sim a_2$, then also their results are, $R(a_1) \sim R(a_2)$. Bireachability requires propagation of distinctions, albeit in a special way. Given two elements b_1, b_2, denote their pre-images under R by $P_1 = R^-(b_1)$ and $P_2 = R^-(b_2)$. Two such sets are 'related' by (an extension of) $\sim$ iff $\forall p_1 \in P_1 \exists p_2 \in P_2 : p_1 \sim p_2$ and vice versa. We then have that if $P_1 \not\sim P_2$ then also $b_1 \not\sim b_2$. (But note that there may be many mutually unrelated elements, i.e., $s_1 \in P_1$ and $s_2 \in P_2$ with $s_1 \not\sim s_2$.)

Fact 2.38 *If $\phi : A \to B$ is OT then so is its kernel $\sim_\phi = ker(\phi) = \phi; \phi^-$.*

PROOF:
$$\begin{array}{rcll} \phi^-; R^A &=& R^B; \phi^- & (\phi\ is\ OT) \\ \phi; \phi^-; R^A &=& \phi; R^B; \phi^- & \\ \sim_\phi; R^A &=& \phi; R^B; \phi^-. & \end{array}$$

On the other hand, we also have:
$$\begin{array}{rcll} \phi^-; R^A &=& R^B; \phi^- & (\phi\ is\ OT) \qquad . \\ \phi; \phi^-; R^A; \phi; \phi^- &=& \phi; R^B; \phi^-; \phi; \phi^- & \\ \sim_\phi; R^A; \sim_\phi &=& \phi; R^B; \phi^- & (since\ \phi^-; \phi; \phi^- = \phi^-) \end{array}$$

which gives the conclusion when combined with the above. □

The inverse does not hold generally; even if $ker(\phi)$ is OT, ϕ itself may be not, even if it is surjective. (The mapping

$$\begin{tikzcd} & a_2 & \overset{\phi}{\rightarrow} & b_2 & \\ a_3 & a_1 \arrow[u, "R"] \arrow[l, "R"] & & b_1 \arrow[u, "R"] & b_3 \end{tikzcd}$$

defined as $\phi(a_i) = b_i$ has the kernel id_A, which is OT, but ϕ is not an OT-homomorphism.) We have a slightly weaker claim.

Fact 2.39 *If $\sim$ is an OT-congruence then the mapping $q : A \to Q = A/_\sim$, $q(a) = [a]$, is an OT-homomorphism.*

PROOF: (The operations in Q are defined by $R^Q = q^-; R^A; q$.)
$$\begin{array}{rcll} q; q^-; R^A; q; q^- &=& q; q^-; R^A & assumption,\ since\ \sim\ =\ \sim_q = q; q^- \\ q; R^Q; q^- &=& q; q^-; R^A & def.\ of\ Q \\ q^-; q; R^Q; q^- &=& q^-; R^A & \\ id_Q; R^Q; q^- &=& q^-; R^A & q\ is\ surjective \end{array}$$
□

[2] We are not addressing any details concerning bisimulations. For the sake of analogy, since OT-congruences are equivalences, it is most convenient to think of bisimulation defined as a symmetric simulation, rather than merely as a simulation with converse being also a simulation. Exact duality obtains between our bireachability and the equivalences satisfying the condition that for every $R : \sim; R^A; \sim\ =\ R^A; \sim$. This characterizes the bisimulation in (2.37) and is the same as the congruence induced by the coalgebraic model of binary relations, referred to in remark 1.12. In [6] such equivalences were said to "preserve the arguments" (in contradistinction to congruences which "preserve the values"). In [18], the relation dual to mere simulation, without the requirement of equivalence, was called "opsimulation," but the name "biopsimulation" does not seem very appealling.

This gives epi-mono factorisation of morphisms in $\mathsf{MAlg}_{OT}(\Sigma)$.

Lemma 2.40 *For every homomorphism $h : A \to B$ there is a (regular) epi $e : A \to Q$ and mono $m : Q \to B$ such that $h = e; m$.*

Proof: We let $\sim$ denote the kernel of h and choose $Q = A/_{\sim}$. By Fact 2.39, $e : A \to Q$ defined by $e(a) = [a]$, is an epi in $\mathsf{MAlg}_{OT}(\Sigma)$. (It is regular by Fact 2.51.) We verify that m, defined by $m([a]) = h(a)$ is OT. (It is trivially injective, and hence mono by 2.13, and makes $h = e; m$ by definition.) Let $b \in B$ and assume first that $m^{-}(b) = \{[a]\} \neq \emptyset$:

$$\begin{array}{rcll}
f^Q(m^{-}(b)) & = & f^Q([a]) & \text{definition of } m \\
& = & \{[c] \mid c \in f^A(a) : h(a) = b\} & \text{definition of } Q \text{ with } a : h(a) = b \\
& = & \{[c] \mid c \in f^A(h^{-}(b))\} & \\
& = & \{[c] \mid c \in h^{-}(f^B(b))\} & h \text{ is OT} \\
& = & e(h^{-}(f^B(b))) & \text{definition of } e \\
& = & e(e^{-}(m^{-}(f^B(b)))) & \text{since } h^{-}(x) = e^{-}(m^{-}(x)) \\
& = & m^{-}(f^B(b)) & \text{since } e^{-}; e = id_Q
\end{array}$$

The same argument applies also when $m^{-}(b) = \emptyset$, since this implies that $h^{-}(b) = \emptyset$. □

Corollary 2.41 *1) For an epi $\phi : A \to B$ with kernel $\sim$, $A/_{\sim} \simeq B$.*
2) If $e_i : A \to B_i$, $i \in \{1, 2\}$, are epis with equal kernels, $\sim_1 = \sim_2$, then $B_1 \simeq B_2$.

Proof:

1) By 2.40, we have an epi-mono factorisation of $\phi = e; m$. But since ϕ is epi, so is m. As m is also mono, it is bijective and thus iso by proposition 2.13.
2) By 1, we have $A/_{\sim_1} \simeq B_1$ and $A/_{\sim_2} \simeq B_2$. But $A/_{\sim_1} = A/_{\sim_2}$. □

Hence the epi-mono facorization from 2.40 coincides with the factorisation mentioned after lemma 2.32, in the sense that $A/_{\sim} \simeq \phi[A]$.

Remark 2.42 *Recall remark 2.8 in which topology on a multialgebra reflected the possible observations of its elements by means of the results of the operations. In the present context, bireachability can be seen as topological indistinguishability – albeit, not as in the topological tradition, of topological spaces or features invariant under homeomorphisms, but of actual elements of a given topological space.*

As an immediate corollary of the fact that the quotient morphism is OT and that such homomorphisms are continuous, remark 2.8, we obtain that, for instance, pre-image of an Q-open is A-open, i.e., that $q^{-}(f^Q([x]))$ can be written as (possibly union of intersections, and possibly of different symbols but, as it turns out, simply as) $\bigcup_{x \in [x]} f^A(x)$. (This can be verified directly, for $f^Q([x]) = \{[y] \mid y \in f^A([x])\}$, i.e., its pre-image $q^{-}(f^Q([x])) = \{y' \mid \exists y \sim y' : y \in \bigcup_{x \in [x]} f^A(x)$ which, by OT, is equal to $\bigcup_{x \in [x]} f^A(x)$.)

However, the topology obtained by our quotient construction according to remark 2.8, is not exactly the same as the standard quotient topology on the quotient space, i.e., one according to which $Y \subseteq Q$ is open iff $q^{-}(Y)$ is A-open. For instance:

$$\begin{tikzcd}
a_1 & a_2 & \arrow[r, "q"] & & {[a_1]} & {[a_2]} \\
b_1 \arrow[u, "f"'] & b_2 \arrow[u, "f"'] & \arrow[r, "q"'] & & {[b_1, b_2]} \arrow[u, "f"] \arrow[ur, "f"'] &
\end{tikzcd}$$

The pre-image $q^{-}([a_1]) = \{a_1\} = f^A(b_1)$ and hence is open, but $[a_1]$ is not since the only open set in Q (besides $\emptyset$) is the whole carrier of the a-sort $\{[a_1], [a_2]\} = f^Q([b_1, b_2])$.

2.3.1 The complete lattice of OT-congruences on an algebra

The condition (2.36) is trivially preserved by taking unions of bireachabilities and so, for any algebra, there is the maximal (with respect to $\subseteq$) bireachability, namely, the union of all bireachabilities. We address now the more specific question of the existence of maximal OT-congruence.

Given a collection $C = \{\sim_i \mid i \in I\}$ of equivalences (on a set/algebra A), one obtains their supremum $\sim = \bigvee_i \sim_i$ as the transitive closure of their union, i.e., $\bigvee_i \sim_i = (\bigcup_i \sim_i)^*$. Explicitly, one lets $a \sim a'$ iff there exists a finite sequence $a = a_0 a_1 ... a_n = a'$ and a respective sequence of the equivalences from C, $\sim_1 \sim_2 \ ... \ \sim_n$, such that $a_i \sim_{i+1} a_{i+1}$ for all $0 \leq i < n$. As all members of C are equivalences, then so is the transitive closure of their union by the standard argument (e.g., [14], §5, th.2). The construction applies also to OT-congruences.

Lemma 2.43 *Given a collection $C = \{\sim_i \mid i \in I\}$ of OT-congruences on a multialgebra A, then $\sim = \bigvee_{i \in I} \sim_i$ is an OT-congruence.*

PROOF: Assume that for each $i : \sim_i; R^A; \sim_i \ = \ \sim_i; R^A$. We have to show that then $\sim; R^A; \sim \ = \ \sim; R^A$. The inclusion $\sim; R^A; \sim \ \supseteq \ \sim; R^A$ is trivial, so we show the opposite.
Assume $\langle a, b \rangle \in \ \sim; R^A; \sim$, i.e., there are the respective sequences such that $a \sim_{a1} a_1 \sim_{a2} a_2 ... \sim_{an} a_n R^A b_0 \sim_1 b_1 \sim_2 b_2 ... \sim_m b$. By induction on m we show that then also $\exists a' : a \sim a' R^A b$ which will establish the claim. The basis for $m = 0$ is trivial, so assume IH

$$\begin{array}{llcl} IH \ \ \forall a, a_0, b_0, ..., b_m : & a \sim a_0 R^A b_0 \sim_1 b_1 ... \sim_m b_m & \Rightarrow & \exists a' : a \sim a' R^A b_m, \\ & \quad\quad\quad\quad \textit{and} & & \\ & a \sim a_0 R^A b_0 \sim_1 b_1 ... \sim_m b_m \sim_{m+1} b_{m+1} & & \end{array}$$

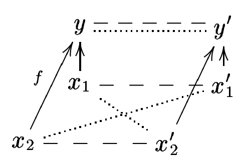
From the latter we obtain, by IH, $a \sim a' R^A b_m$, and $b_m \sim_{m+1} b_{m+1}$. Since $\sim_{m+1}$ is OT, there is an $a'' \sim_{m+1} a'$ such that $a'' R^A b_{m+1}$. But then we can just extend the chain $a \sim a' \sim_{m+1} a''$ obtaining $a \sim a'' R^A b_{m+1}$. □

In particular, performing this construction on the collection of *all* OT-congruences on a given multialgebra A yields the maximal OT-congruence on A. Notice, however, that it need not be the standard universal relation. For instance, for the algebra $\begin{tikzcd} b_1 & b_2 \\ a_1 \arrow[u, "R"] \end{tikzcd}$ the elements b_1 and b_2 cannot be related by any OT-congruence, since it would violate the condition (2.36).

One verifies easily that the construction yields, in fact, the least upper bound – with respect to subset relation – of the argument congruences. Thus, the collection of all OT-congruences on a multialgebra is a complete upper semilattice – with respect to the subset relation – with identity being the least element. And so, by the standard result (e.g., [15], p.24), we have the useful fact.

Fact 2.44 *The collection of all OT-congruences on an algebra is a complete lattice.*

Infima are not, however, obtained as mere intersections. In the following algebra:



both relations:

$$\begin{array}{lcll} \overline{\approx} & = & id \cup \{\langle y, y' \rangle, \langle y', y \rangle, \langle x_1, x_1' \rangle, \langle x_1', x_1 \rangle, \langle x_2, x_2' \rangle, \langle x_2', x_2 \rangle\} & \text{marked with the dashed lines} \\ \dot{\sim} & = & id \cup \{\langle y, y' \rangle, \langle y', y \rangle, \langle x_1, x_2' \rangle, \langle x_2', x_1 \rangle, \langle x_2, x_1' \rangle, \langle x_1', x_2 \rangle\} & \text{marked with the dotted lines} \end{array}$$

are OT-congruences. Their intersection, however, $id \cup \{\langle y, y' \rangle, \langle y', y \rangle\}$, is not an OT-congruence. In fact, the infimum of the two will be identity.

Given two OT-congruences, $\sim_a, \sim_b$, on an algebra A, their infimum $\sim \ = \ \sim_a \wedge \sim_b$ can be constructed by "propagating the distinctions" (cf. the remark following (2.37)) as follows:

$$\begin{array}{llcl} 1. & \sim_0 & = & \sim_a \cap \sim_b \\ 2. & \sim_{i+1} & = & \sim_i \setminus D_i, \textit{ where } D_i \textit{ is the set of all } \langle y_1, y_2 \rangle, \langle y_2, y_1 \rangle \in \sim_i \\ & & & \textit{such that} : \exists x_1 \in f^-(y_1) \ \forall x_2 \in f^-(y_2) : x_1 \not\sim_i x_2 \\ 3. & \sim & = & \bigcap_{i \in \omega} \sim_i \end{array} \tag{2.45}$$

As $\sim_0$ is an equivalence relation, so $\sim$ is obviously reflexive and symmetric. To see that it is transitive, we show that each $\sim_{i+1}$ is transitive. $\sim_0$ is transitive, so assume $\sim_i$ be transitive and let $y_1 \sim_{i+1} y_2 \sim_{i+1} y_3$. (Hence also $y_1 \sim_i y_2 \sim_i y_3$ and, since $\sim_i$ is transitive, so $y_1 \sim_i y_3$.) Then $\forall x_1 \in f^-(y_1)\ \exists x_2 \in f^-(y_2) : x_1 \sim_i x_2$, and likewise $\forall x_2 \in f^-(y_2)\ \exists x_3 \in f^-(y_3) : x_2 \sim_i x_3$. But since $\sim_i$ is transitive, this implies that also $\forall x_1 \in f^-(y_1)\ \exists x_3 \in f^-(y_3) : x_1 \sim_i x_3$ (and vice versa), i.e., $y_1 \sim_{i+1} y_3$.

The condition in step 2 amounts to removing all pairs which violate the bireachability requirement (2.36). That is, any OT-congruence contained in $\sim_0$ must not contain any of these pairs. On the other hand, the resulting $\sim$ is indeed an OT-congruence. By the above argument, it is transitive and hence an equivalence. Moreover, whenever $y_1 \sim y_2$, then the negation of condition 2 holds, i.e., $\forall x_1 \in f^-(y_1)\ \exists x_2 \in f^-(y_2) : x_1 \sim x_2$ (and vice versa). Thus, indeed, $\sim\ =\ \sim_a \wedge \sim_b$. (Infimum of a set of congruences $\{\sim_i \mid i \in I\}$ is constructed in the same way, only starting with $\bigcap_{i\in I} \sim_i$.)

Let $\approx_A$ denote the maximal OT-congruence and $\sim_A$ the maximal bireachability (i.e., union of all bireachabilities) on A.

Proposition 2.46 *The maximal bireachability on A is an equivalence, in fact: $\approx_A\ =\ \sim_A$.*

PROOF: Since OT-congruence is bireachability, we obviously have $\approx_A\ \subseteq\ \sim_A$. The opposite inclusion follows because every bireachability is included in some OT-congruence. Namely, given a bireachability $\sim$, its converse $\sim^-$ is also bireachability, which follows trivially by inspecting the definition (2.36). Likewise, union of bireachabilities is a bireachability, in particular, the reflexive, symmetric closure of $\sim$, i.e., $\simeq\ =\ \sim \cup \sim^- \cup id_A$ is a bireachability. One also verifies easily that transitive closure of a bireachability is a bireachability:

$$\begin{array}{lll} b \sim b_1 \sim b_2 & \Rightarrow & \forall aRb \quad \exists a_1 : a \sim a_1 \wedge a_1 R b_1 \\ & \Rightarrow & \forall aRb \quad \exists a_1 : a \sim a_1 \wedge a_1 R b_1 \wedge \exists a_2 : a_1 \sim a_2 \wedge a_2 R b_2 \\ & \Rightarrow & \forall aRb \quad \exists a_1, a_2 : a \sim a_1 \sim a_2 \wedge a_2 R b_2 \end{array}$$

Consequently, the equivalence closure $\simeq_A^*$ of $\sim_A$ is OT-congruence, and so: $\sim_A\ \subseteq\ \simeq_A^*\ \subseteq\ \approx_A$. □

Fact 2.47 *Let $B \sqsubseteq A$, $\sim_A$ be an OT-congruence on A, and $\sim_B\ \subseteq\ \sim_A$ be restriction of $\sim_A$ to the carrier of B, i.e., $\sim_A \cap B \times B$. Then*

1) $\sim_B$ is OT-congruence on B and

2) $\sim_B \cup id_A$ is OT-congruence on A.

PROOF: 1. Let $a, a_1, b_1, b \in B$ be such that $a \sim_B a_1 R^B b_1 \sim_B b$. Then also $a \sim_A a_1 R^A b_1 \sim_A b$ and since $\sim_A$ is OT-congruence on A, so $\exists a_0 \in A : a \sim_A a_0 R^A b$. Since $B \sqsubseteq A$, B is closed under pre-images of A-operations, so $b \in B$ implies $a_0 \in B$, and then $a \sim_B a_0 R^B b$.

2. Let us write $\sim_B$ for the union $\sim_B \cup id_A$. To show that $\sim_B; R^A; \sim_B\ \subseteq\ \sim_B; R^A$, consider the cases of $a_0 \sim_B a_1 R^A a_2 \sim_B a_3$:

– when all $a_i \in B$, there exists an $a' : a_0 \sim_B a' R^A a_3$ since $\sim_B$ is bireachability on B;

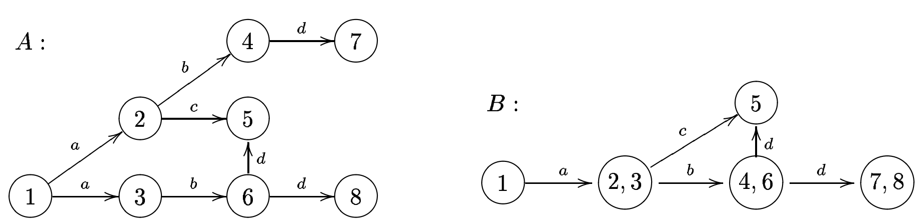
– if $a_0 \notin B$ while $a_1 \in B$, then $a_0 = a_1$ and the result follows when $a_2, a_3 \in B$;

– if $a_2 \notin B$ or $a_3 \notin B$, then $a_2 = a_3$ and the result follows trivially;

– if $a_1 \notin B$ then also $a_2 \notin B$ since $B \sqsubseteq A$ (i.e., inclusion is OT), and so $a_0 = a_1$ and $a_3 = a_2$. □

As an example of quotienting an algebra by a (maximal) OT-congruence, we can consider a kind of minimization of a nondeterministic automaton.

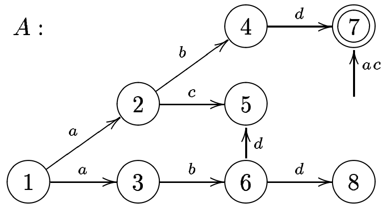
Example 2.48 *The automaton (multialgebra) A from example 2.30, quotiented by the largest OT-congruence yields the automaton (multialgebra) $B = A/_\sim$:*



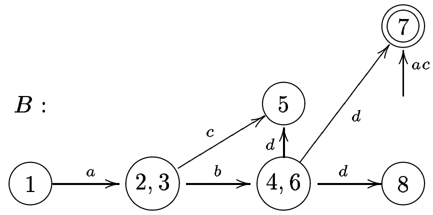
We cannot have $5 \sim 8$ because, although both can be reached by d from 6, i.e., $5, 8 \in d^A(6)$, so $5 \in c^A(2)$ while $8 \notin c^A(2)$ and $8 \notin c^A(3)$ and there are no more states $s \sim 2$.

To model accepting states, we introduce additional constant ac. (Likewise, we can introduce a constant st for identifying the initial state.) If we let only 7 in A be the accepting state, the picture will be modified accordingly:

A :



B :



Obviously, with respect to the accepted language, the obtained automaton is not minimal (we could, for instance, safely remove states 8 and 5). It remains to determine what – if any – known or useful construction on automata is represented by the quotient by OT-congruence.

Example 2.49 *A more refined notion of OT-congruence on automata can be obtained by an alternative model in which an automaton is represented as one operation $tr : S \times Alph \to S$, taking a state and an alphabet symbol and returning the set of possible resulting states. In this case, we can, in addition, consider also various bireachabilities on the alphabet symbols. When it is identity, we obtain the same result as in the previous example. On the other hand, if it is the total relation (no operations returning Alph-elements leaves us full freedom in determining OT-congruence on this sort), $Alph^A \times Alph^A$, the maximal OT-congruence identifies states s, t iff for each number of steps in which s can be reached from some state s', t can be reached in the same number of steps from a state t' which is bireachable with s', and vice versa.*

Thus, for instance, if we represent the search space by a multialgebra A with the subset $X \subseteq A$ of goals, the subalgebra A_X represents, as at the end of example 2.30, the states from which some goal in X is reachable, and then, quotient by the maximal OT-congruence (identifying all symbols), will yield, roughly, a collection of paths (possibly with loops and common nodes) leading to X and having distinct lengths.

2.3.2 Σ-structure of OT-congruence

Just like classical Σ-congruence has algebraic Σ-structure, so OT-congruence on a Σ-multialgebra has itself a multialgebraic Σ-structure. In fact, we define such a structure for an arbitrary bireachability and all the results apply to OT-congruences as special cases.

Definition 2.50 *For a bireachability $\sim$ on $A \in \mathsf{MAlg}_{OT}(\Sigma)$, we define $A^{\sim} \in \mathsf{MAlg}_{OT}(\Sigma)$:*

- $A^{\sim} = \{\langle a_1, a_2\rangle \mid a_1, a_2 \in A \wedge a_1 \sim a_2\}$, *and*
- $f^{A^{\sim}}(\langle a_1, b_1\rangle ... \langle a_n, b_n\rangle) = \{\langle x, y\rangle \mid x \in f^A(a_1...a_n) \wedge y \in f^A(b_1...b_n) \wedge x \sim y\}$, *i.e., for constants* $c^{A^{\sim}} = \{\langle x, y\rangle \mid x, y \in c^A \wedge x \sim y\}$.

Fact 2.51 *Given a bireachability $\sim$ on A.*

1) The projections $\pi_1, \pi_2 : A^{\sim} \to A$, $\pi_i(\langle a_1, a_2\rangle) = a_i$ are OT.

2) $A/_{\sim}$ with the quotient homomorphism $q : A \to A/_{\sim}$ is their coequalizer.

Proof: 1. We verify that π_1 is OT. $\pi_1^-(a) = \{\langle a, x\rangle : x \sim a\}$, and thus:

(i) $\quad \pi_1^-(f^A(a)) \quad = \quad \{\langle b, y\rangle \mid b \in f^A(a), y \sim b\}$, while

(ii) $\quad f^{A^{\sim}}(\pi_1^-(a)) \quad = \quad f^{A^{\sim}}(\{\langle a, x\rangle \mid x \sim a\}) = \{\langle b, y\rangle \mid b \in f^A(a), y \sim b, y \in f^A(x), x \sim a\}$

Obviously (ii) $\subseteq$ (i). The opposite inclusion holds because $\sim$ is bireachability: if $b \in f^A(a)$ and $y \sim b$ then, by (2.36), $\exists x \sim a : y \in f^A(x)$. But this is exactly the restriction in (ii).

$$\begin{tikzcd}
A^{\sim} \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "q"] \arrow[dr, "h"'] & A/_{\sim} \arrow[d, "c"] \\
& & C
\end{tikzcd}$$

2. For every $\langle a_1, a_2\rangle \in A^{\sim}$, we have $q(a_1) = q(a_2)$, so $\pi_1; q = \pi_2; q$. Assume some other $h : A \to C$ with $\pi_1; h = \pi_2; h$. Define $c : A/_{\sim} \to C$ by $c([a]) = h(a)$. It is well defined, for

if $a \sim a'$, i.e., $\langle a, a' \rangle \in A^{\sim}$, then $h(a) = h(a')$ by assumption. Obviously $q; c = h$ and this equality forces also its uniqueness.
To see that c is OT, consider:
(i) $f^{A/\sim}(c^-(c_1)...c^-(c_n)) = f^{A/\sim}(q(h^-(c_1))...q(h^-(c_n)))$ since $c^-(c_i) = q(h^-(c_i))$ and
(ii) $c^-(f^C(c_1...c_n)) = q(h^-(f^C(c_1...c_n))) = q(f^A(h^-(c_1)...h^-(c_n)))$ since h is OT.
To see that (i)=(ii), we observe that the h pre-image of any $c_i \in C$ consists of one or more $\sim$-equivalence classes: (*) $h^- = h^-; q; q^-$, simply because $h^- = c^-; q^-$ and $c^- = h^-; q$. So, since q is OT, we have the first equality, and since $q^-; q = id_{A/_\sim}$ and h is OT, the last one:

$$\begin{array}{rcl}
q^-\Big(f^{A/\sim}(q(h^-(c_1))...q(h^-(c_n)))\Big) & = & f^A\Big(q^-(q(h^-(c_1)))...q^-(q(h^-(c_n)))\Big) \\
(*) & = & f^A\Big(h^-(c_1)...h^-(c_n)\Big) \\
q\Big(q^-\Big(f^{A/\sim}(q(h^-(c_1))...q(h^-(c_n)))\Big)\Big) & = & q\Big(f^A\Big(h^-(c_1)...h^-(c_n)\Big)\Big) \\
(i) \; = \; f^{A/\sim}\Big(q(h^-(c_1))...q(h^-(c_n))\Big) & = & q\Big(h^-(f^C(c_1...c_n))\Big) \; = \; (ii)
\end{array}$$

□

Corollary 2.52 *All epis in* $\mathsf{MAlg}_{OT}(\Sigma)$ *are regular.*

Proof: Given an epi $e : A \to B$ with kernel $\sim$, we have by corollary 2.41 an isomorphism $i : A/_\sim \simeq B$ and such that $q; i = e$ where $q : A \to A/_\sim$. But since q is coequalizing, so is e. □

Strictly speaking, congruence on A is a (special kind of morphism) $i : R \to A \times A$. But we will not verify the existence of products in our category until section 4.2, and so we abbreviate the respective $i; \pi_i$ as the span $r_1, r_2 : R \to A$. In the standard way, given any relation $p_1, p_2 : P \to A$, its congruence closure is the equalizer of $p_1; q$ and $p_2; q : A \to A/_P$, where $(A/_P, q)$ is coequalizer of p_1 and p_2. Assuming that all such equalizers and coequalizers exist (which will be shown first in proposition 3.12), we also obtain the following standard result.

Fact 2.53 *Given OT-congruences* P, R *on* $A : P \subseteq R \Rightarrow \exists h : A/_P \to A/_R$ *with* $q_P; h = q_R$.

Proof: We consider the diagram:

$$\begin{tikzcd}
 & P \arrow[dl, dotted, hook', "i"'] \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "p_1"] & \\
R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & A \arrow[r, "q_R"] \arrow[d, "q_P"] & A/_R \\
 & A/_P \arrow[ur, dotted, "h"'] &
\end{tikzcd}$$

As P, R are congruences on A, so $(A/_R, q_R)$, resp. $(A/_P, q_P)$, coequalize r_1, r_2, resp., p_1, p_2. Assume $P \subseteq R$, i.e., for $k \in \{1, 2\} : p_k = i; r_k$, where i is the inclusion. First, since q_R coequalizes r_1 and r_2, we obtain that also $i; r_1; q_R = i; r_2; q_R$, i.e., $p_1; q_R = p_2; q_R$. But as $(A/_P, q_P)$ is coequalizer of p_1, p_2, we obtain a unique $h : A/_P \to A/_R$ making $q_P; h = q_R$. □

2.3.3 Bireachabilities between algebras

The notion (2.36) of bireachability on an algebra is a special case of the following notion of bireachability *between* algebras.

Definition 2.54 *Bireachability* between *two algebras* A *and* B *is a subset* $\sim \subseteq A \times B$ *satisfying the following bireachability condition:*

$$\begin{array}{rl}
\forall a, b, a_1 : & a \sim b \wedge a \in f^A(a_1) \Rightarrow \exists b_1 \in B : b \in f^B(b_1) \wedge a_1 \sim b_1 \\
\& \;\; \forall a, b, b_1 : & a \sim b \wedge b \in f^B(b_1) \Rightarrow \exists a_1 \in A : a \in f^A(a_1) \wedge a_1 \sim b_1
\end{array}$$

Relation $\sim$ is a bireachability on A according to (2.36) iff it is a bireachability between A and A according to the above definition.

A bireachability $\sim$ between A and B can be given a natural Σ-structure, generalising definition 2.50, as follows

$$f^{\sim}(\langle a_1, b_1\rangle ... \langle a_n, b_n\rangle) = f^A(a_1...a_n) \times f^B(b_1...b_n) \cap \sim . \tag{2.55}$$

When addressing algebra structure of some bireachability we will always mean the above condition unless explicitly stated otherwise. An equivalent formulation of $\sim$ being a bireachability is then as follows.

Lemma 2.56 *$\sim \subseteq A_1 \times A_2$ (with the Σ-structure given by (2.55)) is a bireachability iff the projections $\pi_i : \sim \to A_i$, $\pi_i(\langle a_1, a_2\rangle) = a_i$, are homomorphisms.*

PROOF: $\Rightarrow$) We verify that $f^{\sim}(\pi_1^-(a_1)) = \pi_1^-(f^{A_1}(a_1))$. If $\langle a, b\rangle \in \pi_1^-(f^{A_1}(a_1))$ then $a \in f^{A_1}(a_1)$ and $a \sim b$ so, by 2.54, $\exists b_1 : a_1 \sim b_1$ and $b \in f^{A_2}(b_1)$. But then $\langle a_1, b_1\rangle \in \pi_1^-(a_1)$ and by (2.55) $\langle a, b\rangle \in f^{\sim}(\langle a_1, b_1\rangle)$.
Conversely, if $\langle a, b\rangle \in f^{\sim}(\pi_1^-(a_1))$ then, by (2.55), $a \in f^{A_1}(a_1)$. But then obviously $\langle a, b\rangle \in \pi_1^-(f^{A_1}(a_1))$.
$\Leftarrow$) Assume both π_i are OT and let $a \in f^{A_1}(a_1)$ and $a \sim b$. Since π_1 is OT we have $\pi_1^-(f^{A_1}(a_1)) = f^{\sim}(\pi_1^-(a_1))$ and since $\langle a, b\rangle \in \pi_1^-(a) \subseteq \pi_1^-(f^{A_1}(a_1))$ so also $\langle a, b\rangle \in f^{\sim}(\pi_1^-(a_1))$, i.e., $\exists b_1 \in A_2 : \langle a, b\rangle \in f^{\sim}(\langle a_1, b_1\rangle)$. But by (2.55) this last fact means that $\exists b_1 \in A_2 : b \in f^{A_2}(b_1)$ and $a_1 \sim b_1$. $\square$

As a special case, and in analogy to the case of coalgebras whose homomorphisms are functional bisimulations, the OT-homomorphisms are functional bireachabilities.

Fact 2.57 *A function $\phi : A \to B$ is OT-homomorphism iff its graph $Gr(\phi) = \{\langle a, \phi(a)\rangle : a \in A\}$ is a bireachability between A and B.*

PROOF: Denote the projections by $\pi_A, \pi_B : Gr(\phi) \to A, B$, i.e., $\pi_i(\langle x_1, x_2\rangle) = x_i$. We have that $\pi_A; \phi = \pi_B$ and π_A is a bijection.
$\Leftarrow$) If π_i's are OT then, π_A being iso, so is its converse π_A^-. But since $\phi = \pi_A^-; \pi_B$, so ϕ is OT.
$\Rightarrow$) Assume ϕ to be OT, and define Σ-structure on $Gr(\phi)$ by letting $f^{Gr}(\langle a, b\rangle) = \{\langle a', b'\rangle \in Gr(\phi) \mid a' \in f^A(a)\}$. Since $\pi_B = \pi_A; \phi$, it sufficies to verify that π_A is OT. $f^{Gr}(\pi_A^-(a)) = f^{Gr}(\langle a, b\rangle) = \{\langle a', b'\rangle \in Gr(\phi) \mid a' \in f^A(a)\} = \pi_A^-(f^A(a))$. $\square$

Finally, we can also generalise lemma 2.56 as follows.

Lemma 2.58 *An arbitrary span $A_1 \overset{\phi_1}{\leftarrow} B \overset{\phi_2}{\rightarrow} A_2$, induces a bireachability between A_1 and A_2 given by $\sim = \{\langle \phi_1(b), \phi_2(b)\rangle \mid b \in B\}$.*

PROOF: We verify that the bireachability condition is satisfied. Assume $a_1 \sim a_2$, i.e., for some $b : \langle \phi_1(b), \phi_2(b)\rangle = \langle a_1, a_2\rangle$ and $a_1 \in f^{A_1}(x_1)$. Since ϕ_1 is OT, we then have $x_1 \in \phi_1[B]$, i.e., for some $y \in B : \phi_1(y) = x_1$ and $b \in f^B(y)$. But then, since ϕ_2 is OT (and hence also weak), $\phi_2(f^B(y)) \subseteq f^{A_2}(\phi_2(y))$, i.e., $a_2 \in f^{A_2}(\phi_2(y))$ and we have the required witness $x_2 = \phi_2(y)$ with $x_1 \sim x_2$. $\square$

Thus, any bireachability is a span (according to lemma 2.56) and, conversely, any span induces a bireachability according to the above lemma.

An unpleasant fact is that, given a bireachability $\sim$ induced by a span as above in lemma 2.58, with the algebra structure given by the equation (2.55), there need not be any homomorphism $B \to \sim$. We will address this problem in section 4 considering products.

Example 2.59 *Consider two isomorphic algebras over $\Sigma = \langle \{s_1, s_2\}, \{f : s_1 \to s_2\}\rangle$:*

$$\begin{tikzcd}
A & a_0 & & & & a'_0 & A' \\
a_1 \arrow[ur] & & a_2 \arrow[ul] & & a'_1 \arrow[ur] & & a'_2 \arrow[ul]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & b_0 & & & & \langle a_0, a'_0\rangle & & \sim \\
B & b_1 \arrow[ur] & b & b_2 \arrow[ul] & & \langle a_1, a'_1\rangle \arrow[ur] & \langle a_2, a'_1\rangle \arrow[u] & \langle a_2, a'_2\rangle \arrow[ul] &
\end{tikzcd}$$

and two homomorphisms:

- $p : B \to A$, *given by* $p(b_i) = a_i$ *and* $p(b) = a_2$, *and*
- $p' : B \to A'$, *given by* $p'(b_i) = a'_i$ *and* $p'(b) = a'_1$.

The induced bireachability $\sim$, *with its algebraic structure, is shown to the right. There is no homomorphism* $\phi : B \to \sim$ *since sending* $\phi(b_0) = \langle a_0, a'_0 \rangle$ *requires all the three arguments to be in the image of* ϕ, *in which case the OT-property of* ϕ *fails for* $x = \phi(b)$, *i.e.,* $f^B(\phi^-(x)) = f^B(b) = \emptyset \neq \{b_0\} = \phi^-(f^\sim(x))$.

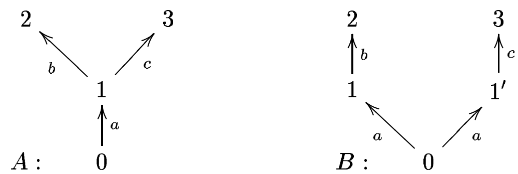
It is easy to see that the following fact holds.

Fact 2.60 *The condition from definition 2.54 is preserved by unions.*

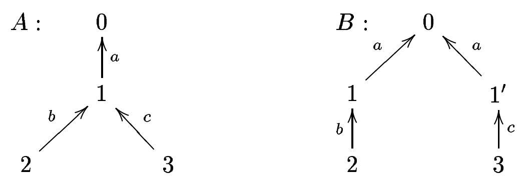
Consequently, for any two algebras, there is always the maximal (with respect to $\subseteq$) bireachability between them, namely, the union of all bireachabilities. In case this maximal bireachability is empty, we will say that the algebras are not bireachable.

The following examples illustrate further the duality of bireachability and bisimilarity.

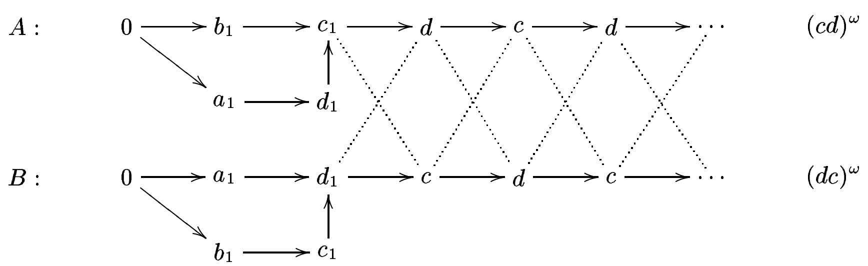
Example 2.61 *Assume three operations* $a, b, c : s \to s$ *and consider the following standard example from process theory:*



A and B are not bisimilar but are both trace equivalent and bireachable. In fact, A is a quotient of B by the bireachability $1 \sim 1'$. *As might be expected, we have a dual situation: bisimulation distinguishes states with respect to differences which 'come after' while bireachability with respect to what 'comes before'. The following two algebras are trace equivalent and bisimilar but not bireachable (as any bireachability on B containing* $\langle 1, 1' \rangle$ *must also contain* $\langle 2, 3 \rangle$*):*



Example 2.62 *The duality of 'after' and 'before' – and at least occasional naturality of the latter – can be illustrated by the following. Let now* $0, a, b, c, d : \to s$ *be constants, and let the arrow represent the only operation* $tr : s \to s$. *(Subscripts serve only reference purposes.)*



A bisimilarity between A and B is given by the pairs $\langle i_1, i_1 \rangle$, *for* $i \in \{a, b, c, d\}$ *and those indicated by the dotted lines. One might feel a bit uneasy about this bisimilarity since A satisfies the formula: "the first* d_1 *occurs before the first* c_1*" (or else: "the first* c_1 *is reachable from the first* d_1*") while B does not.*

Unlike bisimilarity reflecting the relation of 'coming after', bireachability is exactly the relation of 'coming before'. The greatest bireachability $\sim \subseteq A \times B$ *is simply the relation* $\{\langle 0, 0 \rangle, \langle a_1, a_1 \rangle, \langle b_1, b_1 \rangle\}$. *We can not possibly get* $c_1 \sim c_1$ *as this would require* $d_1 \sim b_1$ *(since in* $A : c_1 \in tr^A(b_1) \cap tr^A(d_1)$ *while in B we only have* $c_1 \in tr^B(b_1)$*). But* $d_1 \sim b_1$ *is impossible as it, in turn, would require* $a_1 \sim 0$ *which cannot obtain because while* $a_1 \in tr^A(0)$ *there* *is no $b \in B : 0 \in tr^B(b)$. (Two states can be bireachable here if they have the same label and are reachable in the same number of steps from states with the same labels – compare example 2.49.)*

Example 2.63 *The following two structures are modally indistinguishable but not bisimilar:*

$$
A:\quad
\begin{tikzcd}[column sep=small]
 & 1 & 0 \arrow[l] \arrow[dl] \arrow[d] \\
 & 1 \arrow[dl] & 1 \arrow[d] \\
2 & & 2 \arrow[d] \ \cdots\cdots \\
 & & 3
\end{tikzcd}
\qquad\qquad
B:\quad
\begin{tikzcd}[column sep=small]
 & 1 & 0 \arrow[l] \arrow[dl] \arrow[d] \arrow[dr] & & & \\
 & 1 \arrow[dl] & 1 \arrow[d] & 1 \arrow[dr] & & \\
2 & & 2 \arrow[d] \ \cdots\cdots & & 2 \arrow[dr] & \\
 & & 3 & & & 3 \ \cdots\cdots
\end{tikzcd}
$$

The natural attempt would be to relate all nodes reachable in the equal number of steps, i.e., $\sim = \{\langle n^A, n^B\rangle \mid n^A = n^B\}$. This does not work for the well-known reason that, attempting to set any node n^B on the infinite path of B bisimilar to some n^A on a finite branch of A of length $m > n$, leads to the impossibility of relating the last element m^A to the m-th element on the infinite path of B, since from the latter there is a further transition to $(m+1)^B$.

The above relation yields, however, a bireachability since it "measures" only from which elements any element on the infinite path is reachable and not what elements lie ahead of it. Thus, even taking only the one infinite branch of B, yields the structure bireachable with A.

Recall from remark 2.9 that a coalgebra $\alpha : A \to \Sigma(A)$ over a polynomial functor Σ can be represented as a multialgebra $\alpha^- : \Sigma(A) \to \mathcal{P}(A)$ with the special property that for all $a_1 \neq a_2 : \alpha^-(a_1) \cap \alpha^-(a_2) = \emptyset$. Denote the coalgebras by A, B and the corresponding multialgebras by A^-, B^-.

Fact 2.64 $\sim \subseteq A^- \times B^-$ *is a bireachability between A^- and B^- iff $\sim$ is a bisimilarity between A and B.*

Proof: Follows trivially since each operation in A^- is converse of the respective function in A, while the bireachability condition from 2.54 is just the converse of the bisimilarity condition (as illustrated already in (2.37)). □

The bireachability remains, however, a wider notion as it applies to all multialgebras, also those which do not represent any coalgebra.

2.4 Final objects in $\mathsf{MAlg}_{OT}(\Sigma)$

In general, final objects do not exist in $\mathsf{MAlg}_{OT}(\Sigma)$ due to the usual cardinality reasons. Imagine a simple case of one sort and operation $f : s \to s$. In a multialgebra Z this requires $f^Z : s^Z \to \mathcal{P}(s^Z)$ (essentially a coalgebra for power-set functor) and, when Z is to be final, moreover isomorphism $s^Z \simeq \mathcal{P}(s^Z)$.

As stated in the introduction $\mathsf{MAlg}_{OT}(\Sigma)$ is finitely cocomplete but the existence of final objects has been shown only for a very special case. We show here two such special cases, mainly to illustrate the interesting features of the final objects. The required extension and the general construction is given in the following section 3.

Example 2.65 *Let $\Sigma = \langle\{s_1, s_2\}, \{c :\to s_1; f : s_1 \to s_2\}\rangle$. The final object Z in $\mathsf{MAlg}_{OT}(\Sigma)$ can be described as follows. (Expressions like "$\emptyset_1$" or "$fc\emptyset$" are simple names – mnemonic devices – not any sets or function applications.)*

$$
\begin{tikzcd}
s_2^Z : & fc & fc\emptyset & f\emptyset & \emptyset_2 \\
s_1^Z : & c \arrow[u] \arrow[ur] & & \emptyset_1 \arrow[ul] \arrow[u] &
\end{tikzcd}
$$

In words, each sort contains only elements needed to distinguish any combination of operations returning the elements of this sort. In s_1^Z it is enough with one element to interpret the *constant,* $c^Z = \{c\}$*. In addition, there is always an element not belonging to the result of any operation,* $\emptyset_1$*.* s_2^Z *contains one such element,* $\emptyset_2$*, one element characteristic for* $f^Z(c) \ni fc$*, one for* $f^Z(\emptyset_1) \ni f\emptyset$ *and one for* $f^Z(c) \cap f^Z(\emptyset_1) \ni fc\emptyset$*.*

If we had two constants of sort s_1*, we would obtain corresponding collection* $\{c, d, cd, \emptyset_1\}$ *in* s_1^Z*, while* s_2^Z *would now contain characteristic element for every possible* $f^Z(x)$ *when* $x \in s_1^Z$*, as well as for every intersection* $\bigcap_{x \in X} f^Z(x)$ *for every possible* $X \subseteq s_1^Z$*.*

Viewing results of an operation as possible (or nondeterministic) observations of its arguments, the construction amounts to providing the minimum needed for every series and every (possible intersection of a) set of observations to have its unique characteristic result. Recalling the topology we defined on (an arbitrary) multialgebra in remark 2.8, in case of the final multialgebra, it will amount to each set S of the basis (obtained as arbitrary intersections of the subbasis sets, i.e., sets of the form $f^Z(\overline{x})$) having a unique characteristic element z_S. Alternatively, we can say that the only bireachability on a final algebra is identity and just like final morphisms of coalgebras identify bisimilar states, so here final morphisms will identify bireachable elements.

The most general form of this construction can be obtained when signature does not contain any "loops". Call a signature "acyclic" if there is no derived operator t with target sort occurring also among the argument sorts. More precisely, we can define an ordering on sort symbols by taking the transitive closure of the relation: $s_1 < s_2$ iff $\exists f : \ldots s_1 \ldots \to s_2$. Σ is *acyclic* if there are no two (possibly the same) sort symbols such that $s_1 < s_2$ and $s_2 < s_1$. We then have a well-founded partial ordering of all sort symbols with the minimal elements MIN for which there are at most some constants.

Proposition 2.66 *If* Σ *is acyclic then* $\mathsf{MAlg}_{OT}(\Sigma)$ *has final objects.*

PROOF: Constructions and arguments will depend heavily on the ordering $<$ of sort symbols. We define the carriers of the final algebra Z in this way. $\mathcal{T}(\Sigma)$ denotes all ground Σ-terms, $\mathcal{T}(\Sigma)_s$ all ground terms of sort s, and $\mathcal{T}(\Sigma, X)_s$ all ground terms of sort s relative to a set of additional constants X.

1) For each sort $s \in MIN : s^Z = \mathcal{P}(\mathcal{T}(\Sigma)_s)$ – notice that $\mathcal{T}(\Sigma)_s$ will contain in this case at most some constants.
2) For each sort $s \notin MIN$, let F be the set of all non-constant operations with s as the target sort. For each such $f \in F$, $f : s_1 \ldots s_n \to s$, we have, by induction, constructed s_i^Z for all argument sorts. Let X be the (disjoint) union of all elements from all the argument sorts for all operations from F. We then consider all terms relative to this set, $\mathcal{T}(\Sigma, X)_s$, and define $s^Z = \mathcal{P}(\mathcal{T}(\Sigma, X)_s)$.

Notice that for each sort s, we will obtain the element $\emptyset_s$ – this will represent the element(s) of the respective sort which are "absolute junk", i.e., not in the image of any operation (for any choice of arguments). An element (a set) $p \in s^Z$ is intended to represent the unique point which belongs to the intersection of all terms $t \in p$. The operations in Z are defined as:

3) $p \in c^Z \iff c \in p$
4) $p \in f^Z(p_1 \ldots p_n) \iff f(p_1 \ldots p_n) \in p$

and the definition is extended pointwise to the sets of p's. Notice that, in the last point, the argument p_i's are all from lower levels, i.e., from sorts $s_i < s$, where s is the target sort of f. Given any Σ-algebra A, we define a homomorphism $\phi_A : A \to Z$ by induction on sort ordering:

5) $s \in MIN : a \in s^A : \phi_A(a) = \{c \mid a \in c^A\}$
6) $s \notin MIN : a \in s^A : \phi_A(a) = \{c \mid a \in c^A\} \cup \{f(p) \mid \exists x\ p = \phi_A(x) \wedge a \in f^A(x)\}$

It is an OT homomorphism:

$$\begin{array}{rcl}
\text{for constants:} \quad \phi_A^-(c^Z) & = & \phi_A^-(\{p \mid c \in p\}) \\
& = & \{a \mid c \in \phi_A(a)\} \\
5)\ 6) & = & \{a \mid a \in c^A\} \\
& = & c^A \\
\text{and for operations:} \quad a \in \phi_A^-(f^Z(p)) & \iff & \phi_A(a) \in f^Z(p) \\
4) & \iff & f(p) \in \phi_A(a) \\
6) & \iff & \exists x : p = \phi_A(x) \wedge a \in f^A(x) \\
& \iff & a \in f^A(\phi_A^-(p))
\end{array}$$

Finally, assume another $\psi : A \to Z$, where for some $a \in A : \phi_A(a) \neq \psi(a)$. We show that then ψ cannot be an OT-homomorphism, by induction on the sort ordering:

- $s \in MIN, a \in s^A$ and $\{c \mid a \in c^A\} = \phi_A(a) \neq \psi(a) \Rightarrow \exists c$ such that either
  i) $a \in c^A \wedge c \notin \psi(a)$ – then $a \notin \psi^-(c^Z)$, so ψ wouldn't be OT; or else
  ii) $a \notin c^A \wedge c \in \psi(a)$ – then $a \in \psi^-(c^Z)$ so, again, ψ wouldn't be OT
- $s \notin MIN, a \in s^A$ and $\phi_A(a) \neq \psi(a)$. If the difference from definition 6) concerns some constant c, the argument is the same as above. So assume that it concerns some f, p, i.e., $\exists f \in \Sigma\ p \in Z$ such that either
  iii) $f(p) \in \phi_A(a) \wedge f(p) \notin \psi(a)$; or else
  iv) $f(p) \notin \phi_A(a) \wedge f(p) \in \psi(a)$

  By IH, for any $x : \phi_A(x) = p$ we also have $\psi(x) = p$ since given such an f, the sort of x must be $<$ then the sort of a. Thus also $\phi_A^-(p) = \psi^-(p)$. Then we have:

  $$a \in \psi^-(f^Z(p)) \overset{OT}{\iff} a \in f^A(\psi^-(p)) \overset{IH}{\iff} a \in f^A(\phi_A^-(p)) \overset{6)}{\iff} f(p) \in \phi_A(a)$$

  so neither iii) nor iv) can be the case if ψ is OT. □

If Σ is cyclic, we simply can not stop in point 2) but have to keep constructing new power-sets *ad infinitum*. The construction can terminate for arbitrary Σ if we impose some limitations on the power-set functor. Unlike in the usual case, for instance, of coalgebras where $\mathcal{P}^{fin}$ is restricted to return only finite sets, we need a "converse" restriction, namely, that every element is reachable in at most finitely many ways. In particular, for an operation $f : s \to t$ and an element $x \in t^M$, we require that there is at most finitely many elements $y \in s^M$ such that $x \in f^M(y)$, i.e., that the pre-image of any element for every operation is finite. Generally, we require that

$$\text{given an element } x \in s^M\text{, there is at most a finite number of derived operators } t \text{ (of sort } s\text{) and elements } \overline{y} \text{ such that } x \in t^M(\overline{y}). \tag{2.67}$$

In particular, the condition implies that for any element $x \in M$ and derived operator t, there are no elements $\overline{y}$ such that $x \in t^M(x, \overline{y})$. (For if so, then $x \in (t^n)^M(x, \overline{y})$ for all $n \geq 1$.) Put differently, the set $\Downarrow^x(=)\{\langle t, \overline{y}\rangle \mid x \in t^M(\overline{y})\}$ is finite for every $x \in M$ and the ordering "result of" on M given by $y \prec z \iff \exists t, \overline{y} : \langle t, y\overline{y}\rangle \in \Downarrow^z()$ is well-founded. In fact, the ordering on sorts in the previous proposition, ensures well-foundedness of this very ordering. (It was not, however, a special case of condition (2.67), since the set of pre-images for an element could be infinite.)

Let $\mathsf{MAlg}_{OT}^{fin}(\Sigma)$ denote the category with objects being multialgebras satisfying the above condition (2.67) and morphisms being OT-homomorphisms.

Proposition 2.68 *For any Σ, $\mathsf{MAlg}_{OT}^{fin}(\Sigma)$ has final objects.*

Proof: The construction and proof follow the same schema as the proof of the previous proposition with some technical variations. We iterate now ω times (the second point of) the following construction (in each step considering all sorts s in parallel):

1) $s_0 = \mathcal{P}^{fin}(\mathcal{T}(\Sigma)_s)$ and let $T_0 = \bigcup_s s_0$ (and for all $i : T_i = \bigcup_s s_i$);
2) $s_{i+1} = \mathcal{P}^{fin}(\mathcal{T}(\Sigma, T_i)_s)$ – always flattening the sets by identifing $\{\emptyset, \overline{x}\} = \{\overline{x}\}$ and $\{\{\overline{y}\}, \overline{x}\} = \{\overline{y}, \overline{x}\}$ (e.g., $\{\emptyset_s\} \rightsquigarrow \emptyset_s$, $\{\{\emptyset_s\}, \emptyset_s\} \rightsquigarrow \{\emptyset_s\}$, $\{\{a\}, \{b, c\}\} \rightsquigarrow \{a, b, c\}$);

3) $s^Z = \bigcup_{i<\omega} s_i$.

The iterative construction differs from just taking $\mathcal{P}^{fin}(\mathcal{T}(\Sigma, X)_s)$; for instance, we obtain thus $g(\{a, b\})$ (for appropriate constants a, b and unary operation g: the characteristic element of the result of the application of g to the intersection of a and b), which is different from the set $\{g(a), g(b)\}$ (i.e., the set with the characteristic elements of the results of the applications of g to a and b, respectively.) As before, an element (a set) $p \in s^Z$ is intended to represent the unique point which belongs to the intersection of all terms $t \in p$. The operations in Z are defined as:

4) $p \in c^Z \iff c \in p$

5) $p \in f^Z(p_1 \dots p_n) \iff f(p_1 \dots p_n) \in p$

and the definition is extended pointwise to the sets of p's. By the use of only finite sets in 1-2) and the definiton 4-5), Z obviously satisfies the condition that for every $x \in Z$ there is at most a finite number of $\overline{y} \in Z$ and derived operators t such that $x \in t^Z(\overline{y})$. (There may be, however, elements y for which some result set $f^Z(y)$ is infinite.)
For any $A \in \mathsf{MAlg}^{fin}_{OT}(\Sigma)$, we define a homomorphism $\phi_A : A \to Z$ by induction on $\prec$:

6) $\phi_A(a) = \{c \mid a \in c^A\} \cup \{f(p) \mid \exists x\; p = \phi_A(x) \wedge a \in f^A(x)\}$.

Since $A \in \mathsf{MAlg}^{fin}_{OT}(\Sigma)$, the condition (2.67) implies that each $\phi_A(a)$ (induced by the collection of various x's and f's) will be finite and hence belong to the carrier of Z. The $\prec$-minimal elements will be mapped either to $\emptyset_s$ or to some set of constants $\{c \mid a \in c^A\}$. (The condition $a \in f^A(x)$ in the second disjunct amounts to $\langle f, x\rangle \in \Downarrow^a()$, i.e., $x \prec a$.) In fact, the relation $\sim = \phi_A;\phi_A^- \subseteq A \times A$ is a bireachability. If $a \sim a'$, i.e., $\phi_A(a) = \phi_A(a')$, and $a \in f^A(x)$ then, since $\phi_A(x) = p$ for some p, so $f(p) \in \phi_A(a) = \phi_A(a')$. But then 6) implies the existsence of an $x' : a' \in f^A(x')$ with $\phi_A(x') = p$, which is exactly what is needed for $\sim$ to be bireachability by (2.36) – provided that it is an equivalence which it is since ϕ_A is a function. Essentially, ϕ_A is the quotient homomorphism $A \to A/_\sim \sqsubseteq Z$, but we verify that it is OT directly:

$$\begin{array}{rrcl}
\text{for constants:} & \phi_A^-(c^Z) & = & \phi_A^-(\{p \mid c \in p\}) \\
 & & = & \{a \mid c \in \phi_A(a)\} \\
 & 6) & = & \{a \mid a \in c^A\} \\
 & & = & c^A \\
\text{and for operations:} & a \in \phi_A^-(f^Z(p)) & \iff & \phi_A(a) \in f^Z(p) \\
 & 5) & \iff & f(p) \in \phi_A(a) \\
 & 6) & \iff & \exists x : p = \phi_A(x) \wedge a \in f^A(x) \\
 & & \iff & a \in f^A(\phi_A^-(p))
\end{array}$$

Uniqueness of ϕ_A follows from the following claim:

7) The only bireachability on Z is identity.

Assume a bireachability $\sim \subseteq Z \times Z$ and $a \sim b$. We show that then $a = b$ by induction on the ordering $\prec$. If $a = \emptyset_s$ then, for $a \sim b$ we must have $b = \emptyset_s$ and the same holds if a is a collection of some constants (if $a \in c^Z$, i.e., $c \in a$ then also $b \in c^Z$ and $c \in b$ and, iterating this for all constants, yeilds $a = b$.) So let $a \in f^Z(p)$ for some $f(p)$. i.e., $f(p) \in a$. Then there must exist a $p' \sim p$ with $f(p') \in b$ but, by IH, $p = p'$.

If there are two morphisms $A \to Z$, we obtain a span $Z \overset{\phi_A}{\leftarrow} A \overset{\psi}{\to} Z$ and, by lemma 2.58, a bireachability on $Z : C = \{\langle\phi_A(a), \psi(a)\rangle \mid a \in A\}$. But by the above claim 7), we thus get that $C = id_Z$, i.e., $\phi_A(a) = \psi(a)$ for all $a \in A$. □

The condition (2.67) seems more complex than its dual which in the case of coalgebras restricts results to finite sets. On the one hand, we exclude algebras where an operation $f : s \to t$ might "collapse" an infinite carrier of s to just a finite (subset of) t. But the limitation affects not only one step aplications but all derived operators. Thus, for instance, also the algebra with one element $\bullet$ and operation $f(\bullet) = \bullet$ is not a member of $\mathsf{MAlg}^{fin}_{OT}(\Sigma)$ – as indicated, $\prec$ is well-founded and so no algebra with (infinite) "loops" belongs to $\mathsf{MAlg}^{fin}_{OT}(\Sigma)$. The meaning and implications of this condition might merit further investigations but will not be pursued here.

The category $\mathsf{MAlg}_{OT}(\Sigma)$ is cocomplete and has equalizers and products. However, as it may not possess final objects, we do not prove these claims here. We first extend the category $\mathsf{MAlg}_{OT}(\Sigma)$ to allow for the existence of final objects without any cardinality limits nor restrictions on the signature. As in the case of coalgebras, we have to leave the set-based categories and allow algebras with carriers being classes. The constructions of colimits, equalizers and products to be given in the following section, can be applied also in $\mathsf{MAlg}_{OT}(\Sigma)$.

3 The category Outer-Tight with classes, $\mathsf{MAlg}^*_{OT}(\Sigma)$

Given a Σ with sort symbols $\{s_1...s_n\}$, we allow algebras where carrier of each sort is a class. Constants can denote proper classes and so can operations applied to single elements return proper classes, i.e., the power-set used in definition 1.2, denotes the collection of all subcollections (also proper subclasses) of the argument collection.[3] But we have to require here one restriction. We will need a form of representability of large algebras by small ones, essentially, that any algebra can be obtained as a colimit of its small subalgebras. This, however, may in general be impossible. Assume that $X \subseteq A$ is a proper class and that, for some operation $f : f^A(X) = \{x\}$. Whenever $\phi : B \to A$ is an OT-homomrphism, with $x \in \phi(B)$, B can not be small since it has to be surjective (at least) on the whole class X (this follows from outer-tightness; it is condition 2) from figure 2.1). We therefore limit our category to only special kind of algebras with carriers being proper classes.

Definition 3.1 *A Σ-multialgebra A is set-reflecting iff for every $a \in A$ and every (relevant) $f \in \Sigma$, there exists at most a set $X \subseteq A$ such that $a \in \bigcap_{x \in X} f^A(x)$.*

Put differently, for every f and a, a's pre-image $\left(f^A\right)^{-}(a) = \{x \mid a \in f^A(x)\}$ is a set. (This is not to be confused with the "set-based" functors from [2], even though both restrictions serve the same purpose.) The definition implies – and derives the name from the fact – that if $f^A(X)$ is a set, so is X, i.e., no function collapses a class to a set. (If $Z = f^A(X)$ is a set then, for every $z \in Z$, there is at most a set X_z, such that $z \in \bigcap_{x \in X_z} f^A(x)$. Then also $X_Z = \bigcup_{z \in Z} X_z$ is a set – but $X \subseteq X_Z$.)

$\mathsf{MAlg}^*_{OT}(\Sigma)$ considered in the following is the category of all set-reflecting multialgebras with OT-homomorphisms. Saying algebra we mean from now on a set-reflecting multialgebra.

3.1 Set-reflecting algebras are colimits of small subalgebras

The apparent "inversion" of the condition in definition 3.1 (one might expect it to require $f^A(X)$ to be a set, whenever X is) reflects the inverted direction of bireachability with respect to bisimulation, (2.37). It is crucial in the point 2) of the proof of the following result which extends fact 2.29 to the present category.

Lemma 3.2 *For every (set-reflecting) $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ and every subset $sX \subseteq A$, there is a small subalgebra $sA \sqsubseteq A$ with $sX \subseteq sA$.*

Moreover, there exists a smallest such sA, namely, such that for every other subalgebra $B \sqsubseteq A$ with $sX \subseteq B$, we have $sA \subseteq B$.

PROOF: sX is sorted, and the construction extends in each step each sort (if at all):

1) $X_0 = sX$
2) For all $x \in A$, if $f^A(x) \cap X_i \neq \emptyset$ then include into X_{i+1} also all such x.
3) $X_\omega = \bigcup_{i \in \omega} X_i$

The argument showing that the construction indeed yields a smallest subalgebra containing sX is exactly as in fact 2.29. The only additional observation to be made is that, since X_0 is a set then so is every X_i. For, given in step 2) a set X_i, $f^A(x) \cap X_i$ is a set, and so is

[3] This might cause some foundational worries since functions returning classes, and hence also indexed families of classes, are not legal objects in NBG. This signals that we must rather work with Grothendieck's hierarchy of universes, in which set-algebras reside at the first level, $\mathcal{U}_1$, while all our objects at the second one, $\mathcal{U}_2$. (As will be commented in the appendix 6, we actually end up in $\mathcal{U}_3$.) We will use the words "small"/"set" and "large"/"class" in the sense of being a member of the lowest level $\mathcal{U}_1$ versus of any higher level $\mathcal{U}_i \setminus \mathcal{U}_1$ (for $i \geq 2$), respectively.

$\bigcup_{x\in A} f^A(x) \cap X_i$. Hence, since A is set-reflecting, the elements added to X_{i+1} will form at most a set. Iterating this extension ω times yields X_ω which is indeed a set. $\square$

In particular, given an OT-homomorphism $\phi : A \to B$ and a small subalgebra $sA \sqsubseteq A$, there is also a small subalgebra $sB \sqsubseteq B$ such that the restriction $\phi|_{sA}$ of ϕ to sA is an OT-homorphism $\phi|_{sA} : sA \to sB$.

By the above lemma, each set-reflecting algebra with carrier being a proper class has small subalgebras, and it is used to show:

Lemma 3.3 *Every (set-reflecting) algebra in* $\mathsf{MAlg}^*_{OT}(\Sigma)$ *is a colimit of its small subalgebras.*

PROOF: Given an A, take all its small subalgebras and form the diagram with all the inclusions $\iota_{ij} : A_i \hookrightarrow A_j$ between these subalgebras. (By fact 2.22, these inclusions are OT-monomorphisms.) A is colimit of this diagram with the inclusions $\iota_i : A_i \hookrightarrow A$. Since all morphisms are inclusions, the commutativity condition is trivially satisfied. Assume that there is another algebra B with $\beta_i : A_i \to B$ such that $\beta_i = \iota_{ij};\beta_j$ whenever this composition is defined (i.e., whenever there exists ι_{ij}). We define the unique $u : A \to B$ using the fact that $\forall a \in A\ \exists A_i : a \in A_i$ (by lemma 3.2) – $u(a) := \beta_i(a)$. It is well-defined because the collection of all small subalgebras is directed. If $a \in A_j$ for some other small $A_j \sqsubseteq A$, then there is also a small $A_k \sqsubseteq A$ with $A_i \cup A_j \subseteq A_k$, and since $\beta_i = \iota_{ik};\beta_k$ and $\beta_j = \iota_{jk};\beta_k$ we have, in particular, that $\beta_i(a) = \beta_k(\iota_{ik}(a)) = \beta_k(a) = \beta_k(\iota_{jk}(a)) = \beta_j(a)$.

• $\beta_i = \iota_i;u$: for every $a \in A_i$ we have, by definition of u and the above argument, that $u(a) = \beta_i(a)$, which verifies this claim.

• u is unique: for if some $u' : A \to B$ makes $\iota_i;u' = \beta_i$ for all i then, for every $a \in A$ and A_i such that $a \in A_i$, we must have $u'(a) = \beta_i(a) = u(a)$.

• u is OT: Assume not, i.e., for some f and $b \in B : f^A(u^-(b)) \neq u^-(f^B(b))$. There are two cases. 1) $a \in f^A(u^-(b)) \setminus u^-(f^B(b))$: Let A_i be small subalgebra containing a and $u^-(b)$. Since β_i is OT, we have that $a \in f^{A_i}(\beta_i^-(b)) = \beta_i^-(f^B(b))$ and substituting $\iota_i;u$ for $\beta_i : a \in f^{A_i}(\iota_i^-(u^-(b))) = \iota_i^-(u^-(f^B(b)))$. But then also $a \in \iota(\iota^-(u^-(b))) \subseteq u^-(b)$.
2) $a \in u^-(f^B(b)) \setminus f^A(u^-(b))$. Let A_i be as above. Since $\beta_i = \iota_i;u$ is OT, we have $a \in \iota_i^-(u^-(f^B(b))) = f^{A_i}(\iota_i^-(u^-(b)))$, and since ι_i is OT, $f^{A_i}(\iota_i^-(u^-(b))) = \iota_i^-(f^A(u^-(b)))$. But then $a \in \iota_i^-(f^A(u^-(b)))$ implies that also $a \in f^A(u^-(b))$. $\square$

Notice, however, that the diagram can be large, as $\mathsf{MAlg}^*_{OT}(\Sigma)$ is not well-powered. (An A with s^A being a class and $f^A : s^A \to s^A$ an identity, has a proper class of subobjects – one for each subset, and subclass, of s^A.)

We also have the opposite fact.

Fact 3.4 *If A is colimit of small algebras then A is set-reflecting.*

PROOF: Let the components of the colimit be $\iota_i : A_i \to A$, with all A_i small, and consider an arbitrary $a \in A$ and operation f from the signature. We have to show that $(f^A)^{\neg}(a)$ is a set. As the collection of ι_i's is jointly epi, i.e., surjective (fact 2.13.2), there is some small A_i with $a \in \iota_i[A_i]$. Let $a^- = \iota_i^-(a)$. Since $\iota_i : A_i \to A$ is OT, so $\forall b \in (f^A)^{\neg}(a)\ \exists b_i \in (f^{A_i})^{\neg}(a^-)$. But as A_i is small so a^- as well as $(f^{A_i})^{\neg}(a^-)$ is a set, and hence also $(f^A)^{\neg}(a)$ must be a set. $\square$

3.1.1 Congruences and quotients

Concerning the OT-congruences, we make first the following observation.

Fact 3.5 *Given a bireachability $\sim$ on a set-reflecting $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$, the corresponding congruence-algebra $A^\sim$, as defined in 2.50, is also set-reflecting.*

PROOF: Let double-letters symbols, like XY, denote sets of (some) pairs $\langle x, y\rangle$ where $x \in X, y \in Y$, i.e., $XY \subseteq X \times Y$.

Let XY be the pre-image under f of some element $\langle z, u\rangle$, i.e., $XY = (f^{A^\sim})^{\neg}(\langle z, u\rangle)$. By definition 2.50 of $A^\sim$, $f^{A^\sim}(XY) = \{\langle z_k, u_k\rangle \mid \langle x, y\rangle \in XY \wedge z_k \in f^A(x) \wedge u_k \in f^A(y) \wedge z_k \sim x_k\}$, so that $XY \subseteq (f^A)^{\neg}(z) \times (f^A)^{\neg}(u)$. But both these pre-images are sets since A is set-reflecting, and so XY is a set, too. $\square$

Lemma 2.43 applies unchanged when the collection is a proper class of small OT-congruences. Performing the same standard construction on the collection of *all small* OT-congruences on a given multialgebra yields the following lemma.

Lemma 3.6 *On every $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ there exists a (unique) maximal OT-congruence $\sim_A$.*

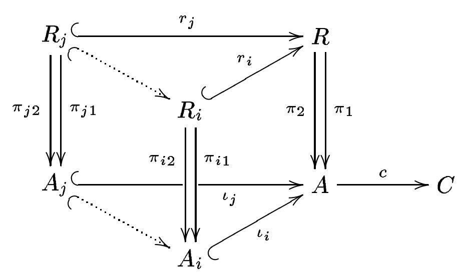
PROOF: Let $C = \{\sim_i \mid i \in I\}$ be the class of all small OT-congruences on A, then $\sim_A = \bigvee_i \sim_i$ is an OT-congruence, by lemma 2.43. It is, in fact, the maximal such.

Suppose that $\approx$ is an OT-congruence, i.e., $\approx; R^A; \approx\ =\ \approx; R^A$. For any $a_1 \approx a_2$, there is, by Lemma 3.2, a small subalgebra $sA \sqsubseteq A$, with $a_1, a_2 \in sA$. Consider the restriction of $\approx$ to sA, i..e., let $\sim_s\ =\ \approx \cap\ (sA \times sA)$. By Fact 2.47, $\sim_s$ is an OT-congruence and thus, any two elements related by $\approx$, are related already by some small OT-congruence in C. Hence $\approx\ \subseteq\ \sim_A$. □

The following easy technicality will be needed it in the proof of the next lemma.

Fact 3.7 *Let $\{A_i \mid i \in I\}$ be the class of small subalgebras of A (A being their colimit), R be the maximal OT-congruence on A and R_i the respective resitriction of R to A_i. Then $\{r_i : R_i \to R \mid i \in I\}$ is jointly epi and, for every $c : A \to C$, if $\forall i \in I : \pi_{i1}; \iota_i; c = \pi_{i2}; \iota_i; c$ then $\pi_1; c = \pi_2; c$.*

PROOF: We have the following diagram



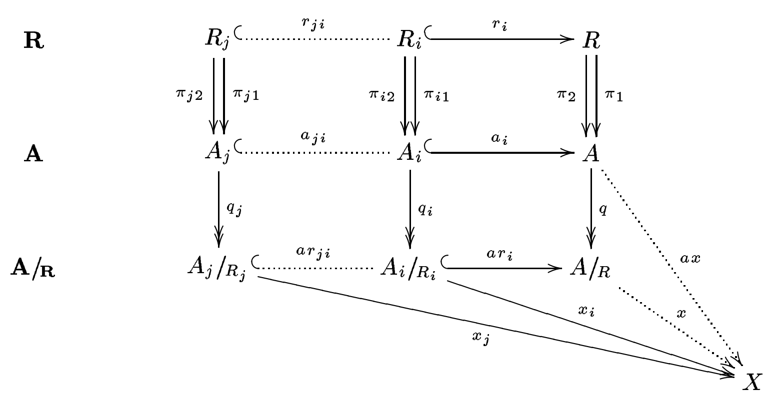
A, with inclusions ι_i, is colimit of the diagram containing all A_i's which is indicated by the dotted arrow. Also various R_i's are related by inclusions, which is indicated by the corresponding dotted arrow. All ι_i and r_i are inclusions.

That all r_i's are jointly epi follows from the proof of the previous lemma. If $\langle a_1, a_2\rangle \in R$ then there is a small subalgebra $A_i \sqsubseteq A$ containing a_1, a_2, and so $\langle a_1, a_2\rangle \in R_i = R \cap A_i \times A_i$.

Assume that $\pi_1; c \neq \pi_2; c$, i.e., for some $\langle a_1, a_2\rangle \in R : c(\pi_1(\langle a_1, a_2\rangle)) = c(a_1) \neq c(a_2) = c(\pi_2(\langle a_1, a_2\rangle))$. Let R_i be one such that $\langle a_1, a_2\rangle \in R_i$. By definition of R_i, for each $i \in I$: $r_i; \pi_k = \pi_{ik}; \iota_i$, for $k \in \{1, 2\}$. Thus we would obtain $c(\iota_i(\pi_{i1}(\langle a_1, a_2\rangle))) = c(a_1) \neq c(a_2) = c(\iota_i(\pi_{i2}(\langle a_1, a_2\rangle)))$, i.e., $\pi_{i1}\iota_i; c \neq \pi_{i2}\iota_i; c$. □

Lemma 3.8 *Given an $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ and an OT-congruence R on A, the quotient $A/_R$ is a colimit of its small subalgebras, and hence $A/_R \in \mathsf{MAlg}^*_{OT}(\Sigma)$.*

PROOF: We consider the following (schema of the) diagram:



$\mathbf{A}$, resp. $\mathbf{R}$, stand for the whole diagrams consisting of the respective small subalgebras A_i of A and $R_i = R \cap A_i \times A_i$ (by fact 3.5, R and all $R_i \in \mathsf{MAlg}^*_{OT}(\Sigma)$, while by fact 2.47, $R_i \sqsubseteq R$) with the inclusion arrows a_{ji}, resp. r_{ji}. A with inclusions a_i is colimit of $\mathbf{A}$. The collection of all r_i's, resp., all a_i's is jointly epi. All q_i's are epi.

The diagram $\mathbf{A}/_{\mathbf{R}}$ contains all quotient algebras $A_i/_{R_i}$ and inclusion arrows between them. Since for each $i : R_i = R \cap A_i \times A_i$, we have an inclusion $a_{ji} : A_j \hookrightarrow A_i$ iff $r_{ji} : R_j \hookrightarrow R_i$. But then, this implies the existence of a mono $ar_{ji} : A_j/_{R_j} \hookrightarrow A_i/_{R_i}$. For each $A_i/_{R_i}$, we can obtain an isomorphic algebra by replacing every element $[a]^{R_i}$ by $[a]^R$ (though $[a]^{R_i} \subseteq [a]^R$ and the inclusion can be proper, whenever $R(a_1, a_2)$ and $a_1, a_2 \in A_i$, then also $R_i(a_1, a_2)$). This will make all monos ar_i, as well as all ar_{ij}, into inclusions.

We want to show that $A/_R$ with all inclusions ar_i is colimit of $\mathbf{A}/_{\mathbf{R}}$. Obviosuly, for each (existing) ar_{ji} we do have that $ar_j = ar_{ji}; ar_i$, since all arrows are inclusions. So assume an X with arrows $x_i : A_i/_{R_i} \to X$ such that $x_j = ar_{ji}; x_i$ for all (relevant) i, j.

1. Since $q_j; ar_{ji} = a_{ji}; q_j$, we obtain that for all (relevant) $j, i : x_j = ar_{ji}; x_i \Rightarrow q_j; x_j = q_j; ar_{ji}; x_i = a_{ji}; q_i; x_i$. That is, X with $q_i; x_i$ is a commutative cocone over $\mathbf{A}$. Since A is colimit of $\mathbf{A}$, we obtain a unique arrow $ax : A \to X$ such that for all $i : q_i; x_i = a_i; ax$.

2. For every i, since $\pi_{i1}; q_i = \pi_{i2}; q_i$, so also $\pi_{i1}; q_i; x_i = \pi_{i2}; q_i; x_i$ and by 1, $\pi_{i1}; a_i; ax = \pi_{i2}; a_i; ax$. By Fact 3.7, we thus have $\pi_1; ax = \pi_2; ax$.

3. By Fact 2.51, $(A/_R, q)$ is coequalizer of π_1, π_2, and thus we obtain a unique arrow $x : A/_R \to X$ making $q; x = ax$. This is the arrow we are looking for:

4. Commutativity: $q_i; ar_i; x = a_i; q; x \overset{3.}{=} a_i; ax \overset{1.}{=} q_i; x_i$. But q_i is epi and so $ar_i; x = x_i$.

5. Uniqueness: assume another arrow $y : A/_R \to X$ with $ar_i; y = x_i$ for all i. Then also, $q_i; x_i = q_i; ar_i; y = a_i; q; y$ and thus, for every $i : a_i; q; y = a_i; q; x$. Since a_i are jointly epi, this means that $q; y = q; x$ and now, since q is epi, $x = y$.

Since $A/_R$ is a colimit of its small subalgebras, it is set-reflecting, i.e., $A/_R \in \mathsf{MAlg}^*_{OT}(\Sigma)$, by fact 3.4. □

3.2 Cocompleteness

Given two functors, $F, G : \mathsf{SET} \to \mathsf{SET}$, one forms dialgebras $\delta : F(X) \to G(X)$ as suggested in section 1, p.3, obtaining the category SET^F_G. Theorem 13 in [40] states that the forgetful functor $\mathsf{SET}^F_G \to \mathsf{SET}$ creates and preserves all kinds of colimits that are preserved by F. (In case of coalgebras, $F = id_{\mathsf{SET}}$, and so creation of colimits (e.g., theorem 4.5 in [37]) follows immediately.) Although we have moved from SET to CLASS, we might be tempted to retain this theorem and apply it to our case, where F is the (polynomial) signature functor. But, of course, this is not possible because we are working with different homomorphisms than those induced by the definition of dialgebras. Nevertheless, although the theorem does not apply to our case, its conclusion does: colimits in $\mathsf{MAlg}^*_{OT}(\Sigma)$ are indeed created by the forgetful functor. The following results apply also to $\mathsf{MAlg}_{OT}(\Sigma)$ and these are given in square brackets.

Proposition 3.9 $\mathsf{MAlg}^*_{OT}(\Sigma)$ *[and* $\mathsf{MAlg}_{OT}(\Sigma)$*] has initial objects and all coproducts [of small diagrams].*

PROOF: Empty algebra is trivially an initial object.

Consider first a class $\{A_i \mid i \in I\}$ of small algebras. We define their coproduct $\coprod_{i \in I} A_i$ to be the algebra CP whose carrier is the disjoint union of the carriers of all A_i, i.e., the class $\biguplus_i A_i = \{\langle a, i\rangle \mid i \in I,\ a \in A_i\}$, with the operations defined as follows:

$$f^{CP}(\langle a_1, i_1\rangle ... \langle a_n, i_n\rangle) = \begin{cases} f^{A_i}(a_1...a_n) \times \{i\} & \text{if } i_1 = ... = i_n = i \\ \emptyset & otherwise \end{cases} \tag{3.10}$$

and constants as: $c^{CP} = \biguplus_i c^{A_i}$.

The injections $\iota_i : A_i \hookrightarrow CP$ are obviously OT-homomorphisms.

Assume an object X with arrows $\psi_i : A_i \to X$, for every $i \in I$, with the OT-arrows, i.e., satisfying for every f:

$$f^{A_i}(\psi_i{}^-(x)) = \psi_i{}^-(f^X(x)) \tag{3.11}$$

The mediating arrow $u : CP \to X$ defined by $u(\langle a,i\rangle) = \psi_i(a)$ trivially satisfies $\iota_i; u = \psi_i$ for every i. We show that u is an OT-homomorphism: $f^{CP}(u^-(x)) = u^-(f^X(x))$.

$$\begin{array}{rcll} f^{CP}(u^-(x)) & = & f^{CP}(\biguplus_i {\psi_i}^-(x)) & \text{def. of } u \\ & = & \biguplus_i f^{A_i}({\psi_i}^-(x)) & \text{by (3.10)} \\ & = & \biguplus_i {\psi_i}^-(f^X(x)) & \text{by (3.11)} \\ & = & u^-(f^X(x)) & \text{def. of } u \end{array}$$

u is unique: Assume $u \neq u_2 : CP \to X$, which also satisfies: $\iota_i; u_2 = \psi_i$ for all i. Then there is a $\langle c,i\rangle \in CP$ such that $u(\langle c,i\rangle) \neq u_2(\langle c,i\rangle)$. But then $\iota_i; u_2(\langle c,i\rangle) \neq \psi_i(c)$.

CP is trivially set-reflecting by the definition of operations in (3.10), as all A_i are small.

If now class $\{A_i \mid i \in I\}$ contains arbitrary set-reflecting algebras, the construction and verification of universality proceed in the same way as above, and we only check that the resulting CP is still set-reflecting. It is, in fact, colimit of small subalgebras. (Just replace each large A_i (in the discrete coproduct diagram) by the diagram of its small subalgebras (or isomorphic ones, with the elements of CP). CP, with the arrows $a_{ik} : A_{ik} \hookrightarrow A_i$ (for each large A_i and all small $A_{ik} \sqsubseteq A_i$) replaced by the respective compositions $a_{ik}; \iota_i$, is colimit of this expanded diagram.) Hence CP is set-reflecting by fact 3.4.

[It is clear that the construction works for small diagrams in the category $\mathsf{MAlg}_{OT}(\Sigma)$.] □

Proposition 3.12 $\mathsf{MAlg}^*_{OT}(\Sigma)$ *[and* $\mathsf{MAlg}_{OT}(\Sigma)$*] has all coequalizers.*

PROOF: Given two arrows $\phi_1, \phi_2 : A \to B$, we start as usual by considering the equivalence closure $\sim$ on B of the relation $E = \{\langle\phi_1(a), \phi_2(a)\rangle \mid a \in A\}$.[4] Equivalence classes induced by this relation are denoted $B_1, B_2 \ldots$. Assuming the global axiom of choice, we can choose the representatives $b_i \in B_i$, and the carrier of the coequalizer object CE is the collection of such representatives. We may occasionally write $[b_i]$ for B_i.[5] Operations are defined by:

$$b_2 \in f^{CE}(b_1) \iff B_2 \subseteq f^B(B_1) \tag{3.13}$$

which for constants specializes to: $b_i \in c^{CE} \iff B_i \subseteq c^B$. The arrow $ce : B \to CE$ is the usual $\forall x \in B_i : ce(x) = b_i$. By the definition of $\sim$, it makes $\phi_1; ce = \phi_2; ce$. It is also OT. Let $b_2' \sim b_2$:

$$\begin{array}{rcl} b_2' \in ce^-(f^{CE}(b_1)) & \Rightarrow & b_2 \in f^{CE}(b_1) \\ (3.13) & \iff & B_2 \subseteq f^B(B_1) \\ ce^-(b_1) = B_1 & \Rightarrow & b_2' \in f^B(ce^-(b_1)) \\ \text{and other way:} & & \\ b_2' \in f^B(ce^-(b_1)) & \Rightarrow & b_2' \in f^B(B_1) \\ (3.15) & \Rightarrow & B_2 \subseteq f^B(B_1) \\ (3.13) & \iff & b_2 \in f^{CE}(b_1) \\ ce^-(b_2) = B_2 & \Rightarrow & B_2 \subseteq ce^-(f^{CE}(b_1)) \\ b_2' \in B_2 & \Rightarrow & b_2' \in ce^-(f^{CE}(b_1)) \end{array} \tag{3.14}$$

The transition marked (3.15) needs a more involved justification. The claim we are making is even stronger, namely, (we write now b_2 instead of b_2' since this choice does not matter here):[6]

$$b_2 \in f^B(b_1) \Rightarrow B_2 \subseteq f^B(B_1) \tag{3.15}$$

So assume that (3.15) does not hold, i.e.,

a. $b_2 \in f^B(b_1)$ but

b. $\exists b_2' \in B_2 : \forall b_1' \in B_1 : b_2' \notin f^B(b_1')$.

[4]If this relation is a class, we can perform the needded closure even if we worked in NBG, as their definitions do not require any quantification over classes. E.g., $ref(E) = E \cup \{\langle a,a\rangle \mid a \in A\}$, $sym(E) = E \cup \{\langle a,b\rangle \mid \langle b,a\rangle \in E\}$, $X; E = \{\langle a,b\rangle \mid \exists c : aXc \wedge cEb\}$, and the last operation can be iterated ω times startig with $X = id$.

[5]In case some of B_i's are proper classes, we have to follow the trick of Dana Scott (quoted in [1], Appendix B) in order to obtain the quotient, i.e., to consider as B_i only its subset of the elements having the least possible rank in the cumulative hierarchy.

[6]This, as a matter of fact, is a general property implied by outer-tightness.

Then, certainly, $b_2' \neq b_2$ and since these two elements end up in the same equivalence class, they both must be in the image of either ϕ_1 or ϕ_2. Moreover, a. and b. mean that we can divide B_2 into two non-empty subclasses: $Y = B_2 \cap f^B(B_1)$ and $N = B_2 \setminus Y$ (with $b_2 \in Y$ and $b_2' \in N$). Since $B_2 = N \cup Y$ so, by definition of $\sim$, there must exist an $a \in A : \phi_1(a) \in N \wedge \phi_2(a) \in Y$. Let us, without loss of generality, call these elements $\phi_1(a) = b_2' \in N$ and $\phi_2(a) = b_2 \in Y$ (ambiguously, since these need not be the same as b_2, b_2' used so far). We now have:

$$\begin{array}{ccc}
b_2' \notin f^B(B_1) & \text{and} & b_2 \in f^B(B_1) \\
\Downarrow & & \Downarrow \\
\phi_1(a) \notin f^B(B_1) & & \phi_2(a) \in f^B(B_1) \\
\Downarrow & & \Downarrow \\
a \notin \phi_1^-(f^B(B_1)) & & a \in \phi_2^-(f^B(B_1)) \\
\Downarrow & \text{since } \phi_i \text{ are } OT & \Downarrow \\
a \notin f^A(\phi_1^-(B_1)) & & a \in f^A(\phi_2^-(B_1)) \\
\Downarrow & \phi_1^-(B_1) = X = \phi_2^-(B_1) & \Downarrow \\
a \notin f^A(X) & \text{and} & a \in f^A(X)
\end{array}$$

The equality $\phi_1^-(B_i) = \phi_2^-(B_i)$ holds for all equivalence classes B_i by definition of $\sim$. This contradiction establishes (3.15) and hence the equality (3.14), so ce is OT-homomorphism.

To show universality, assume a $\psi : B \to X$ with $\phi_1;\psi = \phi_2;\psi$. We define the mediating arrow $u : CE \to X$ in the standard way: $u([b]) = \psi(b)$. By the standard argument (since ψ coequalizes ϕ_1, ϕ_2), we have that $[b] \subseteq [b]^\psi$ (where $[b]^\psi = \{b' \in B \mid \psi(b') = \psi(b)\} = \psi^-(\psi(b))$) which, in turn, implies that u is well defined and unique making $\psi = ce; u$. (We use the notation $[b]$ ambiguously: whenever followed by $[b] \in ...$ it stands for the chosen representative, while in $[b] \subseteq ...$ it stands for the whole class.)

We show that u is OT-homomorphism. First the inclusion $f^{CE}(u^-(x)) \supseteq u^-(f^X(x))$:

$$\begin{array}{rcl}
[b] \in u^-(f^X(x)) & \Rightarrow & u([b]) \in f^X(x) \\
\text{def. of } u & \Rightarrow & \psi(b) \in f^X(x) \\
& \Rightarrow & \psi^-(\psi(b)) \subseteq \psi^-(f^X(x)) \\
\psi \text{ is } OT & \Rightarrow & [b]^\psi \subseteq f^B(\psi^-(x)) \\
[b] \subseteq [b]^\psi & \Rightarrow & [b] \subseteq f^B(\psi^-(x)) \\
& \Rightarrow & ?
\end{array}$$

What we want now is that $[b] \in f^{CE}(u^-(x))$ but this requires a more involved argument. We have that $\exists b' : \psi^-(x) = [b']^\psi$ and also that $[b']^\psi = \bigcup_{[b_i] \in u^-(x)} [b_i]$ by definition of u (i.e., $[b]^\psi$ may comprise several distinct $[b_i]$.) Rewriting the conclusion of the above implications, we thus have

$$[b] \subseteq f^B(\bigcup_{[b_i] \in u^-(x)} [b_i]) = \bigcup_{[b_i] \in u^-(x)} f^B([b_i]). \tag{3.16}$$

We want to show that $[b]$ is actually included in $f^B([b_i])$ for some particular $[b_i] \in u^-(x)$. Now, from (3.16) we certainly have then that $\exists [b_i] \in u^-(x) : b \in f^B([b_i])$. The desired fact, namely, $\exists [b_i] \in u^-(x) : [b] \subseteq f^B([b_i])$, follows now by outer-tightness of ce or, more specifically, by (3.15). The overall conclusion, that $[b] \in f^{CE}(u^-(x))$, follows now by (3.13).

We show the other inclusion $f^{CE}(u^-(x)) \subseteq u^-(f^X(x))$:

$$\begin{array}{rcl}
[b] \in f^{CE}(u^-(x)) & \Rightarrow & [b] \subseteq ce^-(f^{CE}(u^-(x))) \\
ce \text{ is } OT & \Rightarrow & [b] \subseteq f^B(ce^-(u^-(x))) \\
\psi^- = u^-; ce^- & \Rightarrow & [b] \subseteq f^B(\psi^-(x)) \\
\psi \text{ is } OT & \Rightarrow & [b] \subseteq \psi^-(f^X(x)) \\
\psi^- = u^-; ce^- & \Rightarrow & [b] \subseteq ce^-(u^-(f^X(x))) \\
& \Rightarrow & [b] \in u^-(f^X(x))
\end{array}$$

So, CE is a coequalizer object with the OT-homomorphism ce.

The equivalence $\sim$ we have started with is the kernel of ce and so, since ce is OT, $\sim$ is OT-congruence by fact 2.38. Thus CE, being a quotient of $B \in \mathsf{MAlg}^*_{OT}(\Sigma)$ by this congruence, is set-reflecting, i.e., $CE \in \mathsf{MAlg}^*_{OT}(\Sigma)$, by lemma 3.8.

[It is clear that the construction for small algebras can be applied to obtain coequalizers in $\mathsf{MAlg}_{OT}(\Sigma)$.] $\square$

This fact shows why we must admit operations in multialgebras returning proper classes, and not only sets. Let A have sorts s^A and t^A, both proper classes, and a function $f^A : s^A \to t^A$ which is bijective. The relation $\sim$ given by $s^A \times s^A$ and id_{t^A} is OT-congruence on A, and a coequalizer of the (projection) arrows from the congruence algebra $A^\sim$ to A, is C with $t^C = t^A$ and $s^C = \{\bullet\}$, and with $f^C(\bullet) = t^C$.

Thus, we conclude that $\mathsf{MAlg}^*_{OT}(\Sigma)$ [and $\mathsf{MAlg}_{OT}(\Sigma)$] is cocomplete. (This, of course, strengthens the initial lemmata 3.3-3.4 which only showed equivalence of being set-reflecting and being colimit of small subalgebras without either claiming nor demonstrating the actual existence of all such colimits.)

3.3 Completeness

Theorem 9 from [40], corresponding to the one quoted at the beginning of the previous subsection, states that the forgetful functor $\mathsf{SET}^F_G \to \mathsf{SET}$ creates and preserves all kinds of limits that are preserved by G. (In case of algebras, $G = id_{\mathsf{SET}}$, and so completeness follows from this general statement.) In our case, G is the power-set functor which preserves weak pullbacks (and hence intersections) or pullbacks with at least one arrow being injective but, unfortunately, neither products nor equalizers. Thus, even if we could apply the theorem, it would not yield any positive result. As we will show, constructions of limits are challenging and novel and offer new insights into the structure of our category. In particular, in case of final objects and products, we will see close but intricate relationship to the notion of bireachability.

Proposition 3.17 *$\mathsf{MAlg}^*_{OT}(\Sigma)$ [and $\mathsf{MAlg}_{OT}(\Sigma)$] has all equalizers.*

PROOF: We show first the claim only for small algebras, namely, the existence of an equalizer object E and arrow $e : E \to A$ for a pair of arrows $\phi_1, \phi_2 : A \to B$, where A is small. It is constructed in the more or less standard way.

We let $E_0 = \{a \in A \mid \phi_1(a) = \phi_2(a)\}$ and let E be the largest subalgebra of A contained in E_0. I.e., following the construction from fact 2.31, given E_i, we obtain E_{i+1} by removing all elements $e \in E_i$ such that for some $a_0 \in A \setminus E_i : e \in f^A(a', a_0, a)$. $E = \bigcap_{i \in N} E_i$. The operations are defined by $f^E(x) = f^A(x) \cap E$ for all $x \in E$, and the arrow $e : E \to A$ is inclusion (which is OT, by fact 2.31).

We verify the universal property. Assume $\psi : X \to A$ with $\psi; \phi_1 = \psi; \phi_2$. We define the arrow $u : X \to E$ by $u(x) = \psi(x)$. This will do the job (yielding unique u such that $u; e = \psi$) whenever $\psi(x) \in E$, so we have to show that this will be the case for all $x \in X$, i.e., that $\psi[X] \subseteq E$. Since ψ equalizes ϕ_1, ϕ_2, we certainly have $\psi[X] \subseteq E_0$. Assume contrapositively that $\psi[X] \not\subseteq E$, and let i be the least number such that $\psi[X] \subseteq E_i$ while $\psi[X] \not\subseteq E_{i+1}$. Choose an arbitrary element $e = \psi(x) \in \psi[X] \setminus E_{i+1}$ (and hence also $e \in E_i \setminus E_{i+1}$). By construction of E, there is some operation with $f^{A-}(e) \not\subseteq E_i$ and, since $\psi[X] \subseteq E_i$, also $f^{A-}(e) \not\subseteq \psi[X]$. Let $a_0 \in f^{A-}(e) \setminus \psi[X]$. $e \in f^A(a_0)$ and $e = \psi(x)$ so, in order for ψ to be OT, we must have $x \in \psi^-(f^A(a_0)) \Rightarrow x \in f^X(\psi^-(a_0)) = f^X(\emptyset) = \emptyset$. The consequent fails, and the contradiction shows that $\psi[X] \subseteq E_i \Rightarrow \psi[X] \subseteq E_{i+1}$ and so, eventually, $\psi[X] \subseteq E$.

In the general case, when A is set-reflecting, it is colimit of its small subalgebras, $\{A_k \mid k \in I\}$, over some diagram D. Take equalizer (E_k, e_k) of each pair $\iota_k; \phi_1$ and $\iota_k; \phi_2$ and then the colimit E of the diagram D with each A_k replaced by E_k (Colimit exists since $\mathsf{MAlg}^*_{OT}(\Sigma)$ is cocomplete, and the shape of D remains the same since, if for some $k, l : A_l \sqsubseteq A_k$, then both $E_l, E_k \sqsubseteq A_k$ and thus, by fact 2.22, also $E_l \sqsubseteq E_k$.) Denote the arrows from E_k to E by i_k (since, for each $k : i_k; e = e_k; \iota_k$ and both latter arrows are inclusions, each i_k must be injective.) The arrows $e_k; \iota_k : E_k \to A$ imply the existence of unique universal arrow

$e : E \to A$, and we show that (E, e) is equalizer of ϕ_1, ϕ_2.

$$\begin{tikzcd}
E_1 \arrow[rr, hook, "e_1"] \arrow[dr, hook, "i_1"'] & & A_1 \arrow[dr, hook, "\iota_1"] & & \\
& E \arrow[rr, hook, "e"] & & A \arrow[r, shift left, "\phi_1"] \arrow[r, shift right, "\phi_2"'] & B \\
E_k \arrow[rr, hook, "e_k"] \arrow[ur, hook, "i_k"] & & A_k \arrow[ur, hook, "\iota_k"'] & &
\end{tikzcd}$$

Since $e_k; \iota_k = i_k; e$ for every k, and each $e_k; \iota_k$ equalizes ϕ_1, ϕ_2, we also have $i_k; e; \phi_1 = i_k; e; \phi_2$. Let $x \in E$ be arbitrary. If for some k and $x' \in E_k : x = i_k(x')$, we obtain that $\phi_1(e(x)) = \phi_2(e(x))$. But since E is colimit of all E_k, all $x \in E$ must satisfy this condition (i.e., by the construction of coproducts and coequalizers, $\forall x \in E\ \exists E_k, x' \in E_k : x = i_k(x')$), and so $e; \phi_1 = e; \phi_2$.

We verify the universal property. Given an X with an arrow $\psi : X \to A$ such that $\psi; \phi_1 = \psi; \phi_2$. If X is small, the arrow ψ can be factored through some small subalgebra $\psi : X \xrightarrow{\psi_k} A_k \xrightarrow{\iota_k} A$, and since E_k is an equalizer with respect to $\iota_k; \phi_1$ and $\iota_k; \phi_2$, we obtain a unique arrow $u_k : X \to E_k$ with $u_k; e_k = \psi_k$, yielding also $u_k; e_k; \iota_k = \psi$ and hence also (since i_k is mono) a unique $u_k; i_k = u : X \to E$ with $u; e = \psi$. If X is not small (but set-reflecting) it is a colimit of its small subalgebras and the above construction follows for each such $X_k \sqsubseteq X$. We obtain the collection of (unique) arrows $u_k; i_k : X_k \to E$ which, by the colimit property of X, give a unique arrow $u : X \to E$. Chasing the diagram yields the required fact that $u; e = \psi$.

Since E is colimit of its small subalgebras, it is set-reflecting, i.e., $E \in \mathsf{MAlg}^*_{OT}(\Sigma)$, by fact 3.4. [It is clear that the construction for small algebras will yield equalizers also in $\mathsf{MAlg}_{OT}(\Sigma)$.] $\square$

To show the existence of final objects, reported in [41], we first state a simple lemma.

Lemma 3.18 *For a multialgebra A, let $\sim_A$ be the maximal OT-congruence on A (existing by Lemma 3.6). For any algebra B there is at most one OT-homomorphism $B \to A/_{\sim_A}$.*

PROOF: By the construction of coequalizers in $\mathsf{MAlg}^*_{OT}(\Sigma)$, Fact 3.12. If there were two distinct $\phi_1, \phi_2 : B \to A/_{\sim_A}$, there would be a non-trivial coequalizing arrow $ce : A/_{\sim_A} \to CE$, making $\phi_1; ce = \phi_2; ce$. Its non-triviality means that its kernel $\sim_{ce} \neq id_{A/_{\sim_A}}$ and, since ce is OT so, by Fact 2.38, $\sim_{ce}$ is an OT-congruence. But then we can use $\sim_{ce}$ to obtain a larger OT-congruence on A than $\sim_A$, contradicting the assumption that $\sim_A$ was the largest such. $\square$

Theorem 3.19 $\mathsf{MAlg}^*_{OT}(\Sigma)$ *has final objects.*

PROOF: Let CP be a coproduct of all small algebras in $\mathsf{MAlg}^*_{OT}(\Sigma)$ (which exists and is set-reflecting by Fact 3.9). Let $\sim_{CP}$ be the maximal OT-congruence on CP (existing by Lemma 3.6), and let $Z = CP/_{\sim_{CP}}$. (By Lemma 3.8, Z is set-reflecting and so $Z \in \mathsf{MAlg}^*_{OT}(\Sigma)$.)

For every small algebra $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$, there is (at least one) morphism $A \to CP$ and then, composing it with the quotient morphism $CP \to Z$, exactly one (by lemma 3.18) morphism $a : A \to Z$.

Any other (large) $A \in \mathsf{MAlg}^*_{OT}(\Sigma)$ is colimit of its small subalgebras, with the inclusions $\iota_i : A_i \to A$. Since there is also (exactly) one morphism $a_i : A_i \to Z$ for each small subalgebra $A_i \sqsubseteq A$, the colimit property yields a (unique) morphism $u : A \to Z$ (making $\iota_i; u = a_i$). But then, since there is such a morphism $A \to Z$ so, by lemma 3.18, it is unique. $\square$

Construction of products is a more complicated task and we devote the next section to it.

4 Products

We comment first on the relationship between product and (maximal) bireachability between algebras in order to signal potential complications. The actual construction and proofs are given in subsection 4.1. Subsection 4.2 shows that this construction can be utilized also in the category $\mathsf{MAlg}_{OT}(\Sigma)$.

In the case of coalgebras, preservation of mono-sources (by the signature functor) is equivalent with the coincidence of product and maximal bisimulation (theorem 8.6 in [16]). Thus, by the duality from remark 2.9 and that between bisimilarity and bireachability, if we considered only the subcategory of multialgebras obtained from coalgebras (over a given polynomial functor), we could conclude the existence of products, namely, of maximal bireachabilities between the arguments (with the algebraic structure defined in (2.55)). However, our case is more general and also more complicated since, in a given $\mathsf{MAlg}_{OT}(\Sigma)/\mathsf{MAlg}^*_{OT}(\Sigma)$, there are multilagebras which are not converses of coalgebras over (the respective) polynomial functor Σ. The problems and counterexamples will be provided exactly by such multialgebras. (The category of colagebras for power-set functor is isomorphic to $\mathsf{MAlg}_{OT}(\Sigma)$ for a Σ with a single operation $S \to \mathcal{P}(S)$. But power-set functor does not preserve mono-sources, and so, by the just quoted theorem, products do not coincide with maximal bisimulations. Our results will yield also a construction of products for coalgebras over power-set functor.)

Recall the definition 2.54 of bireachability between two algebras as a subset $C \subseteq A_1 \times A_2$ satisfying the bireachbility condition:

$$\begin{array}{rl} \forall a, b, a_1: & C(a,b) \wedge a \in f^{A_1}(a_1) \Rightarrow \exists b_1 \in A_2 : b \in f^{A_2}(b_1) \wedge C(a_1, b_1) \\ \& \ \forall a, b, b_1: & C(a,b) \wedge b \in f^{A_2}(b_1) \Rightarrow \exists a_1 \in A_1 : a \in f^{A_1}(a_1) \wedge C(a_1, b_1) \end{array} \tag{2.54}$$

As noted earlier, this condition is preserved under arbitrary unions and thus, collecting all small bireachabilities between two algebras A and B, we obtain the counterpart of lemma 3.6. We also register the counterpart of fact 3.5 (with essentially the same proof.)

Fact 4.1 *For every $A, B \in \mathsf{MAlg}^*_{OT}(\Sigma)$ there exists a (unique) maximal bireachability between A and B. Moreover, any bireachability between A, B (with operations defined according to (2.55)) is set-reflecting.*

This maximal bireachability need not, however, be the product of A, B.

Example 4.2 *Consider two algebras over $\Sigma = \langle \{s_1, s_2\}, \{f : s_1 \to s_2\} \rangle$ (as in example 2.59):*

$$\begin{tikzcd}[column sep=small] A & a & & & b & B \\ a_1 \arrow[ur] & & a_2 \arrow[ul] & b_1 \arrow[ur] & & b_2 \arrow[ul] \end{tikzcd}$$

Following are examples of bireachabilities between A and B:

$$\begin{tikzcd}[column sep=small] & R_1: & \langle a,b \rangle & & R_2: & \langle a,b \rangle & \\ R_0 : \langle a_1, b_1 \rangle & \langle a_1, b_1 \rangle \arrow[ur] & & \langle a_2, b_2 \rangle \arrow[ul] & \langle a_1, b_2 \rangle \arrow[ur] & & \langle a_2, b_1 \rangle \arrow[ul] \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] R_3: & \langle a,b \rangle & & R_4 = R_1 \cup R_2 : & & \langle a,b \rangle & \\ \langle a_1, b_1 \rangle \arrow[ur] & \langle a_1, b_2 \rangle \arrow[u] & \langle a_2, b_2 \rangle \arrow[ul] & \langle a_1, b_1 \rangle \arrow[urr] & \langle a_1, b_2 \rangle \arrow[ur] & \langle a_2, b_1 \rangle \arrow[u] & \langle a_2, b_2 \rangle \arrow[ul] \end{tikzcd}$$

R_4 is the maximal bireachability between A and B – every other bireachability is a subset of it. However, only $R_0 \sqsubseteq R_4$, while niether R_1, R_2 nor R_3 is a subalgebra of R_4: the inclusions are not OT-homomorphisms. Consequently, R_4 can not possibly be the product of A, B, as the projections from, say, R_2 would not factor through it.

In fact, even taking colimit of all bireachabilities between the given algebras is not sufficient...

4.1 Construction of products

We will conduct the proof for the binary products only, but it will be easy to see that all the constructions and results generalize to products of arbitrary sets of objects.

Our category $\mathsf{MAlg}^*_{OT}(\Sigma)$ has all colimits and, given A_1, A_2, we proceed as follows:

4.1.1 show that it has epi-monosource factorisation, namely, proposition 4.7; moreover, that the monosource is a quotient of the domain of the span by a unique congruence

4.1.2 take colimit $\mathbf{P}$ of the diagram $\mathcal{D}$ of all non-isomorphic monosources for A_1, A_2 with arrows commuting with the span arrows (uniqe monos)

4.1.3 show that $\mathbf{P}$ is a monosource and hence (since it is colimit of monosources) any span has a uniqe factorisation through $\mathbf{P}$.

For the rest of this section, we fix some A_1, A_2. All considerations are relative to these two objects, in particular, all spans and monosources have them as their codomain.

Definition 4.3 *1. A (binary) monosource is a span* $A_1 \xleftarrow{m_1} M \xrightarrow{m_2} A_2$ *such that for any* $\phi, \psi : X \to M$ *we have:* $(\forall i : \phi; m_i = \psi; m_i) \Rightarrow \phi = \psi$ *(we also say,* m_1, m_2 *are jointly mono.)*
2. A morphism between monosources (M, m_i) *and* (N, n_i) *with the same codomain is a morphism* $h : M \to N$ *such that* $m_i = h; n_i$.
3. Monosources (M, m_i) *and* (N, n_i) *are isomorphic iff there exists an isomorphism* $h : M \simeq N$ *which is monosource morphism.*

Non-isomorphic monosources can still have isomorphic domains. Obviously, if (M, m_i) is a monosource and $N \sqsubseteq M$ (with mono $\iota : N \to M$), then also $(N, \iota; m_i)$ is a monosource. (We will often skip ι in the notation for monosources/spans from subobjects, i.e., we will usually write simply (N, m_i).) Given two objects $A_1, A_2 \in \mathsf{MAlg}^*_{OT}(\Sigma)$, we obtain the category $\mathsf{MS}(A_1, A_2)$ with all monosources with the codomain A_1, A_2 as objects and monosource morphisms as morphisms.

Fact 4.4 *1. Given a span* $A_1 \xleftarrow{n_1} N \xrightarrow{n_2} A_2$ *and a morphism* $h : M \to N$*, if* $(M, h; n_i)$ *is a monosource, then* h *is mono.*
2. For any monosource morphism $h : (M, m_i) \to (N, n_i)$*,* $h : M \to N$ *is mono.*

PROOF: 1. If for two morphisms $f, g : X \to M$ we have $f; h = g; h$ then also $f; h; n_i = g; h; n_i$, and so $f = g$ since $(M, h; n_i)$ is a monosource.
2. follows from 1. since $(M, m_i) = (N, h; n_i)$. $\square$

The following fact helps in establishing some uniqueness results.

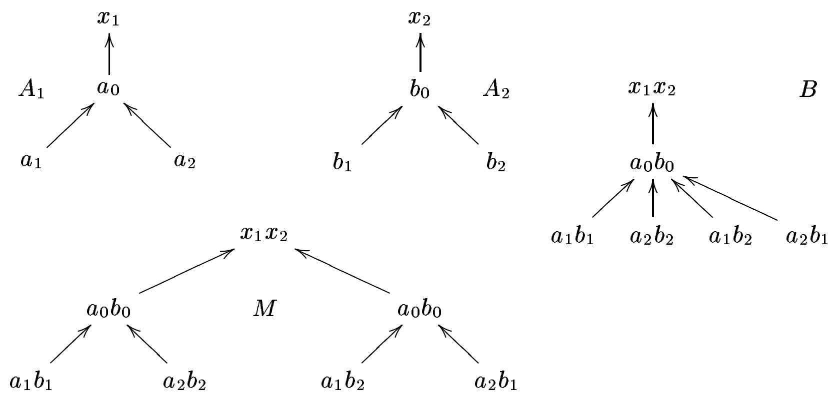
Fact 4.5 *Given a span* (M, m_i)*, let* $\sim_i$ *be the kernel of* m_i*.* (M, m_i) *is a monosource iff the greatest lower bound* $\sim_1 \wedge \sim_2 = id_M$.

PROOF: $\Rightarrow$) Let $\sim = \sim_1 \wedge \sim_2$ denote the greatest lower bound of m_is' kernels and $M^\sim$ be as in definition 2.50. By fact 2.51, there is a coequalizer diagram: $M^\sim \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} M \xrightarrow{n} M/_\sim$.
Now $\sim \subseteq \sim_i$ ensures the existence of a unique $n_i : M/_\sim \to A_i$ such that $n; n_i = m_i$.

$$\begin{tikzcd}
 & & & A_i \\
M^\sim \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & M \arrow[urr, "m_i"] \arrow[r, "n"] & M/_\sim \arrow[ur, "n_i"'] &
\end{tikzcd}$$

This entails $\pi_1; m_i = \pi_1; n; n_i = \pi_2; n; n_i = \pi_2; m_i$ and so $\pi_1 = \pi_2$ since (M, m_i) is a monosource. Hence $\sim \subseteq id_M$ with the opposite inclusion following trivially since $\sim$ is a congruence.
$\Leftarrow$) Conversely, assume $\sim_1 \wedge \sim_2 = id_M$, and let (*) $\pi_1; m_i = \pi_2; m_i$ for some $\pi_i : M^\sim \to M$ as in the diagram above. If $\pi_1 \neq \pi_2$, let $n : M \to M/_\sim$ be their coequalizer, with a non-trivial kernel $\sim \neq id_M$. The coequalizer property with the assumption (*) give then unique $n_i : M/_\sim \to A_i$ such that $n; n_i = m_i$. But this means that $ker(n; n_i) = \sim_i$ which implies, in particular, that $\sim = ker(n) \subseteq \sim_i$. But since $\sim \neq id_M$, this contradicts the assumption that $\sim_1 \wedge \sim_2 = id_M$. $\square$

Example 4.6 *Note that monosource need not have domain which is a subset of the cartesian* *product of the codomains. E.g.:*



The names of M elements identify the images under $m_i : M \to A_i$. B is the bireachability induced between A_1 and A_2 by this span. (It is also the maximal bireachability between them.) Using the criterion from the above fact, it is easy to convince oneself that M is a monosource. In particular, the obvious mapping $M \to B$ is not a homomorphism (i.e., the relation relating (only) the two copies of a_0b_0 is not a congruence on M.) B, with the indicatedd projections, is also a monosource.

4.1.1 Epi-monosource factorisation

The first main partial result is:

Lemma 4.7 *For any span $A_1 \xleftarrow{\phi_1} C \xrightarrow{\phi_2} A_2$ there is a monosource $A_1 \xleftarrow{m_1} M \xrightarrow{m_2} A_2$ and an epi $e_M : C \to M = C/_\sim$ such that $\phi_i = e_M;m_i$ and $ker(e_M) = ker(\phi_1) \wedge ker(\phi_2)$.*

Proof: Let $\sim_i$ denote the kernel of ϕ_i and $\sim = \sim_1 \wedge \sim_2$ their greatest lower bound (which exists by fact 2.44). We show first that we have m_i making $M \simeq C/_\sim$ a monosource.
Since $\sim \subseteq \sim_i$, we obtain by the coequalizer property of $e_M : C \to C/_\sim$, unique $m_i : C/_\sim \to A_i$ such that $\phi_i = e_M;m_i$:

$$\begin{tikzcd}
 & & & A_i \\
C \arrow[r, two heads, "e_M"] \arrow[urrr, two heads, "\phi_i"] & C/_\sim \arrow[urr, two heads, "m_i"] \arrow[rr, "n"] & & N \arrow[u, "n_i"'] \\
X \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & & &
\end{tikzcd}$$

To show that (M, m_i) is a monosource, assume given $g_1, g_2 : X \to M$ with $g_1;m_i = g_2;m_i$. Let $n : C/_\sim \to N$ be their coequalizer. As coequalizer arrow (*) n is epi and, moreover, $\sim = ker(e_M) \subseteq ker(e_M;n)$.
By coequalizer property, there exist unique $n_i : N \to A_i$ making $n;n_i = m_i$. Then, since $e_M;n;n_i = e_M;m_i = \phi_i$, so $ker(e_M;n) \subseteq ker(e_M;n;n_i) = \sim_i$ and thus $ker(e_M;n) \subseteq \sim$. Together with (*) we obtain $\sim = ker(e_M;n)$ which implies that n is injective. Since n is also epi, it is iso by proposition 2.13. But then $g_1 = g_1;n;n^- = g_2;n;n^- = g_2$. □

We strengthen the above lemma showing that $C/_\sim$ is the unique monosource satisfying the conditions. To do this, we first verify that our category has the diagonal fill-in property.

Lemma 4.8 *Given a span $g_i : G \to A_i$, an epi $e : C \to G$, a monosource (N, n_i) and an $f : C \to N$ such that $e;g_i = f;n_i$, there exists a unique $k : G \to N$ such that $e;k = f$ and $k;n_i = g_i$.*

PROOF: By corollary 2.52, e is reqular, so let $\pi_i : X \to C$ be coequalized by e.

$$\begin{tikzcd}
X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & C \arrow[r, "e"] \arrow[d, "f"'] & G \arrow[d, "g_i"] \arrow[dl, dotted, "k"'] \\
& N \arrow[r, "n_i"'] & A_i
\end{tikzcd}$$

By assumption, $\pi_1; f; n_i = \pi_1; e; g_i = \pi_2; e; g_i = \pi_2; f; n_i$ and thus $\pi_1; f = \pi_2; f$ since N is a monosource. Since $e : C \to G$ is a coqualizer of π_is, we obtain a unique $k : G \to N$ with $e; k = f$. Now, $e; k; n_i = f; n_i = e; g_i$ and so $k; n_i = g_i$ since e is epi. $\square$

Corollary 4.9 *$(C/_{\sim}, m_i)$ in lemma 4.7 is unique (up to monosource isomorphism).*

PROOF: Consider a span $A_1 \xleftarrow{\phi_1} C \xrightarrow{\phi_2} A_2$, and let $(M, m_i), (N, n_i)$ be two monosource factorisations of this span.
By lemma 4.8 there exists a unique $k : M \to N$ such that $e_M; k = e_N$ and $m_i = k; n_i$ and, dually, a uniqe $g : N \to M$ with the respective $e_N; g = e_M$ and $n_i = g; m_i$. Then $k; g; m_i = k; n_i = m_i = id_M; m_i$ and, since (M, m_i) is a monosource, $k; g = id_M$. Analogously, we get $g; k = id_N$. $\square$

4.1.2 Colimit of monosources...

Given a pair $A_1, A_2 \in \mathsf{MAlg}^*_{OT}(\Sigma)$, we obtain the category $\mathsf{MS}(A_1, A_2)$ of monosources with the codomain A_1, A_2 and with monosource morphisms. We let $\mathcal{D}$ be its skeleton. As $\mathsf{MAlg}^*_{OT}(\Sigma)$ has all colimits, consider the colimit $\mathbf{P}$ of the "base" of the diagram $\mathcal{D}$, namely $(\mathbf{P}, \{\iota_{(M,m_i)} : M \to \mathbf{P} \mid (M, m_i) \in \mathcal{D}\})$. (We will skip the morphisms in the notation, i.e., will write ι_M rather than $\iota_{(M,m_i)}$.) Thus, for any $g : (M, m_i) \to (N, n_i)$ in $\mathcal{D}$, we have $m_i = g; n_i$ and so the colimit property provides a unique "mediating" span $\pi_i : \mathbf{P} \to A_i$ such that $m_i = \iota_M; \pi_i$ for every $(M, m_i) \in \mathcal{D}$.

According to fact 4.4.1, each ι_M is a mono.

(Typically, given a subobject $S \sqsubseteq \mathbf{P}$, we will identify in the notation the arrows $\pi_i : \mathbf{P} \to A_i$ and $\pi_i : S \to A_i$.)

4.1.3 ... is a product

To show that $(\mathbf{P}, \pi_i)$ is a product, we show first some auxiliary results.

Fact 4.10 *1. For every monosource $(M, m_i) \in \mathcal{D}$, the restriction $\iota_M : M \to \iota_M[M]$ gives a (monosource) isomorphism $(M, m_i) \simeq (\iota_M[M], \pi_i)$.*

2. For every monosource $(N, n_i) \in \mathsf{MS}(A_1, A_2)$ there is a subalgebra $P_N \sqsubseteq \mathbf{P}$ such that $\iota_N : (N, n_i) \to (P_N, \pi_i)$ is a monosource isomorphism.

PROOF: 1. $\iota_M[M] \sqsubseteq \mathbf{P}$ so ι_M is actually a span morphism $\iota_M : (M, m_i) \to (\iota_M[M], \pi_i)$. It is surjective and, by fact 4.4.1, mono so, by 2.13 it is iso. Hence $(\iota_M[M], \pi_i)$ is a monsource, and so ι_M is a monosource iso.
2. By definition, for every monosource $(N, n_i) \in \mathsf{MS}(A_1, A_2)$ there is a unique $(M, m_i) \in \mathcal{D}$ with a monosource iso $j_N : (N, n_i) \to (M, m_i)$. By 1, we can choose $P_N = \iota_M[M]$ and let $\iota_N = j_N; \iota_M$. $\square$

Lemma 4.11 *Let $(M, m_i) \in \mathcal{D}$ and assume given $f : N \to M$ and $g : N \to \mathbf{P}$ such that $(N, g; \pi_i)$ is a monosource and f is a monosource morphism $(N, g; \pi_i) \to (M, m_i)$. Then $g = f; \iota_M$.*

PROOF: We consider the following diagram

$$
\begin{tikzcd}
 & & & M \arrow[dl, phantom] \\
S'_x \arrow[r, "i_{S_x}"'] \arrow[urrr, dashed] \arrow[d, dashed, "i_{S_x};g;\iota^{-}_{N_x}"'] & S_x \arrow[r, "\sqsubseteq"] \arrow[d, "g"] & N \arrow[ur, "f"] \arrow[dd, "g"] & \\
N_x \arrow[r, "\iota_{N_x}"] \arrow[drr, "\iota_{N_x}"'] & \iota_{N_x}[N_x] \arrow[dr, "\sqsubseteq"] & & \\
 & & \mathbf{P} \arrow[from=uuur, "\iota_M"'] &
\end{tikzcd}
\tag{4.12}
$$

Since colimit arrows are jointly epi, for any $x \in N$ there exists a monosource $(N_x, n_i) \in \mathcal{D}$ such that $g(x) \in \iota_{N_x}[N_x] \sqsubseteq \mathbf{P}$. By lemma 2.32.2, $S_x = g^{-}[\iota_{N_x}[N_x]] \sqsubseteq N$. By assumption, $(N, g; \pi_i)$ is a monosource and so is $(S_x, \sqsubseteq; g; \pi_i)$ since $\sqsubseteq: S_x \to N$ is mono.

We have the possibility that (i) S_x and $\iota_{N_x}[N_x]$ are isomorphic monosources, which we leave for the time aside. Otherwise (ii), by definition of $\mathcal{D}$ there exists a monosource $(S'_x, s_i) \in \mathcal{D}$ with a monosource isomorphism $i_{S_x} : (S'_x, s_i) \to (S_x, \sqsubseteq; g; \pi_i)$. Since the restriction ι_{N_x} is a monosource isomorphism by fact 4.10.1, so we have a monosource morphism $i_{S_x}; g; \iota^{-}_{N_x}$: $(S'_x, s_i) \to (N_x, n_i)$ in $\mathcal{D}$ (marked with the dashed arrow). On the other hand, we also have in $\mathcal{D}$ a monosource morphism $i_{S_x}; \sqsubseteq; f : (S'_x, s_i) \to (M, m_i)$. The colimit property entails then $i_{S_x}; g; \iota^{-}_{N_x}; \iota_{N_x} = \iota_{S_x}; \sqsubseteq; f; \iota_M$ which, according to the definition of S_x, implies $i_{S_x}; g; \sqsubseteq = i_{S_x}; \sqsubseteq; f; \iota_M$. We have $g; \sqsubseteq = \sqsubseteq; g$, so we obtain $i_{S_x}; \sqsubseteq; g = i_{S_x}; \sqsubseteq; f; \iota_M$ from which it follows that $\sqsubseteq; g = \sqsubseteq; f; \iota_M$ since i_{S_x} is iso. Since $x \in S_x$, this proves $g(x) = \iota_M(f(x))$.

In case (i) of the monosource isomorphism $S_x \simeq N_x$, we only obtain that $i_{S_x}; g; \iota^{-}_{N_x} = id_{N_x}$, which does not change anything in the argument; similarly if $S'_x = M$. $\square$

Corollary 4.13 *1. Given $k : N \to K$, $g : N \to \mathbf{P}$, $h : K \to \mathbf{P}$ such that $(N; g; \pi_i), (K, h; \pi_i)$ are monosources and k is monosource morphism between them – then $k; h = g$.*

2. For every monosource (M, m_i) there is exactly one monosource $M' \sqsubseteq \mathbf{P}$ such that (M, m_i) and (M', π_i) are isomorphic (as monosources).

PROOF: 1. By definition of $\mathcal{D}$ there is an isomorphism $i : (K, h; \pi_i) \to (M, m_i)$, for some $(M, m_i) \in \mathcal{D}$.

$$
\begin{tikzcd}
N \arrow[r, "k"] \arrow[dr, "g"'] & K \arrow[r, "i"] \arrow[d, "h"] & M \arrow[dl, "\iota_M"] \\
 & \mathbf{P} &
\end{tikzcd}
$$

By 4.11, we obtain $k; i; \iota_M = g$ and $i; \iota_M = h$ so $k; h = g$.

2. The existence of such an M' is given in fact 4.10.1. As to uniqueness, assume two isomorphic subobjects $N, K \sqsubseteq \mathbf{P}$ with an isomorphism $k : N \to K$. In the diagram above, k, g are then inclusions. By 1. $k; \sqsubseteq = \sqsubseteq$, which means that k itself is an inclusion, and likewise is k^{-}. $\square$

The proof of the following lemma follows the same argument as the proof of lemma 4.11.

Lemma 4.14 *Given some $g : C \to \mathbf{P}$, let $(e_g : C \to g[C]; \sqsubseteq: g[C] \to \mathbf{P})$ be epi-mono factorisation of g (existing by lemma 2.40). Furthermore, let (M, m_i) and $e : C \to M$ be epi-monosource factorisation of the span $(C, g; \pi_i)$ (existing by lemma 4.7). Then $e; \iota_M = g$ is also epi-mono factorisation of g.*

Proof: By assumption $g;\pi_i = e_g;\sqsubseteq;\pi_i = e;m_i$

$$
\begin{tikzcd}
C \arrow[r, "e_g"] \arrow[dr, "g"] \arrow[dd, "e"'] & g[C] \arrow[d, "\sqsubseteq"] \arrow[ddl, dotted, "k"'] \\
& \mathbf{P} \arrow[d, "\pi_i"] \\
M \arrow[ur, "\iota_M"] \arrow[r, "m_i"'] & A_i
\end{tikzcd}
$$

Thus, by lemma 4.8, there exists a (unique) $k : g[C] \to M$ such that (*) $e_g;k = e$ and $k;m_i = \sqsubseteq;\pi_i$. That is, k is a span morphism $k : (g[C], \sqsubseteq;\pi_i) \to (M, m_i) = (M, \iota_M;\pi_i)$. Showing that $k;\iota_M = \sqsubseteq$ will yield the claim.

We refer to the diagram (4.12) where we substitute $g[C]$ for N and k for f. We annotate some inclusions to ease the references.

$$
\begin{tikzcd}
& & & M \arrow[ddl, "\iota_M"] \\
S'_x \arrow[urrr, dashed] \arrow[r, "i_{S_x}"'] \arrow[d, dashed, "i_{S_x};g;\iota_{N_x}^{-}"'] & S_x \arrow[r, "\sqsubseteq'_x"] \arrow[d, "\sqsubseteq'"] & g[C] \arrow[ur, "k"] \arrow[dd, "\sqsubseteq"'] & \\
N_x \arrow[r, "\iota_{N_x}"] \arrow[drr, "\iota_{N_x}"'] & \iota_{N_x}[N_x] \arrow[dr, "\sqsubseteq_x"] & & \\
& & \mathbf{P} &
\end{tikzcd}
$$

As in the proof of 4.11, for any $x \in g[C]$ there is a monosource $(N_x, n_i) \in \mathcal{D}$ with $x \in \iota_{N_x}[N_x]$. Then, by fact 2.25, also $g[C] \cap \iota_{N_x}[N_x] = S_x$ is a subalgebra of $g[C]$ and of $\iota_{N_x}[N_x]$. By fact 4.10.1, $(\iota_{N_x}[N_x], \sqsubseteq_x;\pi_i)$ is a monosource, and hence $(S_x, \sqsubseteq';\sqsubseteq_x;\pi_i) = (S_x, \sqsubseteq'_x;\sqsubseteq;\pi_i)$ is a monosource, since $\sqsubseteq'$ is mono.

By definition of $\mathcal{D}$, there is a monosource $(S'_x, s_i) \in \mathcal{D}$ with the isomorphism $i_{S_x} : (S'_x, s_i) \to (S_x, \sqsubseteq'_x;\sqsubseteq;\pi_i)$. As in the proof of 4.11, we can conclude $i_{S_x};\sqsubseteq';\iota_{N_x}^{-};\iota_{N_x} = i_{S_x};\sqsubseteq';k;\iota_M$, i.e., $i_{S_x};\sqsubseteq';\sqsubseteq_x = i_{S_x};\sqsubseteq'_x;\sqsubseteq = i_{S_x};\sqsubseteq';k;\iota_M$. Since i_{S_x} is iso, this implies $\sqsubseteq'_x;\sqsubseteq = \sqsubseteq';k;\iota_M$, and as $x \in S_x$, this shows that $x = \iota_M(k(x))$ and thus (**) $\sqsubseteq = k;\iota_M$ since x was arbitrary.

We now obtain $g = e_g;\sqsubseteq \overset{(**)}{=} e_g;k;\iota_M \overset{(*)}{=} e;\iota_M$. □

Corollary 4.15 *1.* $(\mathbf{P}, \pi_i)$ *is a monosource and 2. every subobject* $C \sqsubseteq \mathbf{P}$ *is a monosource* $(C, \sqsubseteq;\pi_i)$.

Proof: 2. follows from 1. since composition of a mono with a monosource is a monosource. To show 1, assume $f, g : C \to \mathbf{P}$ with $f;\pi_i = g;\pi_i$. Let (F, f_i) with $e_f : C \to F$ be epi-monosource factorisation of f, and similarly (G, g_i) with $e_g : C \to G$ for g.

$$
\begin{tikzcd}
& F \arrow[dd, "k"] \arrow[dr, "\iota_F"] \arrow[drr, "f_i"] & & \\
C \arrow[ur, "e_f"] \arrow[dr, "e_g"'] & & \mathbf{P} \arrow[r, "\pi_i"] & A_i \\
& G \arrow[ur, "\iota_G"'] \arrow[urr, "g_i"'] & &
\end{tikzcd}
$$

By lemma 4.8, there is a unique $k : F \to G$ such that (*) $e_f;k = e_g$ and $k;g_i = f_i$. This latter equality means that $k : (F, f_i) \to (G, g_i)$ is a monosource morphism. By fact 4.10.2, we also have a monosource morphism $\iota_F : (F, f_i) \to (\mathbf{P}, \pi_i)$, hence $(F, f_i) = (F, \iota_F;\pi_i)$ and likewise for $(G, g_i) = (G, \iota_G; \pi_i)$. So k is a monosource morphism $k : (F, \iota_F; \pi_i) \to (G, \iota_G; \pi_i)$. By corollary 4.13.1, we then have $\iota_F = k; \iota_G$, which gives the second of the following equalities, with the first and the last one following from lemma 4.14: $f = e_f; \iota_F = e_f; k; \iota_G \stackrel{(*)}{=} e_g; \iota_G = g$. □

Theorem 4.16 *Colimit* $\mathbf{P}$ *of the diagram* $\mathcal{D}$ *– a skeleton of* $\mathsf{MS}(A_1, A_2)$ *– is a product* $A_1 \times A_2$.

PROOF: Let $\phi_i : C \to A_i$ be a span. By lemma 4.7, there is a morphism $e : C \to M$ into a monosource (M, m_i) such that $\phi_i = e; m_i$. Composed with $\iota_M : M \to \mathbf{P}$ (which commutes with the projections, i.e., $m_i = \iota_M; \pi_i$), this gives an $u = e; \iota_M$ such that $\phi_i = u; \pi_i$. We thus obtain an arrow $u : C \to \mathbf{P}$ which, by corollary 4.15, is unique.

Since $\mathbf{P}$ is colimit object of a diagram over $\mathsf{MAlg}^*_{OT}(\Sigma)$, it is set-reflecting by cocompleteness of the category, i.e., $\mathbf{P} \in \mathsf{MAlg}^*_{OT}(\Sigma)$. □

Extension to products of arbitrary sets of objects, $\prod_{i \in I} A_i$, is straightforward. (The only changes of some significance are to consider I-indexed monosources and taking greatest lower bound of I kernels, in 4.1.1, which is possible since collection of congruences on a given algebra is a complete lattice.)

Theorem 4.17 *For any set* I *and collection of objects* $\{A_i \in \mathsf{MAlg}^*_{OT}(\Sigma) \mid i \in I\}$*, the colimit of the diagram of all non-isomorphic monosources* $(M, m_i : M \to A_i)$ *is the product* $\prod_{i \in I} A_i$.

Notice that we do not obtain products for all class-indexed families. Extending the proof might, for instance, require showing that not only any set but also any class of congruences on an algebra has an infimum. This limitation follows directly from the fact that a category that has all, also class-indexed, products is thin (e.g., theorem 10.32 in [3]), while our category obviously is not thin.

4.2 Products in $\mathsf{MAlg}_{OT}(\Sigma)$

The above construction does not require classes and can be performed also in the category $\mathsf{MAlg}_{OT}(\Sigma)$ of small multialgebras. We only have to ensure that, given a pair/set of such algebras, the diagram $\mathcal{D}$ will be small. (This is the main reason for taking as $\mathcal{D}$ only the skeleton of $\mathsf{MS}(A_1, A_2)$.)

Lemma 4.18 *There is a function* $f : \mathsf{Card} \times \mathsf{Card} \to \mathsf{Card}$ *such that for any* $A_1, A_2 \in \mathsf{MAlg}_{OT}(\Sigma)$ *and any* $(M, m_i) \in \mathsf{MS}(A_1, A_2)$, $|M| \le f(|A_1|, |A_2|)$.

PROOF: Let $\alpha_i = |A_i|$ be the cardinalities of A_i's and $\alpha = \alpha_1 * \alpha_2 = |A_1 \times A_2|$. Assume $(M, m_i) \in \mathsf{MS}(A_1, A_2)$ and let $\simeq_i = ker(m_i)$. We show that there is a cardinal number limiting from above the possible size of M. We use fact 4.5 and show that if cardinality of M is too large then $\simeq_1 \wedge \simeq_2 \neq id_M$. To do this, we apply the construction of infimum of two congruences as given in (2.45), adapting it so that in each step we set a limit on the possible number of obtained equivalence classes.

1. The first step gives $x \sim_0 x' \iff x \simeq_1 x' \wedge x \simeq_2 x'$, i.e., each equivalence class is associated with a unique pair $\langle a_1, a_2 \rangle \in A_1 \times A_2$, namely such that $m_1([x]^{\sim_0}) = a_1$ and $m_2([x]^{\sim_0}) = a_2$. Hence, the number of these classes $\kappa_0 \le \alpha$.

2. The inductive step amounts to propagation of the existing distinctions, i.e., splitting of the equivalence classes obtained so far. A class $[y_1]^{\sim_i}$ is splitted by removing the pairs $\langle y_1, y_2 \rangle \in \sim_i$ for which there exist noncongruent pre-images, i.e., such pairs that (*) $\exists x_1 \in f^-(y_1)\ \forall x_2 \in f^-(y_2) : \langle x_1, x_2 \rangle \notin \sim_i$ for some $f \in \Sigma$.[7] Now, for any y_1 and f, the pre-image $f^-(y_1)$ determines a subset of $\sim_i$-equivalence classes, namely $[f^-(y_1)]^{\sim_i} = \{[x]^{\sim_i} \mid [x]^{\sim_i} \cap f^-(y_i) \neq \emptyset\}$. If y_1, y_2 satisfy (*), i.e., are splitted in the step $i+1$, then also $[f^-(y_1)]^{\sim_i} \neq [f^-(y_2)]^{\sim_i}$. There are 2^{κ_i} such subsets, i.e., there are no more than 2^{κ_i} possible splittings of every one of κ_i equivalence classes at step $i+1$. So we obtain $\kappa_{i+1} \le \kappa_i * 2^{\kappa_i}$. (We have ignored some finite constants which do not increase this estimate when κ_i's are infinite, namely, the number of operations in the signature (which is finite), as well as the arity of the operations (which is finite).)

[7] We write $f^-(y)$ for $(f^M)^-(y)$.

3. In the limit, we obtain $f(\alpha_1, \alpha_2) = \kappa \leq \bigcup_{i \in \omega} \kappa_i$, which gives the upper bound on the number of equivalence classes for the congruence $\sim \, = \, \simeq_1 \wedge \simeq_2$.

So if $|M| > \kappa$, the congruence $\sim$ will yield some equivalence class with more than 1 element, i.e., $\sim \neq id_M$. But then, by fact 4.5, (M, m_i) will not be a monosource. □

Theorem 4.19 *For any set I and collection of objects $\{A_i \in \mathsf{MAlg}_{OT}(\Sigma) \mid i \in I\}$, the diagram $\mathcal{D}$ of all non-isomorphic monosources $(M, m_i : M \to A_i)$ is small and its colimit is a product $\prod_{i \in I} A_i$.*

PROOF: By the above lemma 4.18, the size of monosources is limited by (a function of) the size of the codomain objects, and so there is at most a set of non-isomorphic monosources with codomain $\{A_i \mid i \in I\}$. Since $\mathsf{MAlg}_{OT}(\Sigma)$ is cocomplete, a colimit of $\mathcal{D}$ exists in $\mathsf{MAlg}_{OT}(\Sigma)$. The rest of the proof is exactly the same as the proof of theorem 4.16. □

Recalling the remark 2.12, the category of coalgebras for the (direct image) power-set functor is isomorphic to $\mathsf{MAlg}_{OT}(\Sigma)$ for Σ containing one sort and operation symbol $f : S \to S$. Hence, all our constructions give the respective constructions for this particular category of colagebras.

Corollary 4.20 *The construction from section 4.1 yields products of coalgebras for power-set functor.*

5 Conclusions

Multialgebras lie at the intersection of several research topics. They

- represent relational structures and, generally, Boolean algebras with operators, [24, 25];
- generalise traditional – both total and partial – algebras;
- provide a fundamental instance of power structures;
- provide an example of dialgebras, [17], by combining the general algebraic and specific coalgebraic aspect in the signature (arbitrary products in arguments, only power-set in the result);
- can represent categories of coalgebras for polynomial functors, as well as for power-set functor;
- can represent (nondeterministic) automata, Kripke-frames, topological spaces...

The fact that multialgebras have attracted only limited attention might be the result of the apparently poor algebraic structure and, on the other hand, a multiplicity of choices in defining most of the standard notions. We have shown that, as far as the notion of homomorphism is concerned, the number of choices is limited to 9, and that most of these do not appear very attractive. (Of course, we do not mean that they can not possibly find applications which depend on the specific context. Also, limiting the objects of the category, e.g., to only deterministic or partial algebras, may yield several alternatives which do not obtain in the general situation studied here.) The structural properties as well as most other choices are heavily conditioned by this notion. While the traditional weak homomorphisms yield, indeed, very poor structure, we have shown that, choosing outer-tight homomorphisms (which imply weakness and, in the case of standard deterministic algebras, the classical notion), multialgebras and their category obtain strong algebraic structure: the associated notion of congruence – bireachability – can be seen as a converse of the traditional notion of congruence (and of bisimilarity), requiring propagation of the congruence to the pre-images (e.g., subalgebras are closed under pre-images and not, as in the classical case, under images of the operations) or, equivalently, propagation of the distinctions to the images. The category $\mathsf{MAlg}_{OT}(\Sigma)$ of all Σ-multialgebras is cocomplete and has interesting final objects reflecting the maximal bireachability relation in the way analogous to final coalgebras reflecting maximal bimsimilarity. The category has products, and the given construction yields as a special case construction of products for coalgebras over power-set functor. To ensure the existence of final objects in general, we have to extend the category to $\mathsf{MAlg}^*_{OT}(\Sigma)$ by allowing algebras with carriers being proper classes. We have characterized its objects as set-reflecting algebras which condition is equivalent to every algebra being colimit of its small subalgebras. The category is complete and cocomplete.

We have not addressed the issue of logic and reasoning in the present paper. However, sound and strongly complete logics (i.e., for deriving not only tautologies but also consequences of axiomatic theories) for various variants of multialgebras have been designed [26, 20, 43, 44], the most recent one in [28]. Its primitives contain set-inclusion and deterministic equality which holds when both sides are not merely equal but equal one-element sets. (A different approach, based on membership relation, is developed and studied in [8, 9].) Its main specificity is the lack of substitutitivity property (as variables range only over individuals while terms denote arbitrary sets). This can be seen as a serious drawback (precluding the possibility of algebraization of the logic) or as a feature interesting in itself – representing not so unusual situations when, for some reason, variables range only over a subset of semantic objects (as is also the case, for instance, with partial algebras) or when allowed substitutions are restricted for other reasons (as in first-order logic where one has to avoid variable capture).

The natural next step will be to study the preservation properties of the OT-homomorphisms which may lead to adjustments in the primitive predicates of the logics used so far. Then one would like to investigate the possibilities of lifting (some of) the current results on the existence of (co)limits to the axiomatic classes.

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6 Appendix: classes

We use Grothendieck universes (after [33], 12.1) each satisfying the following axioms (for Zermelo universe):

ax1) $x \in \mathcal{U} \Rightarrow x \subseteq \mathcal{U}$ – $\mathcal{U}$ is transitive;

ax2) $x, y \in \mathcal{U} \Rightarrow \{x, y\}, \langle x, y \rangle \in \mathcal{U}$ – finite sets and pairs of members of $\mathcal{U}$ belong to $\mathcal{U}$;

ax3) $x \in \mathcal{U} \Rightarrow \mathcal{P}(x) \in \mathcal{U} \wedge \bigcup x \in \mathcal{U}$ – collection of all subcollections and unions of members of $\mathcal{U}$ belong to $\mathcal{U}$;

ax4) $\omega \in \mathcal{U}$ natural numbers/finite ordinals belong to $\mathcal{U}$;

ax5) $x \in \mathcal{U}, y \subset \mathcal{U}, f : x \twoheadrightarrow y \Rightarrow y \in \mathcal{U}$ – image of a member of $\mathcal{U}$ under surjection belongs to $\mathcal{U}$.

In addition, one postulates Grothendieck axiom:

ax6) every set/class belongs to some universe,

and obtains thus the hierarchy $\mathcal{U}_1 \in \mathcal{U}_2 \in \mathcal{U}_3 \in \ldots$ which, by transitivity, **ax1**), is cumulative (i.e., $\in$ can be repalced by $\subseteq$). $\mathcal{U}_{i+1}$ can be thought of as $\mathcal{P}(\mathcal{U}_i)$ where $\mathcal{P}(\_)$ forms not only subsets (not only $\mathcal{U}_i$-objects), but all subcollections (also subclasses, i.e., $\mathcal{U}_{i+1}$-objects) of the argument $\mathcal{U}_i$.

For instance, the following facts used at some places, are implied:

- $K \in \mathcal{U}_i, s_k \in \mathcal{U}_i \Rightarrow \bigcup_{k \in K} s_k \in \mathcal{U}_i$ – in particular, set-indexed union of sets is a set,
- $c \in \mathcal{U}_{i+1}, s \in \mathcal{U}_i \Rightarrow c \cap s \in \mathcal{U}_i$ – in particular, intersection with a set is a set,

which are among the axioms of NBG. But we are not working in NBG, for the reasons expressed after fact 3.12 – we need allow in multialgebras operations retuning proper classes. We thus have the following picture:

1) Usual algebras, $A = \langle\{s_1 ... s_n\rangle; f_k \subseteq s_{ik} \times ... \times s_{rk}; ...\rangle$ belong all to $\mathcal{U}_1$.
2) When the collections are proper classes, i.e, $s_i \in \mathcal{U}_2$, then:
   - $\langle s_1 ... s_n \rangle \in \mathcal{U}_2$ and $s_{ik} \times ... \times s_{rk} \in \mathcal{U}_2$ by **ax2**)
   - $f_k : s_{ik} \times ... \times s_{jk} \to \mathcal{P}(s_{rk})$, from definition 1.2 is thus generalised to an operation with the result $\mathcal{P}(s_{rk}) \in \mathcal{U}_2$ – and $f_k \subseteq s_{ik} \times ... \times s_{rk} \in \mathcal{U}_2$ by **ax3**) and **ax1**)

   and so class-algebras, with carriers being proper classes and operations returning proper classes, are also in $\mathcal{U}_2$.
3) Our constructions from section 3 apply thus to $\mathcal{U}_2$-objects; in particular, the diagrams (of limits, colimits) referred to by the word "all" are all $\mathcal{U}_2$ diagrams, but they work in the same way if we were to move higher up in the hierarchy.
4) This, in fact, we have to do. Consider an operation $s \to \mathcal{P}(s)$ and the isomorphism $s \simeq \mathcal{P}(s)$ required by finality. The proof from [2] obtains this bijection by letting s range over classes – objects of $\mathcal{U}_2$ – while $\mathcal{P}(\_)$ constructing only sub*sets*, i.e., objects of $\mathcal{U}_1$. Let us write (confusedly) $\mathcal{U}_i$ also for the cardinality of $\mathcal{U}_i$ (or the i-th (strongly) inaccessible cardinal, if one prefers), and denote by $\mathcal{U}_0$ – $\aleph_0$, by $\mathcal{U}_1$ – the (cardinality of the) class of all sets, by $\mathcal{U}_2$ – the collection of all classes, etc. Just like we have the bijection $\mathbf{N} \simeq \bigcup_{\lambda < \mathcal{U}_0} \mathcal{P}^\lambda(\mathbf{N})$, where $\mathcal{P}^\lambda(X)$ denotes the collection of subclasses of X of cardinality λ, so in [2] we obtain:

   $$s_2 \simeq \bigcup_{\lambda < \mathcal{U}_1} \mathcal{P}^\lambda(s_2) \quad \text{for some } s_2 \in \mathcal{U}_2 \text{ with } s_2 \geq \mathcal{U}_1 \tag{6.1}$$

   which is but another instance of the general fact (e.g., [12], 10.2, p.119), according to which for a (strongly) inaccessible cardinal v:

   $$v = \sum_{\lambda < v} v^\lambda \quad = |\bigcup_{\lambda < v} \mathcal{P}^\lambda(v)|.$$

   Accidentaly, it seems that the bijection (6.1) could be obtained working in NBG with the limitation of size, as choosing s_2 to be a class, i.e., V, the collection of its subsets (with or without subclasses) has the same cardinality being, too, a class.

In our case, we start with $s \subseteq \mathcal{U}_1$, i.e., $s \in \mathcal{U}_2$, and then also $\mathcal{P}(s) \in \mathcal{U}_2$, which might suggest that everything happens at the same level as in (6.1). However, $\mathcal{P}(s)$ forms now not only $\mathcal{U}_1$-objects but, as pointed out at the beginning of section 3 and after fact 3.12, also proper subclasses of s, i.e., $\mathcal{U}_2$-objects, and so, for every $s \in \mathcal{U}_2 : |s| < |\mathcal{P}(s)|$. The desired isomorphism is possible first at the next level, i.e., we must allow carriers at the level $\mathcal{U}_3$:

$$s_3 \simeq \bigcup_{\lambda < \mathcal{U}_2} \mathcal{P}^{\lambda}(s_3) \quad \text{for some } s_3 \in \mathcal{U}_3 \text{ with } s_3 \geq \mathcal{U}_2.$$

According to theorem 3.19, a final object (which satisfies this isomorphism, for a signature with an $f : s \to s$) is set-reflecting and hence is a colimit of small subalgebras – a colimit, as mentioned above in 3, possibly over a diagram of size $\mathcal{U}_2 \in \mathcal{U}_3$.

In addition to the above axioms, we have also used (in the proof of fact 3.12) the global axiom of choice $\exists C : \mathcal{U} \to \mathcal{U}\ \forall x : x \neq \emptyset \Rightarrow C(x) \in x$, or rather its equivalent:

ax7) Every equivalence relation on a class has a system of representatives.