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Spheres of Permutations under the Infinity Norm - Permutations with limited displacement

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Abstract

The following problem is considered: how many permutations p of $\{1, 2, \ldots, n\}$ satisfy $|p_i - i| \leq d$ for all i?

1 Introduction

A class of enumerative combinatorial problems can be formulated as follows. Let S_n denote the set of all permutations of $\{1, \ldots, n\}$. Suppose that for each integer $i \geq 1$ we have a set $S[i]$ of positive integers. How many of the permutations p in S_n satisfy $p_i \in S[i]$ for all i, $1 \leq i \leq n$? Problems of this form have been studied for more than a hundred years, and several methods have been developed to attack such problem. One method is the reformulation of the problem as a problem on permanents. In a paper in 1961, Mendelsohn [6] expressed the following opinion:

> "The method of expressing the number of permutations of a set of elements subject to certain types of restrictions as the value of a permanent is well known, but the literature is practically non-existent mainly because the evaluation of a permanent is a formidable problem."

He then went on to apply the permanent to obtain recursions which could be used to give numerical result and, in some cases, also closed formulas.

The problem considered in this paper is the enumeration of permutations which satisfy $|p_i - i| \leq d$ for all i. The motivation comes from coding theory. The problem can be rephrased as: what is the size of a sphere of radius d in the set of permutations of length n under the distance defined by the l_∞ norm.

More precisely, we consider the distance d_∞ between permutations defined by

$$d_\infty(p, q) = \max_j |p_j - q_j|.$$

Let ι denote the identity permutation in S_n. The object of study in this paper is $V(d, n)$, the number of permutations in S_n within distance d of the identity permutation, that is

$$V(d, n) = |T_{d,n}|,$$

where

$$T_{d,n} = \{p \in S_n \mid |p_i - i| \leq d \text{ for } 1 \leq i \leq n\}.$$

We note that since $l_\infty(\iota, q) = l_\infty(p, pq)$ for any permutation $p \in S_n$, the number of permutations in S_n within distance d of p will also be $V(d, n)$. In [3] this fact was used to obtain a Plotkin type bound for permutation arrays.

In general, no simple expression for $V(d, n)$ is known. As far as I can tell, this particular problem was first discussed by Lagrange [4] in 1962. He limited his study to $d \leq 3$. He found recursions that could be used to compute $V(d, n)$ numerically. The method was ad hoc, and already for $d = 3$, the recursion and the discussion leading up to them became quite complicated. The case $d = 3$ takes over five pages in his paper.

The problem was next considered by Lehmer [5] in 1970. For fixed d, $V(d, n)$ satisfies a linear recurrence in n. Lehmer stated this without giving a detailed proof. He used the expression for $V(d, n)$ as a permanent. His idea was to expand the permanent by the first row or column. Each of the resulting matrices are in turn expanded and "the process continues until no 'new' matrices occur". He carried the process through for $d \leq 3$ and gave generating functions (with no published details), but gave no proof that the process will always work.

A proof that $V(d, n)$ do satisfy a linear recurrence is given in Stanley's textbook [9] (Proposition 4.7.8 on page 246). Stanley considers a wider class of related problems. The permanent is not used in his proof.

Baltić studied the problem more recently. Only a very short abstract of his work has been published, in [1], p. 105. He has computed values of $V(4,n)$ for $n \le 24$, posted in [8] as sequence A072856.

The fact that $V(d,n)$ satisfies a linear recurrence implies that

$$\lim_{n\to\infty} V(d,n)^{1/n} = \mu_d,$$

where μ_d is the largest root of the polynomial corresponding to the shortest linear recurrence of $V(d,n)$. For $d=1$, $d=2$, and $d=3$, these recurrences were determined explicitly by Lehmer [5] and he also determined μ_d (approximately).

2 Notation and basic results

The permanent of an $n \times n$ matrix A is defined by

$$\operatorname{per} A = \sum_{p \in S_n} a_{1,p_1} \cdots a_{n,p_n}. \tag{1}$$

In particular, if A is a $(0,1)$-matrix, then

$$\operatorname{per} A = |\{p \in S_n \mid a_{i,p_i} = 1 \text{ for all } i, 1 \le i \le n\}|.$$

Let $A^{(d,n)}$ be the $n \times n$ matrix with $a_{i,j}^{(d,n)} = 1$ if $|i-j| \le d$ and $a_{i,j}^{(d,n)} = 0$ otherwise. The old idea is the following relation. For completeness, we include a proof.

Lemma 1. $V(d,n) = \operatorname{per} A^{(d,n)}$.

Proof.

$$V(d,n) = |\{p \in S_n \mid |p_i - i| \le d \text{ for all } i\}| = |\{p \in S_n \mid a_{i,p_i}^{(d,n)} = 1 \text{ for all } i\}| = \operatorname{per} A^{(d,n)}.$$

□

For an $n \times n$ $(0,1)$-matrix it is known (see Theorem 11.5 in [13]) that

$$\operatorname{per} A \le \prod_{i=1}^{n} (r_i!)^{1/r_i},$$

where r_i is the number of ones in row i.

For $A^{(d,n)}$ we clearly have $r_i \le 2d+1$ for all i. Hence we have the following upper bound given in [3].

$$V(d,n) \le [(2d+1)!]^{n/(2d+1)} \text{ for all } n \tag{2}$$

and so

$$\mu_d \le [(2d+1)!]^{1/(2d+1)}. \tag{3}$$

Since we have two rows of weight i for $d+1 \le i \le 2d$ and the remaining rows of weight $2d+1$, we actually get the stronger (but more complicated) upper bound

$$V(d,n) \le \frac{\prod_{i=d+1}^{2d} (i!)^{2/i}}{[(2d+1)!]^{2d/(2d+1)}} [(2d+1)!]^{n/(2d+1)}. \tag{4}$$

We note that $[(2d+1)!]^{1/(2d+1)} \sim \frac{2d+1}{e}$ for $d \to \infty$. Lehmer [5] also stated that the "van der Waerden conjecture" (now a theorem) can be used to prove that

$$\frac{\mu_d}{2d+1} \ge \frac{1}{e} \approx 0.36788. \tag{5}$$

He did not give any details. We will discuss this lower bound in some detail at the end of the paper. In particular, we give a proof.

Also, combined with the upper bound, this shows that

$$\frac{\mu_d}{2d+1} \to \frac{1}{e} \text{ when } d \to \infty.$$

In Table 1 we give the values of μ_d and the upper bound in (2) for $d \leq 8$. For $d = 1, 2, 3$, the values of μ_d are due to Lehmer [5]. The values for $d \geq 4$ are new.

Table 1: μ_d and its upper bound.

d	μ_d	$[(2d+1)!]^{1/(2d+1)}$	$\mu_d/(2d+1)$	$[(2d+1)!]^{1/(2d+1)}/(2d+1)$
1	1.61803	1.81712	0.53934	0.60571
2	2.33355	2.60517	0.46671	0.52103
3	3.06177	3.38002	0.43739	0.48286
4	3.79352	4.14717	0.42150	0.46080
5	4.52677	4.90924	0.41152	0.44629
6	5.26082	5.66769	0.40468	0.43598
7	5.99534	6.42342	0.39969	0.42823
8	6.73016	7.17704	0.39589	0.42218

The purpose of this paper is to study the linear recurrences and generating functions for $V(d, n)$ in more detail. In particular we study constructive upper bounds on the length of such recurrences.

There is no simple way to evaluate permanents similar to Gaussian elimination for determinants. However, if A is an $n \times n$ matrix, then per$A$ can be computed by expanding by some row (or column) and repeat the process recursively. For example, expanding a (0,1) matrix by the first row,

$$\text{per}A = \sum_{j|a_{1,j}=1} \text{per}A_j \tag{6}$$

where A_j denotes the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and column j in A.

To find a recursion of $V(d, n)$ for a given d, a first task is to determine a class X_d of matrices such that for each n, the number of $n \times n$ matrices in X_d is a constant which depends only on d, and such that for any $n \times n$ matrix in X_d, the $(n-1) \times (n-1)$ matrices A_j, where $a_{1,j} = 1$ also belong to X_d. Further we require that the matrices $A^{(d,n)}$ belong to X_d for all n. From this we obtain a recursion for the permanents. In the next section, we determine one such class of matrices, in the following section another class that is more complicated to define but which gives a shorter recursion.

3 First class of matrices

The first class of matrices is relatively simple to describe. One reason to include the first class is that the main underlying ideas are simpler to describe for this class. But it will also be used to prove some results that do not follow from the second construction.

First we introduce some notations that will be used for both classes. Throughout, we will assume that d is fixed. Therefore, d will not be included the notations.

For $1 \leq a \leq d+1$ and $1 \leq b \leq d$, let

$$X_{a,b} = \{(x_1, x_2, \ldots, x_d) \mid a = x_1 \geq x_2 \geq \cdots \geq x_b > 0 \text{ and } x_i = 0 \text{ for } i > b\}.$$

For $0 \leq a \leq d$ and $0 \leq b \leq d$, let

$$Y_{a,b} = \{(x_1, x_2, \ldots, x_d) \mid a = x_1 \geq x_2 \geq \cdots \geq x_b \geq 0 \text{ and } x_i = 0 \text{ for } i > b\}.$$

Let

$$Y = \bigcup_{a=0}^{d} Y_{a,d} = \{(x_1, x_2, \ldots, x_d) \mid d \geq x_1 \geq x_2 \geq \cdots \geq x_d \geq 0\}.$$

Lemma 2. *We have*
i) $|X_{a,b}| = \binom{a+b-2}{b-1}$,
ii) $|Y_{a,b}| = \binom{a+b-1}{b-1} = \binom{a+b-1}{a}$,
iii) $|Y| = \binom{2d}{d}$.

Proof. First we note that

$$|X_{a,b}| = |Y_{a-1,b}| \tag{7}$$

by the 1-1 mapping

$$(x_1, x_2, \ldots, x_b, 0, \ldots, 0) \leftrightarrow (x_1 - 1, x_2 - 1, \ldots, x_b - 1, 0, \ldots, 0).$$

Hence, for $a \geq 1$ and $b \geq 1$ we have, by definition of $Y_{a,b}$,

$$|Y_{a,b}| = \sum_{c=1}^{b} |X_{a,c}| = \sum_{c=1}^{b} |Y_{a-1,c}|.$$

Since $|Y_{0,b}| = 1$, ii) follows by induction:

$$|Y_{a,b}| = \sum_{c=1}^{b} \binom{a+c-2}{a-1} = \binom{a+b-1}{a}.$$

Next, i) follows from (7). Finally

$$|Y| = \sum_{a=0}^{d} |Y_{a,d}| = \sum_{a=0}^{d} \binom{a+d-1}{d-1} = \binom{2d}{d}.$$

□

The recursion

For a sequence $\mathbf{x}$, let $A_{\mathbf{x},n}$ be the $n \times n$ matrix $(a_{i,j})$ be defined by

$$\begin{aligned} a_{i,j} &= 0 \text{ for } j > i + d \text{ or } i > j + d, \\ a_{i,j} &= 0 \text{ for } 1 \leq j \leq d \text{ and } j + d - x_j < i \leq j + d \\ a_{i,j} &= 1 \text{ otherwise,} \end{aligned}$$

and let

$$\alpha_{\mathbf{x}}(n) = \text{per } A_{\mathbf{x},n}.$$

We note that if $x_1 = d+1$, then the first column of $A_{\mathbf{x},n}$ is all zero and so $\alpha_{\mathbf{x}}(n) = 0$.

In this notation, $A_{\mathbf{0},n} = A^{(d,n)}$ and $\alpha_{\mathbf{0}}(n) = V(d,n)$. The first class of matrices is

$$\mathcal{M}_1 = \{A_{\mathbf{x},n} \mid \mathbf{x} \in Y, n \geq 1\}.$$

Suppose that $n \geq d+1$. In the first row of $A_{\mathbf{x},n}$, exactly the first $d+1$ elements are 1. We see that for $1 \leq j \leq d+1$, we get $(A_{\mathbf{x},n})_j = A_{\mathbf{x}[j],n-1}$ where

$$\mathbf{x}[j] = (x_1+1, x_2+1, \ldots, x_{j-1}+1, x_{j+1}, x_{j+2}, \ldots, x_d, 0).$$

In particular, $\mathbf{x}[1] = (x_2, x_3, \ldots, x_d, 0)$ and $\mathbf{x}[d+1] = (x_1+1, x_2+1, \ldots, x_d+1)$.

By (6),

$$\alpha_{\mathbf{x}}(n) = \sum_{j=1}^{d+1} \alpha_{\mathbf{x}[j]}(n-1). \tag{8}$$

For $n \leq d$, we similarly get

$$\alpha_{\mathbf{x}}(n) = \sum_{j=1}^{n} \alpha_{\mathbf{x}[j]}(n-1) = \sum_{j=1}^{d+1} \alpha_{\mathbf{x}[j]}(n-1)$$

when we put $\alpha_{\mathbf{y}}(n) = 0$ whenever $\mathbf{y} \in X_{y_1,b}$ for some $b > n$.

For $n = 0$ we get $\alpha_{\mathbf{0}}(0) = 1$ and $\alpha_{\mathbf{x}}(0) = 0$ for $\mathbf{x} \neq \mathbf{0}$. Combining this with (8), we get a recursion which can be used to compute all $\alpha_{\mathbf{x}}(n)$, and in particular $\alpha_{\mathbf{0}}(n)$, that is $V(d,n)$. The values of $V(d,n)$ for $n \leq 30$ and $d \leq 10$, computed in this way, are given in an appendix.

Generating functions

We can also determine a generating function for $V(d,n)$ from this recursion. For convenience we will use the notation $\delta = \binom{2d}{d}$.

For numerical computations is convenient to note that when we order the elements of Y lexicographically and let the position of $\mathbf{0}$ be 0, then the position of $\mathbf{x}$ is

$$\sum_{j=1}^{d} \binom{d-j+x_j}{d+1-j}.$$

Let T be the $\delta \times \delta$ matrix where the rows and columns are indexed by Y (ordered lexicographically), and where

$$\begin{array}{ll} t_{\mathbf{x}[j],\mathbf{x}} = 1 \text{ for } j = 1, 2, \ldots, d+1 & \text{and } t_{\mathbf{y},\mathbf{x}} = 0 \text{ otherwise if } x_1 < d, \\ t_{\mathbf{x}[1],\mathbf{x}} = 1 & \text{and } t_{\mathbf{y},\mathbf{x}} = 0 \text{ otherwise if } x_1 = d, \end{array}$$

and let similarly $\mathbf{v}(n)$ be the vector of length δ with the elements $\alpha_{\mathbf{x}}(n)$ indexed in the same way. In particular, $\mathbf{v}(0) = (1, 0, 0, \ldots, 0)$. Then, for $n \geq 1$,

$$\mathbf{v}(n) = \mathbf{v}(n-1)T, \tag{9}$$

and so

$$\mathbf{v}(n) = \mathbf{v}(0)T^n. \tag{10}$$

Let

$$g(z) = \det(T - zI) = \sum_{i=0}^{\delta} c_i z^i$$

be the characteristic polynomial of T. Then $g(T) = 0$ and so $\sum_{i=0}^{\delta} c_i T^{i+k} = 0$ for all $k \geq 0$. Hence

$$\sum_{i=0}^{\delta} c_i \alpha_{\mathbf{x}}(i+k) = 0 \tag{11}$$

for all $\mathbf{x} \in Y$ and all $k \geq 0$.

Let

$$H(z)=\sum_{j=0}^{\delta} c_{\delta-j}z^j,$$

and let

$$F_{\mathbf{x}}(z)=\sum_{i=0}^{\infty} \alpha_{\mathbf{x}}(i)z^i$$

be the generating function of $\{\alpha_{\mathbf{x}}(i)\}$. From the theory of linear recurrences, we know that this is a rational function in z (see e.g. [9], Chapter 4). More precisely,

$$\begin{aligned} F_{\mathbf{x}}(z)H(z) &= \sum_{n=0}^{\infty} z^n \sum_{i+j=n} \alpha_{\mathbf{x}}(i)c_{\delta-j} \\ &= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\min\{n,\delta\}} \alpha_{\mathbf{x}}(n-j)c_{\delta-j} = \sum_{n=0}^{\infty} b_n z^n. \end{aligned} \tag{12}$$

In particular, if $n \geq \delta$, then

$$b_n = \sum_{j=0}^{\delta} \alpha_{\mathbf{x}}(n-j)c_{\delta-j} = \sum_{l=0}^{\delta} \alpha_{\mathbf{x}}(n-\delta+l)c_l = 0$$

by (11). Hence

$$F_{\mathbf{x}}(z)=\frac{\sum_{j=0}^{\delta-1} b_j z^j}{H(z)}=\frac{h_{\mathbf{x}}(z)}{H(z)}. \tag{13}$$

In particular,

$$F_{\mathbf{0}}(z)=\frac{h_{\mathbf{0}}(z)}{H(z)}=\frac{f_d(z)}{g_d(z)}$$

where $\gcd\{f_d(z), g_d(z)\}=1$. Hence, we get the following theorem.

Theorem 1. *For $d \geq 1$, $V(d,n)$ has a generating function*

$$\sum_{i=0}^{\infty} V(d,n)\, z^n = \frac{f_d(z)}{g_d(z)} \tag{14}$$

where $f_d(z)$ and $g_d(z)$ are polynomials, $\gcd\{f_d(z), g_d(z)\}=1$*, and*

$$\deg g_d(z) \leq \binom{2d}{d}.$$

In the next section we derive another recursion that proves the stronger upper bound $\deg g_d(z) \leq 2^{d-1}+\binom{2d}{d}/2$.

We have computed $F_{\mathbf{x}}(z)$ by the method above for all $\mathbf{x}$ when $d \leq 5$. Using the recursion developed in the next section, we have also computed $F_{\mathbf{0}}(z)$ for $d=6$. In the appendix, we give the $f_d(z)$ and $g_d(z)$ for $d \leq 6$. In Example 1, we give the results for $d=3$.

Table 2: $h_{\mathbf{x}}$ for $d = 3$.

$\mathbf{x}$	$h_{\mathbf{x}}(z)/(1+z+z^2-z^3+z^4+z^5-z^6)$
000	$1-z-2\,z^2-2\,z^4+z^7+z^8$
100	$z+2\,z^4+2\,z^5-2\,z^7-z^8$
110	$2\,z^2+2\,z^3+2\,z^4-2\,z^5-2\,z^6$
111	$6\,z^3+6\,z^4-2\,z^5-13\,z^6-8\,z^7-2\,z^8+4\,z^9+2\,z^{10}+2\,z^{12}+z^{13}$
200	$(z+z^2-z^3-z^4-z^5+z^6+z^7)(1-z)$
210	$2\,z^2$
211	$4\,z^3+4\,z^4-4\,z^6-2\,z^7$
220	$2\,z^2-4\,z^4-2\,z^5-2\,z^6+2\,z^7+2\,z^8$
221	$4\,z^3-2\,z^5$
222	$4\,z^3-9\,z^5-4\,z^6-4\,z^7+2\,z^8+2\,z^9+2\,z^{11}+z^{12}$
300	$z-z^2-2\,z^3-2\,z^5+z^8+z^9$
310	$z^2+2\,z^5+2\,z^6-2\,z^8-z^9$
311	$2\,z^3+2\,z^4+2\,z^5-2\,z^6-2\,z^7$
320	$(z^2+z^3-z^4-z^5-z^6+z^7+z^8)(1-z)$
321	$2\,z^3$
322	$2\,z^3-4\,z^5-2\,z^6-2\,z^7+2\,z^8+2\,z^9$
330	$z^2-z^3-2\,z^4-2\,z^6+z^9+z^{10}$
331	$z^3+2\,z^6+2\,z^7-2\,z^9-z^{10}$
332	$(z^3+z^4-z^5-z^6-z^7+z^8+z^9)(1-z)$
333	$z^3-z^4-2\,z^5-2\,z^7+z^{10}+z^{11}$

Example 1. *For $d = 3$, $\delta = 20$. Calculation shows that the characteristic polynomial is*

$$\begin{aligned} g(z) &= 1+z-3z^2+z^3+3z^4-11z^5-13z^6-4z^7+16z^8 \\ &\quad +12z^9-4z^{10}+8z^{11}+28z^{12}+20z^{13}-21z^{14} \\ &\quad -17z^{15}-9z^{16}-5z^{17}-3z^{18}-z^{19}+z^{20} \\ &= (1-z)(-1+z+z^2-z^3+z^4+z^5+z^6) \\ &\quad \cdot(-1-3z-3z^2-5z^3-9z^4-7z^5+3z^6+19z^7+21z^8 \\ &\qquad +13z^9+3z^{10}+3z^{11}+z^{12}-z^{13}). \end{aligned}$$

We get $H(z) = z^{20}g(1/z) = (1+z+z^2-z^3+z^4+z^5-z^6)g_3(z)$ (where $g_3(z)$ is divisible by $(1-z)$). It turns out that for all $\mathbf{x}$, $h_{\mathbf{x}}(z)$ is divisible by $1+z+z^2-z^3+z^4+z^5-z^6$. We list $h_{\mathbf{x}}(z)/(1+z+z^2-z^3+z^4+z^5-z^6)$ in Table 2. In some cases, $h_{\mathbf{x}}(z)$ is also divisible by $1-z$ and we have marked this by giving the factor $1-z$ explicitly.

By the standard theory of linear recurrences (see e.g. [9] Theorem 4.1.1), $V(d,n) \sim p(n)\zeta^n$, where ζ is the largest positive zero of $g_{\mathbf{x}}(1/z) = 0$ and $p(n)$ is some polynomial of degree one less than the order of ζ (as a zero). This implies that $\mu_d = \zeta$.

The relation (8) implies a similar relation for the generating functions. If $x_1 < d$, we get

$$
\begin{aligned}
F_{\mathbf{x}}(z) = \sum_{n=0}^{\infty} \alpha_{\mathbf{x}}(n) z^n &= \alpha_{\mathbf{x}}(0) + \sum_{n=1}^{\infty} z^n \sum_{j=1}^{d+1} \alpha_{\mathbf{x}[j]}(n-1) \\
&= \alpha_{\mathbf{x}}(0) + \sum_{j=1}^{d+1} \sum_{n=0}^{\infty} \alpha_{\mathbf{x}[j]}(n) z^{n+1} \\
&= \alpha_{\mathbf{x}}(0) + z \sum_{j=1}^{d+1} F_{\mathbf{x}[j]}(z). \qquad (15)
\end{aligned}
$$

If $x_1 = d$, then $\alpha_{\mathbf{x}[j]}(n-1) = 0$ for $j > 1$ and $\alpha_{\mathbf{x}}(0) = 0$ and so

$$
F_{\mathbf{x}}(z) = z F_{\mathbf{x}[1]}(z) = z F_{(x_2, x_3, \ldots, x_d, 0)}(z). \qquad (16)
$$

Note that $x_1 = d$ for exactly half of the vectors in Y. More general, if $x_i = d$ for $1 \leq i < l$ and $x_l < d$, then we can repeat the argument and get

$$
F_{\mathbf{x}}(z) = z^{l-1} F_{(x_l, x_{l+1}, \ldots, x_d, 0, 0, \ldots, 0)}(z). \qquad (17)
$$

Substituting this back into the set of equations (15), we get a set of $\delta/2$ equations for the $F_{\mathbf{x}}(z)$ with $x_1 < d$. Explicitly, if $x_1 < d-1$, the equation (15) is kept. But if $x_i = d-1$ for $i < l$ and $x_l < d-1$, then $\alpha_{\mathbf{x}}(0) = 0$ and (17) gives

$$
\begin{aligned}
F_{\mathbf{x}}(z) &= z F_{(x_2, x_3, \ldots, x_d, 0)}(z) + \sum_{j=2}^{l} z F_{(x_1+1, \ldots, x_{j-1}+1, x_{j+1}, \ldots, x_d, 0)}(z) \\
&\quad + \sum_{j=l+1}^{d+1} z F_{(x_1, \ldots, x_{l-1}+1, x_l+1, \ldots, x_{j-1}+1, x_{j+1}, \ldots, x_d, 0)}(z) \\
&= z F_{(x_2, x_3, \ldots, x_d, 0)}(z) + \sum_{j=2}^{l} z^j F_{(x_{j+1}, \ldots, x_d, 0, \ldots, 0)}(z) \\
&\quad + \sum_{j=l+1}^{d+1} z^l F_{(x_l+1, \ldots, x_{j-1}+1, x_{j+1}, \ldots, x_d, 0, \ldots, 0)}(z). \qquad (18)
\end{aligned}
$$

Example 2. *For $d = 3$ we get the following 10 equations.*

$$
\begin{aligned}
&F_{(000)}(z) = 1 + z\,F_{(000)}(z) + z\,F_{(100)}(z) + z\,F_{(110)}(z) + z\,F_{(111)}(z), \\
&F_{(100)}(z) = z\,F_{(000)}(z) + z\,F_{(200)}(z) + z\,F_{(210)}(z) + z\,F_{(211)}(z), \\
&F_{(110)}(z) = z\,F_{(100)}(z) + z\,F_{(200)}(z) + z\,F_{(220)}(z) + z\,F_{(221)}(z), \\
&F_{(111)}(z) = z\,F_{(110)}(z) + z\,F_{(210)}(z) + z\,F_{(220)}(z) + z\,F_{(222)}(z), \\
&F_{(200)}(z) = (z + z^2)\,F_{(000)}(z) + z^2\,F_{(100)}(z) + z^2\,F_{(110)}(z), \\
&F_{(210)}(z) = z^2\,F_{(000)}(z) + z\,F_{(100)}(z) + z^2\,F_{(200)}(z) + z^2\,F_{(210)}(z), \\
&F_{(211)}(z) = z^2\,F_{(100)}(z) + z\,F_{(110)}(z) + z^2\,F_{(200)}(z) + z^2\,F_{(220)}(z), \\
&F_{(220)}(z) = (z^2 + z^3)\,F_{(000)}(z) + z^3\,F_{(100)}(z) + z\,F_{(200)}(z), \\
&F_{(221)}(z) = z^3\,F_{(000)}(z) + z^2\,F_{(100)}(z) + z^3\,F_{(200)}(z) + z\,F_{(210)}(z), \\
&F_{(222)}(z) = (z^3 + z^4)\,F_{(000)}(z) + z^2\,F_{(200)}(z) + z\,F_{(220)}(z)
\end{aligned}
$$

The set of equations can be reduced by further simple substitutions. For example, we see that for $\mathbf{x} \in X_{d-1,d}$, all the terms $F_{\mathbf{y}}(z)$ in the right hand expressions in (18) have $y_d = 0$. Hence we can substitute these equations back into the remaining equations and thus reduce the number of equations by $|X_{d-1,d}| = \binom{2d-3}{d-1}$. Since

$$
\frac{\binom{2d-3}{d-1}}{\binom{2d}{d}/2} = \frac{(d-1)^2}{2d(2d-1)} \approx \frac{1}{4},
$$

this reduces the number of equation by almost 25%.

This can be carried further. For $\mathbf{x} \in X_{d-1,b}$ (where $1 \le b \le d$) and $\mathbf{y} \in Y$, let

$$\mathbf{y} \prec \mathbf{x} \text{ if } y_1 < d-1 \text{ or } \mathbf{y} \in X_{d-1,c} \text{ where } c < b.$$

We can make simple substitutions for all $\mathbf{x} \in Y_{d-1,d}$ such that

$$\mathbf{x}[j]^* \prec \mathbf{x} \text{ for all } j = 1, 2, \ldots, d. \tag{19}$$

Here $\mathbf{x}[j]^*$ denotes the vector we obtain from $\mathbf{x}[j]$ when removing any initial elements with value d (this is what we get using (17); for example, if $d = 4$ and $\mathbf{x} = (33221)$, then $\mathbf{x}[2] = (42210)$ and $\mathbf{x}[2]^* = (22100)$. For example, we see that for $d = 3$, (19) is satisfied for 5 of the 6 vectors $\mathbf{x} \in Y_{2,3}$, the exception is (210). For the actual computation we first substitute for the $\mathbf{x} \in X_{d-1,d}$ (this is the case we considered above), next all $\mathbf{x} \in X_{d-1,d-1}$ that satisfy (19), then those in $X_{d-1,d-2}$, etc.

For example, for $d = 3$ this leave us with the following set of 5 equations.

$$\begin{aligned}
F_{(000)}(z) = {} & 1 + z\,F_{(000)}(z) + z\,F_{(100)}(z) + z\,F_{(110)}(z) + z\,F_{(111)}(z),\\
F_{(100)}(z) = {} & (z + z^2 + z^3 + z^4 + 3z^5 + 2z^6)F_{(000)}(z) + (2z^3 + z^5 + 2z^6)F_{(100)}(z)\\
& +(z^2 + z^3 + z^5 + z^6)F_{(110)}(z) + z\,F_{(210)}(z),\\
F_{(110)}(z) = {} & (z^2 + 3z^3 + 3z^4 + z^5 + z^6)F_{(000)}(z) + (z + 2z^4 + z^6)\,F_{(100)}(z)\\
& +(z^3 + z^4 + z^6)\,F_{(110)}(z) + z^2\,F_{(210)}(z),\\
F_{(111)}(z) = {} & (2z^3 + 6z^4 + z^5)F_{(000)}(z) + (2z^4 + 3z^5)F_{(100)}(z)\\
& +(z + z^4 + 2z^5)F_{(110)}(z) + z\,F_{(210)}(z),\\
F_{(210)}(z) = {} & (z^2 + z^3 + z^4)F_{(000)}(z) + (z + z^4)F_{(100)}(z)\\
& +z^4\,F_{(110)}(z) + z^2\,F_{(210)}(z),
\end{aligned}$$

In general, we can find a formula for the number of $\mathbf{x} \in Y_{d-1,d}$ that does not satisfy (19), that is, the number of equations that will remain. Assume that $\mathbf{x}$ starts with r elements that equal $d-1$ and ends in s zeros (that is $\mathbf{x} \in X_{d-1,d-s}$). For $1 \le j \le r$, we see that

$$\mathbf{x}[j] = (d, \ldots, d, d-1, \ldots, d-1, x_{r+1}, \ldots, x_{d-s}, 0, \ldots, 0)$$

with $j-1$ initial d's and so

$$\mathbf{x}[j]^* = (d-1, \ldots, d-1, x_{r+1}, \ldots, x_{d-s}, 0, \ldots, 0)$$

ends in $d - s - j$ zeros. Hence (19) is satisfied for those j. It is easy to see that if (19) is not satisfied for some j where $r < j \le d$, then it is not satisfied for $j = d$. Since $\mathbf{x}[d]^* \in X_{x_{r+1}+1,d-r}$, (19) is not satisfied exactly when $x_{r+1} = d-2$ and $r \le s$. For such $\mathbf{x}$, $(x_{r+1}, x_{r+2}, \ldots, x_{d-s}) \in X_{d-2,d-r-s}$. Hence $(x_{r+1}, x_{r+2}, \ldots, x_{d-s}, 0, \ldots, 0) \in Y_{d-2,d-2r}$ and also any element in $Y_{d-2,d-2r}$ gives rise to an $\mathbf{x} \in Y_{d-1,d}$ that do not satisfy (19). The total number of such $\mathbf{x} \in Y_{d-1,d}$ is therefore

$$\nu_d = \sum_{r=1}^{\lfloor (d-1)/2 \rfloor} |Y_{d-2,d-2r}| = \sum_{r=1}^{\lfloor (d-1)/2 \rfloor} \binom{2d-2r-3}{d-2}.$$

There seems not to be a simple closed expression for ν_d. The sequence $\{\nu_d\}$ is sequence A014301 in [8]. It is not too hard to show that $\nu_d/|Y_{d-1,d}| \to 1/6$ when $d \to \infty$. Hence this simple method of repeated substitutions reduces the remaining set of sequences with approximately 41.6% (compared with the 25% for the first step given above). In all, we are left with an equation set of size 29.2% of the original set of $|Y|$ equations.

Another way to reduce the number of equations is to start with the equations (18) for $F_{\mathbf{x}}(z)$ for x_1 small. To elaborate on this, we introduce an alternative notation. For $\mathbf{x} \in Y_{a,d}$, let

$$\rho_j = |\{i \mid x_i = j\}|$$

and denote $F_{\mathbf{x}}(z)$ by $\Phi_{\rho_a,\rho_{a-1},\ldots,\rho_1}$ (we omit the z for convenience). For example $F_{(3311100)}(z) = \Phi_{2,0,3}$. For $F_{\mathbf{0}}(z)$ we just use the notation Φ, and sometimes, by abuse of notation, Φ_0. The first equation in the example above for $d = 3$ then reads

$$\Phi = 1 + z\,\Phi + z\,\Phi_1 + z\,\Phi_2 + z\,\Phi_3.$$

In general, we see that

$$\Phi = 1 + z\,\Phi + z\sum_{r=1}^{d}\Phi_r. \tag{20}$$

For $\mathbf{x} \in Y_{1,d}$ we get the equations

$$\Phi_j = z\,\Phi_{j-1} + z\sum_{r=1}^{j-1}\Phi_{r,j-1-r} + z\sum_{s=0}^{d-j}\Phi_{j,s}, \tag{21}$$

for $1 \le j \le d$. We can solve these equations for the Φ_j in terms of the $\Phi_{r,s}$ and be left with d fewer equations. Then we can repeat this process for the set of equations (18) where $x_1 = 2$, etc.

We will show that the solutions for the equations with $x_1 = 1$ (that is, equation (21) can be given a relatively compact explicit form. We do this in two lemmas. To simplify the expressions, we introduce two more notation:

$$P_r = 1 - z - z^2 - \ldots - z^r = \frac{1 - z^{r+1}}{1 + z}.$$

$$Q_r = 1 + z + \cdots + z^r = \frac{1 - z^{r+1}}{1 - z}.$$

We note that for $b > a$ we get $P_a - P_b = z^{a+1}Q_{b-a-1}$.

Clearly, $P_r + z^r = P_{r-1}$ for $r \ge 1$ and $1 - P_r = zQ_{r-1}$. More general, if $a < b$, then $P_a - P_b = z^{a+1}Q_{b-a-1}$.

Lemma 3. *For $0 \le j \le d$ we have*

$$P_{j+1}\Phi_j = z^j + z^{j+1}\sum_{r=j+1}^{d}\Phi_r + \sum_{r=1}^{j}\sum_{s=0}^{d-r} z^{j+1-r}P_r\Phi_{r,s} + \sum_{r=1}^{j-1}\sum_{s=0}^{j-1-r} z^{j-r-s}P_{r+s+1}\Phi_{r,s}. \tag{22}$$

We prove this by induction on j. For $j = 0$, (22) follows immediately from (20).

Let $j \ge 1$ and suppose that (22) is true for $j - 1$. From (21) and (22) we get

$$\begin{aligned} P_j\Phi_j &= z\,P_j\Phi_{j-1} + z\sum_{r=1}^{j-1}P_j\Phi_{r,j-1-r} + z\sum_{s=0}^{d-j}P_j\Phi_{j,s} \\ &= z\Big\{z^{j-1} + z^j\Phi_j + z^j\sum_{r=j+1}^{d}\Phi_r + \sum_{r=1}^{j-1}\sum_{s=0}^{d-r} z^{j-r}P_r\Phi_{r,s} \\ &\quad + \sum_{r=1}^{j-2}\sum_{s=0}^{j-2-r} z^{j-1-r-s}P_{r+s+1}\Phi_{r,s}\Big\} + z\sum_{r=1}^{j-1}P_j\Phi_{r,j-1-r} + z\sum_{s=0}^{d-j}P_j\Phi_{j,s}. \end{aligned}$$

Hence, noting that $P_j - z^{j+1} = P_{j+1}$, we get

$$\begin{aligned} P_{j+1}\Phi_j &= z^j + z^{j+1}\sum_{r=j+1}^{d}\Phi_r \\ &\quad + \sum_{r=1}^{j-1}\sum_{s=0}^{d-r} z^{j+1-r}P_r\Phi_{r,s} + \sum_{s=0}^{d-j} z\,P_j\Phi_{j,s} \\ &\quad + \sum_{r=1}^{j-2}\sum_{s=0}^{j-2-r} z^{j-r-s}P_{r+s+1}\Phi_{r,s} + \sum_{r=1}^{j-1} z\,P_j\Phi_{r,j-1-r} \end{aligned}$$

which shows that (22) is true for j. QED

Starting from $j = d$ and working downwards, we can now eliminate the Φ_r terms from the right hand side of (22). It is easiest to do this in two steps.

Lemma 4. *For $0 \le j \le d$ we have*

$$\Phi_j = z^{j-d}\Phi_d - \sum_{r=1}^{d-1}\sum_{s=\max\{0,j-r\}}^{d-r-1} z^{j-r-s}\Phi_{r,s} - \sum_{r=j+1}^{d}\sum_{s=0}^{d-r} z^{j+1-r}\Phi_{r,s}. \qquad (23)$$

Proof. Again the proof is by induction, starting with $j = d$. We see that (23) is trivially true for $j = d$ since both double sums are zero.

Let $0 \le j < d$ and suppose that (23) is true for $j + 1$. From (21) and the induction hypothesis we get

$$\begin{aligned} \Phi_j &= z^{-1}\Phi_{j+1} - \sum_{r=1}^{j}\Phi_{r,j-r} - \sum_{s=0}^{d-j-1}\Phi_{j+1,s} \\ &= z^{j-d}\Phi_d - \sum_{r=1}^{d-1}\sum_{s=\max\{0,j+1-r\}}^{d-r-1} z^{j-r-s}\,\Phi_{r,s} - \sum_{r=j+2}^{d}\sum_{s=0}^{d-r} z^{j+1-r}\,\Phi_{r,s} \\ &\quad - \sum_{r=1}^{j}\Phi_{r,j-r} - \sum_{s=0}^{d-j-1}\Phi_{j+1,s}. \end{aligned}$$

Since

$$\sum_{r=1}^{j}\Phi_{r,j-r} + \sum_{r=1}^{d-1}\sum_{s=\max\{0,j+1-r\}}^{d-r-1} z^{j-r-s}\,\Phi_{r,s} = \sum_{r=1}^{d-1}\sum_{s=\max\{0,j-r\}}^{d-r-1} z^{j-r-s}\,\Phi_{r,s}$$

and

$$\sum_{s=0}^{d-j-1}\Phi_{j+1,s} + \sum_{r=j+2}^{d}\sum_{s=0}^{d-r} z^{j+1-r}\,\Phi_{r,s} = \sum_{r=j+1}^{d}\sum_{s=0}^{d-r} z^{j+1-r}\,\Phi_{r,s},$$

(23) is true also for j. □

From Lemma 3 we get in particular that

$$P_{d+1}\Phi_d = z^d + + \sum_{r=1}^{d}\sum_{s=0}^{d-r} z^{d+1-r}\,P_r\Phi_{r,s} + \sum_{r=1}^{d-1}\sum_{s=0}^{d-1-r} z^{d-r-s}\,P_{r+s+1}\Phi_{r,s}. \qquad (24)$$

We can now combine this and Lemma 4 to get the following theorem. We omit the easy proof.

Theorem 2. *For $0 \le j \le d$ we have*

$$P_{d+1}\Phi_j = z^j + \sum_{r=1}^{j}\sum_{s=0}^{d-r} z^{j+1-r}\, P_r \Phi_{r,s} + \sum_{r=j+1}^{d}\sum_{s=0}^{d-r} z^2\, Q_{d-r}\Phi_{r,s} + \sum_{r=1}^{d-1}\sum_{s=0}^{j-r-1} z^{j-r-s}\, P_{r+s+1}\Phi_{r,s} + \sum_{r=1}^{d-1}\sum_{s=\max\{0,j-r\}}^{d-r-1} z^{j+2}\, Q_{d+1-r-s}\Phi_{r,s}.$$

What we have done is essentially Gaussian elimination. We can go on the solve for the $F_{\mathbf{x}}(z)$ for with $x_1 = 2$ in terms of $F_{\mathbf{y}}(z)$ where $y_1 \ge 3$, again with Gaussian elimination. However, to get explicit general expressions, valid for all d, in this case seems to be quite complicated. For a given small d however, solving the equations seems relatively easy.

A bound on $\deg f_d(z)$

The relation (17) can also be used to derive an upper bound on $\deg f_d(z)$.

Theorem 3. *For the polynomial $f_d(z)$ given by (14),*

$$\deg f_d(z) < \deg g_d(z) - d.$$

Proof. From 17) we see that

$$\deg h_{\mathbf{0}}(z) = -d + \deg h_{(d,d,\ldots,d)}(z) < -d + \deg H(z). \tag{25}$$

Since $\deg g_d(z) - \deg f_d(z) = \deg H(z) - \deg h_{\mathbf{0}}(z)$, the theorem follows. □

For $1 \le d \le 6$ it turns out that $\deg f_{\mathbf{x}}(z) = \deg g_{\mathbf{x}}(z) - 2d$. Whether this is true in general is an open question, on the quite limited basis we have, combined with some heuristic argument for the general case, we conjecture that it is.

Conjecture 1. *For all $d \ge 1$ we have*

$$\deg f_d(z) = \deg g_d(z) - 2d. \tag{26}$$

where $f_d(z)$ is given by (14) in Theorem 1.

We now give the heuristic argument for the conjecture. From (15) we see that for $\mathbf{x} \ne \mathbf{0}$ we have

$$\deg h_{\mathbf{x}}(z) \le 1 + \max_{1 \le j \le d+1} \deg h_{\mathbf{x}[j]}(z). \tag{27}$$

Moreover, we have equality, except if there are several j for which $F_{\mathbf{x}[j]}(z)$ have the same degree, that this degree is maximal, and that the sum of the coefficients of this maximal degree is zero. Provided we do have equality in (27) when $\mathbf{x} = (r, r, \ldots, r)$ for $1 \le r \le d$ (which seems likely), we get

$$\deg h_{(r-1,r-1,\ldots,r-1)}(z) = 1 + \deg h_{(r,r,\ldots,r)}(z).$$

By induction,

$$\deg h_{(d,d,\ldots,d)}(z) = -(d-1) + \deg h_{(1,1,\ldots,1)}(z) \le \deg H(z) - d.$$

Combining this with (25), we get

$$\deg h_{\mathbf{0}}(z) = -(2d-1) + \deg h_{(1,1,\ldots,1)}(z)$$

and so

$$\deg h_{\mathbf{0}}(z) \leq \deg H(z) - 2d \tag{28}$$

with equality if

$$\deg h_{(1,1,\ldots,1)}(z) = \deg H(z) - 1. \tag{29}$$

Note that (28) with equality is equivalent to (26). So, why should (29) be true? From (15) with $\mathbf{x} = \mathbf{0}$ we see that

$$\deg H(z) = 1 + \max_{1 \leq j \leq d+1} \deg h_{\mathbf{0}[j]}(z)$$

where $h_{\mathbf{0}[j]}(z) = 1, 1 \ldots, 1, 0, 0 \ldots, 0)$ (with j ones and $d - j$ zeros). Hence

$$\deg H(z) = 1 + \deg h_{h_{\mathbf{0}[j]}(z)}(z) \text{ for at least one value of } j.$$

For $d \leq 5$, this appears for $j = d$, that is, (29) is satisfied. In fact it appear that

$$\deg h_{\mathbf{0}[j]}(z) = \begin{cases} \deg h_{\mathbf{0}}(z) & \text{for } j = 1, \\ \deg h_{\mathbf{0}}(z) - d - 1 + j & \text{for } 2 \leq j \leq d - 1, \\ \deg h_{\mathbf{0}}(z) + 2d - 1 & \text{for } j = d. \end{cases}$$

We showed above that $\deg h_{(d,x_2,\ldots,x_d)}(z) = 1 + \deg h_{(x_2,\ldots,x_d,0)}(z)$.

For $d \leq 5$ we also have

$$\deg h_{(r,r,\ldots,r)}(z) = \deg H - r \text{ for } 1 \leq r \leq d,$$

and

$$\deg h_{\mathbf{x}}(z) \leq \deg H(z) - 2d$$

for all other $\mathbf{x}$ with $x_1 < d$. Further, $\deg h_{\mathbf{x}}(z) = \deg H(z) - 2d$ for these $\mathbf{x}$ exactly when $\mathbf{x} = \mathbf{0}$, $\mathbf{x} = (1, 0, 0, \ldots, 0)$ or when $x_i \geq 2$ for all i.

This may very well be the case for all d.

4 Second recurrence

The first recursion was obtained by in each case expanding the permanent by the first row. We now consider a recursion which is obtained by sometimes expanding by the first row, sometimes by the first column (or equivalently, by the first row of the transposed matrix, since a matrix and its transposed have the same permanent). The task is twofold. We have to introduce a rule for when to expand by the first row and when to expand by the first column. Further we have to describe the class of matrices obtained. We will do this in reversed order, first describe the class of matrices, then the rule. We have to show that the class is closed under expansion using this rule. Also we will determine the number of matrices (of given size), as above this will determine the length of the recursion.

To describe the class of matrices, we first introduce some further notations. Let

$$X = \bigcup_{a=0}^{d-1} Y_{a,d-a}.$$

For $\mathbf{y} = (y_1, y_2, \ldots, y_b, 0, 0, \ldots, 0) \in X_{a,b}$, let

$$\mathbf{y}^- = (y_2 - 1, y_3 - 1, \ldots, y_b - 1, 0, 0, \ldots, 0) \in Y_{y_2-1,b-1}.$$

For a pair $\mathbf{x}, \mathbf{y} \in Y$, let $A_{\mathbf{x},\mathbf{y},n}$ be the $n \times n$ matrix $(a_{i,j})$ be defined by

$$\begin{aligned}
a_{i,j} &= 0 \text{ for } j > i+d \text{ or } i > j+d, \\
a_{i,j} &= 0 \text{ for } 1 \le i \le d \text{ and } i+d-x_i < j \le i+d \\
a_{i,j} &= 0 \text{ for } 1 \le j \le d \text{ and } j+d-y_j < i \le j+d \\
a_{i,j} &= 1 \text{ otherwise,}
\end{aligned}$$

We note that in the first row of this matrix, the first $d+1-x_1$ elements are 1, the remaining are 0.

Let $\le$ denote the lexicographic ordering, that is $\mathbf{y} \le \mathbf{x}$ if $\mathbf{y} = \mathbf{x}$ or $y_i = x_i$ for $1 \le i < j$ and $y_j < x_j$ for some j.

We define three classes of pairs of sequences:

$$\begin{aligned}
Z_1 &= \{(\mathbf{x}, \mathbf{0}) \mid \mathbf{x} \in X\}, \\
Z_2 &= \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in Y_{a,d-a}, \mathbf{y} \in Y_{b,d-a}, \text{ where } 1 \le b \le a \le d-1 \text{ and } \mathbf{y} \le \mathbf{x}\}, \\
Z_3 &= \{(\mathbf{x}, \mathbf{y}^-) \mid \mathbf{x}, \mathbf{y} \in X_{a,d+1-a}, \text{ where } 1 \le a \le d \text{ and } \mathbf{x} \le \mathbf{y}\}.
\end{aligned}$$

Let $Z = Z_1 \cup Z_2 \cup Z_3$. Further, let

$$\zeta_{i,a} = |\{(\mathbf{x}, \mathbf{z}) \in Z_i \mid x_1 = a\}|,$$

and $\zeta_a = \zeta_{1,a} + \zeta_{2,a} + \zeta_{3,a}$.

Lemma 5. *We have*
i) $\zeta_a = \frac{1}{2}\left\{\binom{d}{a}^2 + \binom{d}{a}\right\}$ *for* $0 \le a \le d$,
ii) $|Z| = \frac{1}{2}\binom{2d}{d} + 2^{d-1}$.

Proof. We observe that $\zeta_0 = \zeta_d = 1$. Let $1 \le a \le d-1$. First we have

$$\zeta_{1,a} = |Y_{a,d-a}| = \binom{d-1}{a}.$$

We know, by Lemma 2, that $|X_{a,d+1-a}| = \binom{d-1}{a-1}$. For the largest $\mathbf{x} \in X_{a,d+1-a}$, there is one $\mathbf{y}$ which determines an element of Z_3, for the second largest there are two, etc. Hence

$$\zeta_{3,a} = \sum_{j=1}^{\binom{d-1}{a-1}} j = \frac{1}{2}\binom{d-1}{a-1}\left\{\binom{d-1}{a-1} + 1\right\}.$$

Similarly, for the largest $\mathbf{x} \in Y_{a,d-a}$, there are $\binom{d}{a} - 1$ possible $\mathbf{y}$ that determines an element of Z_2, for the second largest $\mathbf{x}$ there is one less $\mathbf{y}$, etc. Hence

$$\zeta_{2,a} = \sum_{j=\binom{d-1}{a-1}}^{\binom{d}{a}-1} j = \frac{1}{2}\left\{\binom{d}{a} - \binom{d-1}{a-1}\right\}\left\{\binom{d}{a} - 1 + \binom{d-1}{a-1}\right\}.$$

Combining these expressions, we get

$$\zeta_a = \frac{1}{2}\left\{\binom{d}{a}^2 + \binom{d}{a}\right\}.$$

Hence

$$|Z| = \sum_{a=0}^{d} \zeta_a = \frac{1}{2}\sum_{a=0}^{d}\binom{d}{a}^2 + \frac{1}{2}\sum_{a=0}^{d}\binom{d}{a} = \frac{1}{2}\binom{2d}{d} + 2^{d-1}.$$

□

The second class of matrices is defined by

$$\mathcal{M}_2 = \{A_{\mathbf{x},\mathbf{z},n} \mid (\mathbf{x},\mathbf{z}) \in Z\}.$$

Observe that for any matrix $M = (a_{i,j})$ in $\mathcal{M}_2$,

$$a_{i,d+1-x_1} = 1 \text{ for } 1 \leq i \leq 2d+1-x_1,$$

that is, there are no "extra" zeros in column $d+1-x_1$ (when compared to $A^{(d,n)}$). Moreover, the set of extra zeros determined by $\mathbf{x}$ and the set of extra zeros determined by $\mathbf{y}$ are disjoint. Hence, when we remove the first row and j'th column of matrix $A_{\mathbf{x},\mathbf{z},n}$, where $1 \leq j \leq d+1-x_1$, we get a matrix $A_{\mathbf{x}',\mathbf{z}',n-1}$ where

$$\mathbf{x}' = (x_2, x_3, \ldots, x_d, 0)$$

and

$$z_i' = z_i + 1 \text{ for } 1 \leq i < j \text{ and } z_i' = z_{i+1} \text{ for } i \geq j.$$

We will show that when we do this for a matrix in $\mathcal{M}_2$, then we get a matrix in $\mathcal{M}_2$ or the transposed of such a matrix, that is $(\mathbf{x}',\mathbf{z}') \in Z$ or $(\mathbf{z}',\mathbf{x}') \in Z$.

Proof. We split the proof into cases.
Case I) $(\mathbf{x},\mathbf{z}) \in Z_1$ (where $(\mathbf{z} = \mathbf{0})$. Then $\mathbf{x} \in X_{a,l}$ where $1 \leq l \leq d-a$.
Subcase I.a) $j = 1$. Then $\mathbf{z}' = \mathbf{0}$. Hence $(\mathbf{x}',\mathbf{z}') \in Z_1$.
Subcase I.b) $1 < j \leq d+1$. Then $\mathbf{z}' = (1,1,\ldots,1,0,0,\ldots,0) \in X_{1,j-1} \subset X$.
Subsubcase I.b.1) $(\mathbf{x}' = \mathbf{0}$. Then $(\mathbf{z}',\mathbf{x}') \in Z_1$.
Subsubcase I.b.2) $\mathbf{x}' = (1,1,\ldots,1,0,0,\ldots,0) \in X_{1,i}$ where $i \leq j-1$. Then $(\mathbf{z}',\mathbf{x}') \in Z_2$.
Subsubcase I.b.3) $\mathbf{z}' < \mathbf{x}'$. Then $(\mathbf{x}',\mathbf{z}') \in Z_2$.

Case II) $(\mathbf{x},\mathbf{y}) \in Z_2$. Then $\mathbf{x} \in X_{a,l}$ and $\mathbf{y} \in X_{b,m}$ where $1 \leq b \leq a$, $1 \leq l \leq d-a$ and $1 \leq m \leq d-a$. In this case, we get $\mathbf{x}' \in X_{x_2,d-1-a} \subset Y_{x_2,d-x_2}$ since $d-1-a = d_1 - x_1 \leq d-1-x_2 < d-x_2$.
Subcase II.a) $j = 1$. Then $\mathbf{y}' = (y_2,\ldots,y_m,0,\ldots,0) \in X_{y_2,m-1}$. If $\mathbf{y}' \leq \mathbf{x}'$, then $(\mathbf{x}',\mathbf{y}' \in Z_2$ since $X_{y_2,m-1} \subset Y_{y_2,d-x_2}$ (because $m-1 \leq d-a-1 < d-x_2$). On the other hand, if $\mathbf{x}' \leq \mathbf{y}'$, then $x_2 \leq y_2$ and $d-1-a \leq d-1-b \leq d-1-y_2 < d-y_2$ and so $\mathbf{x}' \in Y_{x_2,d-y_2}$ and $(\mathbf{y}',\mathbf{x}' \in Z_2$.
Subcase II.b) $1 < j \leq m$. Then

$$\mathbf{y}' = (y_1+1, y_2+1, \ldots, y_{j-1}+1, y_{j+1}, \ldots, y_m, 0, \ldots, 0) \in X_{y_1+1,m-1}.$$

If $\mathbf{y}' \leq \mathbf{x}'$, then $(\mathbf{x}',\mathbf{y}' \in Z_2$ since $X_{y_1+1,m-1} \subset Y_{y_1+1,d-x_2}$ (because $m-1 \leq d-a-1 < d-x_2$). On the other hand, if $\mathbf{x}' \leq \mathbf{y}'$, we must have $x_2 \leq y_2+1$ and so $d-x_2 \leq d-y_2$, that is, $\mathbf{x}' \in Y_{x_2,d-y_2}$. Hence then $(\mathbf{y}',\mathbf{x}' \in Z_2$.
Subcase II.c) $m+1 \leq j \leq d-a$. We get

$$\mathbf{y}' = (y_1+1, y_2+1, \ldots, y_m+1, 1, \ldots, 1, 0, \ldots, 0) \in X_{y_1+1,j-1}.$$

We have $j-1 \leq d-a-1 < d-x_2$. Hence if $\mathbf{y}' \leq \mathbf{x}'$, then $(\mathbf{x}',\mathbf{y}') \in Z_2$. On the other hand, if $\mathbf{x}' \leq \mathbf{y}'$, then $m-1 \leq d-a-1 \leq d-y_1-1$ and so $\mathbf{x}' \in Y_{x_2,d-y_1-1}$. Hence $(\mathbf{y}',\mathbf{x}') \in Z_2$.
Subcase II.d) $j = d+1-a$. We get

$$\mathbf{y}' = (y_1+1, y_2+1, \ldots, y_m+1, 1, \ldots, 1, 0, \ldots, 0) \in X_{y_1+1,d-a}.$$

Subsubcase II.d.1) $\mathbf{y}' \leq \mathbf{x}'$. Then $(\mathbf{x}',\mathbf{y}') \in Z_2$.
Subsubcase II.d.2) $\mathbf{x}' < \mathbf{y}'$ and $y_1+1 \leq a$. Then $(\mathbf{x}',\mathbf{y}') \in Z_2$.

Subsubcase II.d.3) $y_1 + 1 = a + 1$. Note that $x_1 + 1 = a + 1$. Let $\mathbf{u} = (a + 1, x_2 + 1, \ldots, x_{d-a} + 1, 0, \ldots, 0)$. Then $\mathbf{u}^- = \mathbf{x}'$. Since $\mathbf{y} \le \mathbf{x}$, $\mathbf{y}' \le \mathbf{u}$. Hence $(\mathbf{y}', \mathbf{x}') = (\mathbf{y}', \mathbf{u}^-) \in Z_3$.

Case III) $(\mathbf{x}, \mathbf{z}) \in Z_3$ where $\mathbf{z} = \mathbf{y}^-$, $\mathbf{x}, \mathbf{y} \in X_{a,d+1-a}$ and $\mathbf{x} \le \mathbf{y}$. In this case, we get $\mathbf{x}' \in X_{x_2,d-a} \subset X$. We have $\mathbf{z} \in X_{y_2-1,m}$ for some $m \le d - a$.
Subcase III.a) $j = 1$. If $\mathbf{z}' = \mathbf{0}$, then $(\mathbf{x}', \mathbf{z}') \in Z_1$. Otherwise, $(\mathbf{x}', \mathbf{z}') \in Z_2$ or $(\mathbf{z}', \mathbf{x}') \in Z_2$.
Subcase III.b) $j = 1$. If $\mathbf{z}' = \mathbf{0}$, then $(\mathbf{x}', \mathbf{z}') \in Z_1$. Otherwise, $(\mathbf{x}', \mathbf{z}') \in Z_2$ or $(\mathbf{z}', \mathbf{x}') \in Z_2$.
Subcase III.c) $1 < j \le m$. Then $\mathbf{z}' = (y_2, y_3, \ldots, y_{j-1}, y_{j+1} - 1, \ldots, y_m - 1, 0, \ldots, 0)$. Again, $(\mathbf{x}', \mathbf{z}') \in Z_2$ or $(\mathbf{z}', \mathbf{x}') \in Z_2$.
Subcase III.d) $m + 1 \le j \le d + 1 - a$. Then $\mathbf{z}' = (y_2, y_3, \ldots, y_m, 1, \ldots, 1, 0, \ldots, 0)$.
Subsubcase III.d.1) $j \le d - a$ or $y_2 < a$. Then $(\mathbf{x}', \mathbf{z}') \in Z_2$ or $(\mathbf{z}', \mathbf{x}') \in Z_2$.
Subsubcase III.d.2) $j = d + 1 - a$ and $y_2 = a$. Then $\mathbf{z} \in X_{a,d-a}$ and so $(\mathbf{x}', \mathbf{z}') \in Z_2$ or $(\mathbf{z}', \mathbf{x}') \in Z_2$ also in this case. □

Table 3: Example. The first 12 elements of the sequences $\{\alpha_{\mathbf{x},\mathbf{y}}\}$ for $d = 3$

$\mathbf{x}, \mathbf{y}$												
000, 000	[1,	1,	2,	6,	24,	78,	230,	675,	2069,	6404,	19708,	60216]
100, 000	[0,	1,	2,	6,	18,	60,	184,	560,	1695,	5200,	15956,	48916]
110, 000	[0,	0,	2,	6,	18,	46,	146,	460,	1436,	4352,	13252,	40532]
111, 000	[0,	0,	0,	6,	18,	46,	115,	374,	1204,	3752,	11300,	34324]
100, 100	[0,	1,	2,	6,	14,	48,	152,	476,	1425,	4340,	13288,	40852]
110, 100	[0,	0,	2,	6,	14,	38,	124,	400,	1232,	3712,	11288,	34628]
110, 110	[0,	0,	2,	6,	14,	31,	104,	344,	1084,	3236,	9784,	29964]
200, 000	[0,	1,	2,	4,	12,	42,	138,	414,	1235,	3764,	11604,	35664]
210, 000	[0,	0,	2,	4,	12,	32,	108,	336,	1036,	3120,	9540,	29244]
200, 100	[0,	1,	2,	4,	10,	36,	120,	368,	1089,	3304,	10168,	31312]
210, 100	[0,	0,	2,	4,	10,	28,	96,	304,	928,	2784,	8504,	26124]
220, 100	[0,	0,	2,	4,	7,	20,	72,	240,	722,	2140,	6508,	20077]
200, 200	[0,	1,	2,	3,	8,	30,	102,	308,	905,	2744,	8473,	26112]
300, 000	[0,	1,	1,	2,	6,	24,	78,	230,	675,	2069,	6404,	19708]

Since the length of the recursion for $V(d, n)$ is $|Z| = 2^{d-1} + \frac{1}{2}\binom{2d}{d}$, we get the following theorem.

Theorem 4. *For $d \ge 1$,*

$$\deg g_d(z) \le 2^{d-1} + \frac{1}{2}\binom{2d}{d},$$

where $g_d(z)$ is given by (14) in Theorem 1.

For $1 \le d \le 6$ it turns out that $\deg g_d(z) = 2^{d-1} + \binom{2d}{d}/2$. Whether this is true in general is an open question, on the quite limited basis we have, we conjecture that it is.

Conjecture 2. *For all $d \ge 1$ we have*

$$\deg g_d(z) = 2^{d-1} + \frac{1}{2}\binom{2d}{d}.$$

where $g_d(z)$ is given by (14) in Theorem 1.

5 Complexity

In this section we give rough estimates for the (time) complexity of evaluating $V(d,n)$ and the generating function, using the methods considered in this paper.

For the first recursion, (8), the sum on the right hand side has $d+1$ terms when $x_1 < d$ and one term when $x_1 = d$. Hence d additions are performed for half the $\mathbf{x} \in Y$ no additions for the remaining half. Hence the total number of additions needed when $v(n)$ is computed from $v(n-1)$ is exactly $|Y|/2$. Each element in $v(n)$ has size of the order μ_d^n, that it, they have length approximately $n \log_{10} \mu_d$. Assuming that work needed to add of two numbers of the same length m is approximately cm for some contant c (which is the case for long integers), we see that the total work needed to compute $v(n)$ from $v(n-1)$ is approximately

$$\frac{c|Y|\log_{10}\mu_d}{2} n. \tag{30}$$

Therefore, the work to compute $V(d,n)$ by repeated use of (8) is approximately

$$\frac{c|Y|\log_{10}\mu_d}{4} n^2. \tag{31}$$

For the second recursion, the number of sums with a additions is given by ζ_a in Lemma 5 i). Since $\zeta_a = \zeta_{d-a}$ we see that the average number is $d/2$ also for this class. Hence for the second recusion we get expressions similar to (30) and (31), but with $|Z|$ in stead of $|Y|$. Hence the work need for the second recursion is approximately half of the work needed using the first recurstion.

The work to compute the generating function is harder to estimate. First, we need to compute $V(d,n)$ for $n \le |Y|$, and the work for this is approximately

$$\frac{c\log_{10}\mu_d}{4} |Y|^3 \tag{32}$$

for the first class and similarly, with $|Z|^3$ for the second class.

Further, we need to compute the characteristic polynomial of T. Ordinary Gaussian elimination will involve of the order $|Y|^3$ multiplications. Since T is sparse, many of these multiplications are trivial, others will require relatively much work (of the order a constant times $|Y|$).

The final task is the multiplication in (12) which involves of the order $|Y|^2$ multiplications, the work of each is upper bounded by a constant times $|Y|^3$.

We see that the work to compute the generating function is at least of the order $|Y|^3$ and probably somewhat more (if we use ordinary Gaussian elimination). Using the second recursion we would expect to reduce the work by a factor of approximately 8.

6 Lower bounds

We will discuss lower bounds on $V(d,n)$. In particular, we will give a proof of Lehmer's lower bound (5).

The van der Waerden conjecture (now theorem, see e.g. [10, p.104]) states that for a doubly stochastic $n \times n$ matrix, the permanent is lower bounded by $n!/n^n$. Doubly stochastic means that all the elements are non-negative and that the sum of the element in any row or column is 1. If A is an $n \times n$ matrix where the sum of the element in any row or column is k, then van der Waerdens's theorem shows that the permanent is lower bounded by $n!k^n/n^n$.

In $A^{(d,n)}$, most rows and columns have sum $2d+1$, but not all. For a closely related matrix $B^{(d,n)}$, *all* rows and columns have sum $2d+1$. The matrix $B^{(d,n)}$ is defined as follows:

$$\begin{aligned} b_{i,j} &= 0 \text{ if } i > j+d \text{ or } j > i+d, \\ b_{i,j} &= 2 \text{ if } i+j \le d+1 \text{ or } i+j \ge 2n+1-d, \\ b_{i,j} &= 1 \text{ otherwise.} \end{aligned}$$

We see that $B^{(d,n)}$ is obtained from $A^{(d,n)}$ by changing elements in the upper left and lower right corners from 1 to 2. From the discussion above we see that

$$\operatorname{per} B^{(d,n)} \ge \frac{n!(2d+1)^n}{n^n} > \sqrt{2\pi n}\left(\frac{2d+1}{e}\right)^n. \tag{33}$$

The elements in $B^{(d,n)}$ with value 2 are all located in the first d and the last d columns. Hence from the definition (1), we see that

$$\operatorname{per} B^{(d,n)} \le 2^{2d} \operatorname{per} A^{(d,n)}, \tag{34}$$

and so

$$V(d,n) \ge \frac{\sqrt{2\pi n}}{2^{2d}}\left(\frac{2d+1}{e}\right)^n. \tag{35}$$

We collect the upper bound (4) and the lower bound above in a theorem.

Theorem 5. *For all d and n we have*

$$\frac{\sqrt{2\pi n}}{2^{2d}}\left(\frac{2d+1}{e}\right)^n \le V(d,n) \le \frac{\prod_{i=d+1}^{2d}(i!)^{2/i}}{[(2d+1)!]^{2d/(2d+1)}}[(2d+1)!]^{n/(2d+1)}.$$

We see that (35) implies that

$$(V(d,n))^{1/n} \ge \left(\frac{\sqrt{2\pi n}}{2^{2d}}\right)^{1/n}\frac{2d+1}{e} \to \frac{2d+1}{e}$$

when $n \to \infty$, that is, $\mu_d \ge (2d+1)/e$. Combined with (3) we get the following theorem.

Theorem 6. *For all d we have*

$$\frac{2d+1}{e} \le \mu_d \le [(2d+1)!]^{1/(2d+1)} < ((4d+2)\pi)^{1/(4d+2)}\frac{2d+1}{e}.$$

One way to improve the lower bound in Theorem 35 along the same line is to improve the bound (34), and we (essentially) do this next. Let C be the $d \times 2d$ matrix in the upper left corner of $B^{(d,n)}$, that is, for $1 \le i \le d$ we have

$$\begin{aligned} c_{i,j} &= 2 \quad \text{for } 1 \le j \le d+1-i, \\ c_{i,j} &= 1 \quad \text{for } d+2-i \le i \le d+i, \\ c_{i,j} &= 0 \quad \text{for } d+1+i \le j \le 2d. \end{aligned}$$

Further, let

$$R_d = \{(\rho_1, \rho_2, \dots, \rho_d) \mid 1 \le \rho_i \le d+i,\ 1 \le i \le d, \text{ and } \rho_r \ne \rho_s \text{ for } r \ne s, \}$$

$$\sigma(\rho) = c_{1,\rho_1}c_{2,\rho_2}\cdots c_{d,\rho_d} \text{ for } \rho \in R_d,$$

and

$$\Omega_d = \sum_{\rho \in R_d} \sigma(\rho).$$

For $\rho, \tau \in R_d$, let $\Psi(\rho, \tau, n)$ be the number of permutations $p \in T(d, n)$ such that $p_i = \rho_i$ for $1 \le i \le d$ and $p_{n+1-i} = \tau_i$ for $1 \le i \le d$. Then

$$\operatorname{per} B^{(n+2d,d)} = \sum_{\rho \in R_d} \sum_{\tau \in R_d} \sigma(\rho)\sigma(\tau)\Psi(\rho, \tau, n+2d).$$

The numbers $\Psi(\rho, \tau, n+2d)$ will vary with ρ and τ. However, we have

$$\Psi(\rho, \tau, n+2d) = \operatorname{per} A_{n,\rho,\tau},$$

where $A_{n,\rho,\tau}$ is the $n \times n$ matrix obtained by removing from $A^{(n+2d,d)}$ the d first and d last rows and the columns ρ_i for $1 \le i \le d$ and $n + 2d + 1 - \tau_i$ for $1 \le i \le d$. In particular, $A_{n,\rho,\tau}$ is obtained from $A^{(d,n)}$ by changing some ones to zeros (the number of changes is between 0 and $2d$). Hence $\operatorname{per} A_{n,\rho,\tau} \le V(d, n)$. Therefore

$$\operatorname{per} B^{(n+2d,d)} \le V(d, n) \sum_{\rho \in R_d} \sigma(\rho) \sum_{\tau \in R_d} \sigma(\tau) = V(d, n)\, \Omega_d^2.$$

Let

$$\omega_d = \Omega_d\, e^d/(2d + 1)^d.$$

By (33), we get

Theorem 7.

$$V(d, n) \ge \frac{\sqrt{2\pi(n + 2d)}}{\omega_d^2} \left(\frac{2d + 1}{e}\right)^n. \tag{36}$$

This gives an improvement of (35). To show how large the improvement is, we have to determine or at least estimate ω_d. Our argument is heuristic in that one step in the argument will be based on numerical evidence only.

It is not obvious how we can obtain a useful general formula for Ω_d from its definition. We (first) computed Ω_d by exhaustive search for $1 \le d \le 7$. In the next table we give these values: :

d	1	2	3	4	5	6	7
Ω_d	3	18	170	2200	36232	725200	17095248
$\lfloor\omega_d\rfloor$	3	6	10	19	34	61	110

A search in [8] came up with one sequence, A074932, that coincides with the seven first terms. A general expression for A074932 is given in [8]:

$$\sum_{m=0}^{d} \binom{d}{m} (m + 1)^d.$$

Since the numbers are so large, it is therefore quite likely that this is not a coincidence and that the sequence gives Ω_d all d. We also computed $\Omega_8 = 463936896$ and $\Omega_9 = 14246942336$ and checked that they are given by this formula as well. We therefore make the following conjecture.

Conjecture 3.

$$\Omega_d = \sum_{m=0}^{d} \binom{d}{m} (m + 1)^d. \tag{37}$$

We can not prove this conjecture. We will use this conjectured expression for Ω_d to estimate it.

Let

$$t_m = \binom{d}{m} \left(\frac{(m + 1)e}{2d + 1}\right)^d.$$

From (37) we get

$$\omega_d = \sum_{m=0}^{d} t_m. \tag{38}$$

If we write $m = \lambda d$, we have for, $0 < \lambda < 1$ (see e.g. [7], p. 466):

$$\binom{d}{\lambda d} \approx \frac{1}{\sqrt{2\pi\lambda(1-\lambda)d}} \left(\frac{1}{\lambda^\lambda (1-\lambda)^{1-\lambda}}\right)^d.$$

Let

$$f(\lambda) = \left(\frac{\lambda}{1-\lambda}\right)^{1-\lambda}.$$

Then we get

$$\begin{aligned} t_m &\approx \frac{1}{\sqrt{2\pi\lambda(1-\lambda)d}} \left(\frac{1}{\lambda^\lambda (1-\lambda)^{1-\lambda}}\right)^d \left(\frac{\lambda\, e}{2}\right)^d \\ &= \frac{1}{\sqrt{2\pi\lambda(1-\lambda)d}} \left(\frac{e}{2} f(\lambda)\right)^d. \end{aligned} \tag{39}$$

The function $f(\lambda)$ is maximal for $\lambda = \lambda_0 \approx 0.78219$, and the maximal value is $\psi = f(\lambda_0) \approx 1.32110$. Hence the maximal t_m is

$$t_{\lambda_0 d} \approx \frac{1}{\sqrt{2\pi\lambda_0(1-\lambda_0)d}} \varphi^d \approx \frac{0.96653}{\sqrt{d}} \varphi^d.$$

where

$$\varphi = \frac{\psi e}{2} \approx 1.79556.$$

Since ω_d has $d+1$ terms, clearly, $\omega_d \leq 0.96653 \, \frac{d+1}{\sqrt{d}} \, \varphi^d$. Numerical computations shows that this a quite weak upper bound. We can show the following much stronger bound:

Lemma 6. *We have*

$$\omega_d \lesssim 1.67219 \sqrt{\ln(d) - \ln(\ln(d))}\, \varphi^d.$$

Proof. From the Taylor expansion of $f(\lambda)$ at the point $\lambda = \lambda_0$ we get

$$f(\lambda) = f(\lambda_0)\left\{1 - \eta(\lambda - \lambda_0)^2\right\} + O\big((\lambda - \lambda_0)^3\big),$$

where

$$\eta = \frac{1}{2\lambda_0^2(1-\lambda_0)} \approx 3.75203.$$

Let

$$\lambda_1 = \lambda_0 - \sqrt{\frac{\ln(\sqrt{d/\{\ln(d) - \ln(\ln(d))\}})}{\eta d}} \text{ and } \lambda_2 = \lambda_0 + \sqrt{\frac{\ln(\sqrt{d/\{\ln(d) - \ln(\ln(d))\}})}{\eta d}}.$$

From (38) we get

$$\frac{\omega_d}{\varphi^d} = S_1 + S_2 + S_3,$$

where

$$S_1 = \sum_{m=0}^{\lfloor \lambda_1 d \rfloor} \frac{t_m}{\varphi^d},\ S_2 = \sum_{m=1+\lfloor \lambda_1 d \rfloor}^{\lfloor \lambda_2 d \rfloor} \frac{t_m}{\varphi^d},\ S_3 = \sum_{m=1+\lfloor \lambda_2 d \rfloor}^{d} \frac{t_m}{\varphi^d}.$$

First, we see that

$$\begin{aligned}\ln(\sqrt{d/\{\ln(d)-\ln(\ln(d))\}}) &= \frac{1}{2}\Big\{\ln(d)-\ln(\ln(d))-\ln\Big(1-\frac{\ln(\ln(d))}{\ln(d)}\Big)\Big\}\\ &\approx \frac{1}{2}\{\ln(d)-\ln(\ln(d))\}.\end{aligned}$$

Hence,

$$S_2 \lesssim (\lambda_2-\lambda_1)d\,t_{\lambda_0 d} \lesssim 2\sqrt{\frac{\ln(d)-\ln(\ln(d))}{2\eta}}\cdot 0.96653 \approx 0.70566\sqrt{\ln(d)-\ln(\ln(d))}.$$

For S_1 we get

$$\begin{aligned}S_1 &\lesssim d\lambda_1\frac{1}{\sqrt{2\pi d\lambda_1(1-\lambda_1)}}\frac{f(\lambda_1)^d}{\psi^d}\\ &\approx \frac{\lambda_1\sqrt{d}}{\sqrt{2\pi\lambda_1(1-\lambda_1)}}\left(1-\frac{\ln(\sqrt{d/\{\ln(d)-\ln(\ln(d))\}})}{d}\right)^d\\ &\approx \frac{\lambda_1\sqrt{d}}{\sqrt{2\pi\lambda_1(1-\lambda_1)}}e^{-\ln(\sqrt{d/\{\ln(d)-\ln(\ln(d))\}}}\\ &= \frac{\lambda_1}{\sqrt{2\pi\lambda_1(1-\lambda_1)}}\sqrt{\ln(d)-\ln(\ln(d))}\\ &\approx \frac{\lambda_0}{\sqrt{2\pi\lambda_0(1-\lambda_0)}}\sqrt{\ln(d)-\ln(\ln(d))} \approx 0.75601\sqrt{\ln(d)-\ln(\ln(d))}.\end{aligned}$$

Similarly,

$$S_3 \lesssim 0.21052\sqrt{\ln(d)-\ln(\ln(d))}.$$

□

From Lemma 6 we see that $\omega_d < \sqrt{3(\ln(d)-\ln(\ln(d)))}$ for d sufficiently large, and numerical results shows (or strongly indicate) that this is the case for all $d \geq 3$. Combining this with (36), we get

$$V(d,n) > \frac{2\sqrt{2}}{3\sqrt{\pi}}\cdot\frac{\sqrt{n+2d}}{\{\ln(d)-\ln(\ln(d))\}\,1.79556^{2d}}\left(\frac{2d+1}{e}\right)^n. \tag{40}$$

Hence, (36) clearly improves (35) (provided the conjectured expression (37) is true).

Comments an Conjecture 3

There is an old saying: "If you can't solve it, generalize it". The idea is that a more general problem may shed some light on a problem and even lead to a solution. Let us generalize the matrix C to a matrix C_x that has the value x where C has the value 2 and ask the same question as before. To be precise, let C_x be the $d\times 2d$ matrix defined by

$$\begin{aligned}c_{i,j} &= x \quad \text{for } 1\leq j\leq d+1-i,\\ c_{i,j} &= 1 \quad \text{for } d+2-i\leq i\leq d+i,\\ c_{i,j} &= 0 \quad \text{for } d+1+i\leq j\leq 2d,\end{aligned}$$

and let

$$\Omega_d(x) = \sum_{\rho\in R_d} c_{1,\rho_1}c_{2,\rho_2}\cdots c_{d,\rho_d}.$$

Then $\Omega_d = \Omega_d(2)$. A study of this sum gave the following generalization of Conjecture 3; it has been verified for $d \leq 9$.

Conjecture 4.

$$\Omega_d(x) = \sum_{m=0}^{d} \binom{d}{m} (m+1)^d (x-1)^{d-m}.$$

If we expand $(x-1)^{d-m}$, use the relation $\binom{d}{m}\binom{d-m}{j} = \binom{d}{j}\binom{d-j}{m}$, and rearrange the terms we can get the equivalent expression

$$\sum_{j=0}^{d} x^j \binom{d}{j} \sum_{m=0}^{d-j} \binom{d-j}{m} (m+1)^d (-1)^{d-j-m}.$$

The inner sum can rewritten using the identity given by Gould [2] in his proof of the Govindarajulu-Suzuki identity. For our parameters this identity can be written:

$$\sum_{m=0}^{d-j} \binom{d-j}{m} (-1)^{d-m} (m-z)^d = \sum_{r=0}^{j} \binom{d}{d-j+r} (-1)^r (d-j)! z^{j-r} \left\{ {d-j+r \atop d-j} \right\} \tag{41}$$

for all z. Here, $\left\{ {n \atop k} \right\}$ are the Sterling numbers of the second kind. They have a generating function

$$\prod_{k=0}^{n} (1-kz)^{-1} = \sum_{k=0}^{\infty} \left\{ {n \atop k} \right\} z^k.$$

There are several ways of writing the Sterling numbers as finite sums. For actual computations, the recurrence

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\},$$

with the initial conditions $\left\{ {n \atop n} \right\} = 1$ for all $n \geq 0$, $\left\{ {n \atop 0} \right\} = 0$ for all $n \geq 1$ is often useful. Letting $z = -1$ in (41), we get

$$\sum_{m=0}^{d-j} \binom{d-j}{m} (-1)^{d-m-j} (m+1)^d = (d-j)! \sum_{r=0}^{j} \binom{d}{d-j+r} \left\{ {d-j+r \atop d-j} \right\}.$$

For small j, the right hand side expression is easier to compute, for large j (that is for small $d-j$) the left hand expression is the easier to compute.

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Appendix

This appendix contains the values of $V(d,n)$ for $1 \leq d \leq 10$ and $1 \leq n \leq 30$.

For $1 \leq d \leq 6$ we also give $f_d(z)$ and $g_d(z)$, where

$$\sum_{i=0}^{\infty} V(d,n) z^n = \frac{f_d(z)}{g_d(z)}$$

is the generating function of $V(d,n)$.

$d = 1$

$V(1,n)$: [1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269].
This is sequence A000045 in [8].

$f_1(z) = 1$, $g_1(z) = 1 - z - z^2$.

$d = 2$

$V(2,n)$: [1, 2, 6, 14, 31, 73, 172, 400, 932, 2177, 5081, 11854, 27662, 64554, 150639, 351521, 820296, 1914208, 4466904, 10423761, 24324417, 56762346, 132458006,309097942, 721296815, 1683185225, 3927803988, 9165743600, 21388759708, 49911830577].
This is sequence A002524 in [8].

$f_2(z) = 1 - z$, $g_2(z) = 1 - 2\,z - 2\,z^3 + z^5$.

$d = 3$

$V(3,n)$: [1, 2, 6, 24, 78, 230, 675, 2069, 6404, 19708, 60216, 183988, 563172, 1725349, 5284109, 16177694, 49526506, 151635752, 464286962, 1421566698, 4352505527, 13326304313, 40802053896, 124926806216, 382497958000, 1171122069784, 3585709284968, 10978628154457, 33614073801961, 102918683817146].
This is sequence A002526 in [8].

$f_3(z) = 1 - z - 2\,z^2 - 2\,z^4 + z^7 + z^8$,
$g_3(z) = 1 - 2\,z - 2\,z^2 - 10\,z^4 - 8\,z^5 + 2\,z^6 + 16\,z^7 + 10\,z^8 + 2\,z^9 - 4\,z^{10}$
$- 2\,z^{11} - 2\,z^{13} - z^{14}$.

$d = 4$

$V(4, n)$: [1, 2, 6, 24, 120, 504, 1902, 6902, 25231, 95401, 365116, 1396948, 5316192, 20135712, 76227216, 288878956, 1095937420, 4159450913, 15783649241, 59878012558, 227128287882, 861543171080, 3268198646496, 12398132725784, 47033439463906, 178423731589482, 676852399580847, 2567638859146849, 9740350856272280, 36950160475723944].
This is sequence A072856 in [8].

$$f_4(z) =$$
$$1 - 2\,z - 3\,z^2 - z^3 + 4\,z^4 - 31\,z^5 - 5\,z^6$$
$$+ 32\,z^7 - 21\,z^8 + 129\,z^9 + 94\,z^{10} - 83\,z^{11} + 11\,z^{12}$$
$$- 192\,z^{13} - 59\,z^{14} + 63\,z^{15} - 16\,z^{16} + 3\,z^{17} - 29\,z^{18}$$
$$- 46\,z^{19} - 57\,z^{20} + 253\,z^{21} - 28\,z^{22} - 101\,z^{23} + 17\,z^{24}$$
$$+ 104\,z^{25} - 15\,z^{26} - 29\,z^{27} + 10\,z^{28} - z^{29} + z^{30}$$
$$- z^{32} - 3\,z^{33} + z^{35},$$

$$g_4(z) =$$
$$1 - 3\,z - 2\,z^2 + z^3 + z^4 - 70\,z^5 - 39\,z^6$$
$$+ 31\,z^7 + 114\,z^8 + 522\,z^9 + 184\,z^{10} - 34\,z^{11} - 46\,z^{12}$$
$$- 1444\,z^{13} - 202\,z^{14} + 606\,z^{15} - 1204\,z^{16} - 198\,z^{17} + 804\,z^{18}$$
$$- 542\,z^{19} + 26\,z^{20} + 2372\,z^{21} + 318\,z^{22} - 1582\,z^{23} + 328\,z^{24}$$
$$+ 2018\,z^{25} + 222\,z^{26} - 810\,z^{27} - 184\,z^{28} + 706\,z^{29} + 14\,z^{30}$$
$$- 204\,z^{31} - 70\,z^{32} - 14\,z^{33} - 28\,z^{34} + 22\,z^{35} + 11\,z^{36}$$
$$- 47\,z^{37} + 8\,z^{38} + 11\,z^{39} + z^{40} + 4\,z^{41} - z^{42} - z^{43}.$$

$d = 5$

$V(5, n)$: [1, 2, 6, 24, 120, 720, 3720, 17304, 76110, 329462, 1441923, 6487445, 29555588, 135025756, 615260976, 2791161792, 12618600768, 57008446080, 257708989200, 1166042944564, 5279435858788, 23908888017477, 108262665958797, 490132089640318, 2218641353956314, 10042447508086040, 45456882997868856, 205767975553980624, 93147045735775140 421662694714714 8728].

$$f_5(z) =$$
$$1 - 2\,z - 6\,z^2 - 6\,z^3 - 4\,z^4$$
$$+ 22\,z^5 - 378\,z^6 - 480\,z^7 + 276\,z^8 + 728\,z^9$$
$$+ 408\,z^{10} + 15398\,z^{11} + 22376\,z^{12} + 4096\,z^{13} - 11540\,z^{14}$$
$$- 13114\,z^{15} - 283900\,z^{16} - 262024\,z^{17} + 56670\,z^{18} - 28750\,z^{19}$$
$$- 352318\,z^{20} + 1537342\,z^{21} + 1635762\,z^{22} - 1800918\,z^{23} - 2104130\,z^{24}$$
$$+ 3963102\,z^{25} + 10076316\,z^{26} + 3746874\,z^{27} - 7513590\,z^{28} + 1538536\,z^{29}$$
$$+ 34806172\,z^{30} - 43694200\,z^{31} - 40839352\,z^{32} + 39912634\,z^{33} + 72045960\,z^{34}$$
$$+ 107445760\,z^{35} - 402913756\,z^{36} - 390995478\,z^{37} + 176441940\,z^{38} + 433871232\,z^{39}$$
$$+ 280618882\,z^{40} - 1487917018\,z^{41} - 1448387386\,z^{42} + 399977274\,z^{43} + 1268230975\,z^{44}$$
$$+ 676720348\,z^{45} - 3379646316\,z^{46} - 2529818252\,z^{47} + 721460328\,z^{48} + 1365615500\,z^{49}$$
$$+ 1071433548\,z^{50} - 4195765680\,z^{51} - 2048800840\,z^{52} + 1150167680\,z^{53} - 291299552\,z^{54}$$
$$+ 319720428\,z^{55} - 1126239744\,z^{56} + 58508624\,z^{57} + 1203484328\,z^{58} - 1701184948\,z^{59}$$
$$- 1774111080\,z^{60} + 3049127936\,z^{61} + 1948591548\,z^{62} - 272263964\,z^{63} - 732832604\,z^{64}$$
$$- 1127720996\,z^{65} + 1819368076\,z^{66} + 1467155972\,z^{67} - 1310666356\,z^{68} + 483398188\,z^{69}$$
$$+ 1439280456\,z^{70} - 2005458780\,z^{71} - 993846364\,z^{72} + 419550560\,z^{73} + 118858344\,z^{74}$$
$$+ 277968352\,z^{75} - 439300736\,z^{76} - 424357532\,z^{77} + 792529344\,z^{78} - 382154736\,z^{79}$$
$$- 779966984\,z^{80} + 1291571300\,z^{81} + 452825112\,z^{82} - 755726384\,z^{83} + 400879764\,z^{84}$$
$$+ 439043324\,z^{85} - 742276260\,z^{86} - 160395356\,z^{87} + 238107743\,z^{88} - 206692154\,z^{89}$$
$$- 107502510\,z^{90} + 209221106\,z^{91} + 39196636\,z^{92} - 14120258\,z^{93} + 46775310\,z^{94}$$
$$+ 1989648\,z^{95} - 27234764\,z^{96} - 8578456\,z^{97} - 7153528\,z^{98} - 497714\,z^{99}$$

$+ 4044120\, z^{100} - 1225744\, z^{101} + 858860\, z^{102} + 1109710\, z^{103} - 1487964\, z^{104}$
$- 483832\, z^{105} + 949446\, z^{106} + 204362\, z^{107} + 76666\, z^{108} + 122886\, z^{109}$
$- 55262\, z^{110} - 58062\, z^{111} - 43818\, z^{112} - 21098\, z^{113} + 19932\, z^{114}$
$+ 9410\, z^{115} - 12174\, z^{116} - 3016\, z^{117} - 772\, z^{118} - 1320\, z^{119}$
$+ 504\, z^{120} + 514\, z^{121} + 504\, z^{122} + 176\, z^{123} - 156\, z^{124}$
$- 46\, z^{125} + 84\, z^{126} + 48\, z^{127} + 10\, z^{128} - 2\, z^{129}$
$- 2\, z^{130} + 2\, z^{131} + z^{132},$

$g_5(z) =$
$1 - 3\, z - 5\, z^2 - z^3 + z^4$
$+ 5\, z^5 - 619\, z^6 - 813\, z^7 - 151\, z^8$
$+ 1101\, z^9 + 7131\, z^{10} + 37760\, z^{11} + 36452\, z^{12}$
$+ 12268\, z^{13} + 27864\, z^{14} - 104500\, z^{15} - 909068\, z^{16}$
$- 628052\, z^{17} + 467468\, z^{18} - 875852\, z^{19} - 2986516\, z^{20}$
$+ 4390728\, z^{21} + 7811902\, z^{22} - 5144926\, z^{23} - 6083586\, z^{24}$
$+ 32035730\, z^{25} + 56729442\, z^{26} + 5296722\, z^{27} - 60962238\, z^{28}$
$- 12959610\, z^{29} + 155004506\, z^{30} - 53746134\, z^{31} - 195839134\, z^{32}$
$+ 70957880\, z^{33} + 437243460\, z^{34} + 59895516\, z^{35} - 1971632928\, z^{36}$
$- 1997716868\, z^{37} + 211084516\, z^{38} + 3583523916\, z^{39} + 242527564\, z^{40}$
$- 11167051356\, z^{41} - 8574194260\, z^{42} 446091168\, z^{43} + 12277357107\, z^{44}$
$+ 7659596379\, z^{45} - 31162331539\, z^{46} - 19368038291\, z^{47} - 973520577\, z^{48}$
$+ 15643674819\, z^{49} + 28583038019\, z^{50} - 42257711731\, z^{51} - 27827524321\, z^{52}$
$+ 2896107571\, z^{53} + 1456179529\, z^{54} + 43239063104\, z^{55} - 8375497816\, z^{56}$
$- 21390489288\, z^{57} + 10906591728\, z^{58} - 7068571400\, z^{59} + 13482403656\, z^{60}$
$+ 35071085304\, z^{61} + 5947568120\, z^{62} + 10560167112\, z^{63} + 9420070072\, z^{64}$
$- 22516381360\, z^{65} + 14366178580\, z^{66} + 25518666076\, z^{67} + 6520647012\, z^{68}$
$+ 13867274188\, z^{69} - 12021790788\, z^{70} - 34405335300\, z^{71} + 4345542108\, z^{72}$
$+ 10935106548\, z^{73} - 10054639252\, z^{74} + 515436\, z^{75} - 2365095604\, z^{76}$
$+ 772606192\, z^{77} - 2343355256\, z^{78} - 13908243976\, z^{79} + 4173466240\, z^{80}$
$+ 21182711864\, z^{81} - 2007626424\, z^{82} - 6665549416\, z^{83} + 13101065432\, z^{84}$
$+ 1085146824\, z^{85} - 15722194024\, z^{86} - 4503731712\, z^{87} + 4059378199\, z^{88}$
$- 5935049821\, z^{89} - 3516541275\, z^{90} + 7166582121\, z^{91} + 4314195567\, z^{92}$
$- 1810587141\, z^{93} + 1681967755\, z^{94} + 2212778301\, z^{95} - 2110972073\, z^{96}$
$- 1379269021\, z^{97} + 594002109\, z^{98} - 317773568\, z^{99} - 555399436\, z^{100}$
$+ 331666780\, z^{101} + 140667256\, z^{102} - 49244228\, z^{103} + 47413060\, z^{104}$
$+ 28598620\, z^{105} + 8726268\, z^{106} + 19782660\, z^{107} - 18740644\, z^{108}$
$+ 2348584\, z^{109} + 13323150\, z^{110} - 14683614\, z^{111} - 6586114\, z^{112}$
$+ 3206722\, z^{113} - 2436606\, z^{114} - 2354094\, z^{115} + 1473410\, z^{116}$
$+ 329702\, z^{117} + 193626\, z^{118} - 9846\, z^{119} - 184974\, z^{120}$
$+ 291096\, z^{121} + 132148\, z^{122} - 46932\, z^{123} + 34912\, z^{124}$
$+ 40140\, z^{125} - 20908\, z^{126} - 7396\, z^{127} - 2852\, z^{128}$
$+ 3092\, z^{129} + 4540\, z^{130} - 4384\, z^{131} - 2091\, z^{132}$
$- 27\, z^{133} + 147\, z^{134} + 139\, z^{135} - 207\, z^{136}$
$- 99\, z^{137} - 3\, z^{138} + 3\, z^{139} + z^{140}$
$- 3\, z^{141} - z^{142}.$

$d = 6$

$V(6,n)$: [1, 2, 6, 24, 120, 720, 5040, 30960, 172200, 899064, 4553166, 22934774, 116914351, 610093513, 3222826972, 17101449940, 90706002192, 479654768640, 2527274267136, 13280313508416, 69734129749632, 366283822765632, 1925290900630896, 10126754515065868, 53288497861681452, 280450465518547105, 1475842677769607257, 7765278223667692238, 40852351318766434666, 214904305350092369240].

$f_6(z) =$
$1 - 3\,z - 4\,z^2$
$- 16\,z^3 - 22\,z^4$
$- 32\,z^5 + 130\,z^6$
$- 5182\,z^7 - 6050\,z^8$
$- 11830\,z^9 + 2210\,z^{10}$
$+ 20734\,z^{11} + 205360\,z^{12}$
$+ 2277995\,z^{13} + 3239955\,z^{14}$
$+ 7033148\,z^{15} + 5982696\,z^{16}$
$+ 5602620\,z^{17} - 48680616\,z^{18}$
$- 471687784\,z^{19} - 446079708\,z^{20}$
$- 1043676116\,z^{21} - 973832568\,z^{22}$
$- 3005679876\,z^{23} - 978468680\,z^{24}$
$+ 35807712552\,z^{25} + 32583013850\,z^{26}$
$+ 54766612826\,z^{27} - 16677720844\,z^{28}$
$+ 179210388432\,z^{29} + 689025286712\,z^{30}$
$+ 10935614656\,z^{31} + 773545805308\,z^{32}$
$- 2087737635968\,z^{33} + 3070070175608\,z^{34}$
$+ 9502983281724\,z^{35} - 9824830866232\,z^{36}$
$- 82765993516684\,z^{37} - 66107140980968\,z^{38}$
$- 159550107452058\,z^{39} + 307494866927358\,z^{40}$
$+ 15051768641012\,z^{41} - 1671837218865832\,z^{42}$
$- 621973668255820\,z^{43} - 5759344485910984\,z^{44}$
$- 1610647091042216\,z^{45} + 4140984775227468\,z^{46}$
$- 14344519514963196\,z^{47} - 62180635106219336\,z^{48}$
$+ 98959229332068164\,z^{49} - 46484281630916520\,z^{50}$
$+ 271922376504236616\,z^{51} + 87808417513840141\,z^{52}$
$- 782703582548143023\,z^{53} - 1127999926048213312\,z^{54}$
$+ 5818435859211666528\,z^{55} + 5442047304334097054\,z^{56}$
$+ 18126115354292948624\,z^{57} + 5011402875881396418\,z^{58}$
$- 31517822735428085506\,z^{59} - 13982227471192877270\,z^{60}$
$+ 182292008740601426474\,z^{61} + 174645098944552748518\,z^{62}$
$+ 667162831526531890446\,z^{63} + 175196255202138453640\,z^{64}$
$- 766112510968012268473\,z^{65} - 275321951128691183865\,z^{66}$
$+ 3997276622635500089696\,z^{67} + 2201721769609049126592\,z^{68}$
$+ 15842970441369735238048\,z^{69} + 4282550312676669042880\,z^{70}$
$- 12568926426883755375296\,z^{71} - 8794213143662124115616\,z^{72}$
$+ 63298286470937014208928\,z^{73} + 23991839003772039879 04\,z^{74}$
$+ 240290128466085069573088\,z^{75} + 841428922224183716535 04\,z^{76}$
$- 1524048544339814489637 76\,z^{77} - 1592130788667676141658 40\,z^{78}$
$+ 7041475630645285481615 84\,z^{79} - 3100473007887171211986 88\,z^{80}$
$+ 23220270262396613068327 68\,z^{81} + 11794298285651662273764 48\,z^{82}$
$- 13684302115956992968751 36\,z^{83} - 15560219915966334958712 00\,z^{84}$
$+ 48399348343894114438214 40\,z^{85} - 47884603168614893230879 36\,z^{86}$
$+ 127455314278368787658717 76\,z^{87} + 87532435222317378085556 48\,z^{88}$
$- 81047734506812615001691 84\,z^{89} - 62866888408346986228976 00\,z^{90}$

$$
\begin{aligned}
&+ 1412358141854883763602836 8\, z^{91} - 34520276189592066659531504\, z^{92} \\
&+ 17395201059067744535962016\, z^{93} + 10211875367562225073855296\, z^{94} \\
&- 21274685169694993209213984\, z^{95} + 36740457143839701209828032\, z^{96} \\
&- 66733100007385375705089888 0\, z^{97} - 111245787132978730318955232\, z^{98} \\
&- 254272635927053058629147680\, z^{99} - 378118058230023995216117184\, z^{100} \\
&- 143794810407500804030956 48\, z^{101} + 542981664240856829950163584\, z^{102} \\
&- 659044825166394498746812352\, z^{103} + 1768430778798578123246470 60\, z^{104} \\
&- 1268256497095581625245748220\, z^{105} - 2629593308648771432749507248\, z^{106} \\
&- 230758041736623959787269312\, z^{107} + 16176357383449190718787270 16\, z^{108} \\
&+ 1372530710280515669224642688\, z^{109} + 8181743582565245559734299 92\, z^{110} \\
&+ 4385198111318250724229535880\, z^{111} - 3020823331511235468061413192\, z^{112} \\
&- 3713294406981239034941046008\, z^{113} - 6086371413061388080435767800\, z^{114} \\
&+ 21273485815817788256381765720\, z^{115} - 12647420498501159412963335168\, z^{116} \\
&+ 46589674923679867491996192668\, z^{117} + 26472026146345794548055831292\, z^{118} \\
&+ 1022023915533683718909434192\, z^{119} - 4325585455760106787235643296\, z^{120} \\
&- 60494699606159716880384859248\, z^{121} + 43707718598159840926285223456\, z^{122} \\
&- 95092580377091939716934465632\, z^{123} + 86172607429632421849304704752\, z^{124} \\
&+ 146629864605068789306132582736\, z^{125} + 119598861613733577872570724832\, z^{126} \\
&- 505937992854965467217608216880\, z^{127} + 383714529096649449143927006816\, z^{128} \\
&- 958547900616774311717312346144\, z^{129} + 168446122183388679594749251352\, z^{130} \\
&+ 414261652350553883742797350552\, z^{131} - 75771004470800041257151765392\, z^{132} \\
&+ 2326827741637085249566509649600\, z^{133} - 2153597084275223359564337006816\, z^{134} \\
&+ 4865481829841135810761330955904\, z^{135} - 2928968809786942366462995953904\, z^{136} \\
&- 2519991157982406000567786531200\, z^{137} - 1991929578504827140462291063136\, z^{138} \\
&+ 1136646947696101878394598287248\, z^{139} - 2102757622061287990252996823904\, z^{140} \\
&+ 393766825681349714120500115440\, z^{141} - 7560933519577930303665588538208\, z^{142} \\
&- 1207466372480431479639510546328\, z^{143} + 8326633705020471966511615370760\, z^{144} \\
&- 41336775688948875849154824218512\, z^{145} + 38009710770690376550909668973216\, z^{146} \\
&- 92849628272354332886263617582160\, z^{147} + 101477290111697179887637107177376\, z^{148} \\
&+ 17644073061624427171857995610016\, z^{149} - 13580581282213328934480517102768\, z^{150} \\
&+ 156963884668027234640653470460656\, z^{151} - 185787172692301599225699368143 04\, z^{152} \\
&+ 445820530055794514431390921092016\, z^{153} - 216009846778753064329525170646048\, z^{154} \\
&- 88211485321911480987187347554464\, z^{155} - 132300737797639345806661634003980\, z^{156} \\
&- 237219308300505371358050271822396\, z^{157} - 799297227974670153670923106998624\, z^{158} \\
&- 1064558163856272167335781844940288\, z^{159} - 761900589014946557055632158460648\, z^{160} \\
&+ 830531623770609063002488614035264\, z^{161} + 1150738440680322484207920812836424\, z^{162} \\
&- 248782282297256688309362817746216\, z^{163} + 461935369851408205711991963143437 6\, z^{164} \\
&+ 1264336893805384390217970700617320\, z^{165} + 6466920126130499941286039925593208\, z^{166} \\
&- 4451412712508750923348620879648200\, z^{167} - 4846211112374671904435330334060448\, z^{168} \\
&+ 2333093272196903585124214371792412\, z^{169} - 14509281608877674846296032826648740\, z^{170} \\
&- 608556327980271719020550319173600\, z^{171} - 2253960292976712266920533267809043 2\, z^{172} \\
&+ 13128244822772080730391597657245024\, z^{173} + 15124208630994747127524737985862720\, z^{174} \\
&- 6603297075266237105718256721746240\, z^{175} + 30402405735130624943899217775374240\, z^{176} \\
&+ 3494513765595572488000236271648352\, z^{177} + 50386280861361669437006171217087040\, z^{178} \\
&- 20999656387455048053856965247694688\, z^{179} - 41994457035515718314487431112650752\, z^{180} \\
&+ 11898638778082469877575239479259584\, z^{181} - 44232344204837221112693029348454064\, z^{182} \\
&- 23834119046281166358275747929976112\, z^{183} - 77854611026876890084660186822632480\, z^{184} \\
&+ 6230000437094934918746637741104256\, z^{185} + 105775570848134903095199917669103808\, z^{186} \\
&- 17411484527650435000726907380872320\, z^{187} + 42135137909346981297959959969828576\, z^{188} \\
&+ 76776302536508239207609487954536832\, z^{189} + 81352455993018976383514790591742272\, z^{190} \\
&+ 58152822426008869769665424237641312\, z^{191} - 224209892333925573205538433033197056\, z^{192} \\
&+ 26578115687318492381939209417033824\, z^{193} - 18440585678858979308633015456332864\, z^{194} \\
&- 153106541415318828510528281374306000\, z^{195} - 495171220561805138792574308149381 28\, z^{196} \\
&- 169074209351088047588103340366671904\, z^{197} + 375938452441205562562313978237240512\, z^{198}
\end{aligned}
$$

$-40162500708758350410755902556763360\, z^{199} - 127159299368873001641306918500617 60\, z^{200}$
$+20537171049616020940353650815235 2704\, z^{201} + 749657101899030935135364206242147 2\, z^{202}$
$+268681223938021666287851967020774944\, z^{203} - 4817723127708505505258741650109783 68\, z^{204}$
$+41344312548200155811605816042380512\, z^{205} + 310122615479602868909513621873144 32\, z^{206}$
$-181410563422520321802665435874008384\, z^{207} + 76799234877225968753616243775431 8\, z^{208}$
$-278428732758756876556916355313326634\, z^{209} + 4540796573955979000347740008139588 24\, z^{210}$
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&-529340752087861536\,z^{411}-806965535072375744\,z^{412}\\
&+836479925014907168\,z^{413}+1001142623830543360\,z^{414}
\end{aligned}
$$

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$+ 63230901472614293\, z^{417} + 72427708870960956\, z^{418}$
$+ 20871164990869744\, z^{419} - 37328603136793286\, z^{420}$
$+ 7070441543876320\, z^{421} + 23320321186111410\, z^{422}$
$+ 627145411735954\, z^{423} + 4063364451776590\, z^{424}$
$- 4679012285528518\, z^{425} - 5304443626479886\, z^{426}$
$+ 1705471813958030\, z^{427} + 1043078504846576\, z^{428}$
$- 231372538698477\, z^{429} - 429525749328965\, z^{430}$
$- 47533375150724\, z^{431} + 194685584073960\, z^{432}$
$- 64815830502084\, z^{433} - 92130057588008\, z^{434}$
$+ 8312334913688\, z^{435} - 2322219450076\, z^{436}$
$+ 19174720105644\, z^{437} + 14126070538632\, z^{438}$
$- 4283005289796\, z^{439} - 1290713078152\, z^{440}$
$+ 96919277480\, z^{441} + 1359714752026\, z^{442}$
$- 382999498726\, z^{443} - 824309182412\, z^{444}$
$+ 299794224080\, z^{445} + 200491950904\, z^{446}$
$- 13501727936\, z^{447} - 52031254660\, z^{448}$
$- 32521459968\, z^{449} - 2656128008\, z^{450}$
$- 2253972420\, z^{451} - 2089275896\, z^{452}$
$+ 256195380\, z^{453} + 277216024\, z^{454}$
$+ 1901157542\, z^{455} + 1201919806\, z^{456}$
$- 239437516\, z^{457} - 173972904\, z^{458}$
$- 4127052\, z^{459} + 28188344\, z^{460}$
$- 40378280\, z^{461} - 38212020\, z^{462}$
$+ 7101316\, z^{463} + 6220216\, z^{464}$
$+ 334852\, z^{465} - 731624\, z^{466}$
$+ 361672\, z^{467} + 547765\, z^{468}$
$- 74535\, z^{469} - 83616\, z^{470}$
$- 8736\, z^{471} + 6062\, z^{472}$
$- 752\, z^{473} - 3566\, z^{474}$
$+ 334\, z^{475} + 474\, z^{476}$
$+ 58\, z^{477} - 10\, z^{478}$
$- 2\, z^{479} + 8\, z^{480}$
$- z^{481} - z^{482}.$

$g_6(z) =$
$1 - 4\, z - 2\, z^{2}$
$- 12\, z^{3} - 6\, z^{4}$
$- 14\, z^{5} + 36\, z^{6}$
$- 6786\, z^{7} - 7028\, z^{8}$
$- 20030\, z^{9} - 5716\, z^{10}$
$+ 66552\, z^{11} + 626230\, z^{12}$
$+ 3768002\, z^{13} + 4384620\, z^{14}$
$+ 10680858\, z^{15} + 14115476\, z^{16}$
$+ 15760598\, z^{17} - 140142686\, z^{18}$
$- 888725104\, z^{19} - 752722030\, z^{20}$
$- 1519475616\, z^{21} - 2274448438\, z^{22}$
$- 9858089532\, z^{23} - 6372925096\, z^{24}$
$+ 64667964162\, z^{25} + 67291188687\, z^{26}$
$+ 88982446424\, z^{27} - 18675532860\, z^{28}$
$+ 762635948336\, z^{29} + 2475221798796\, z^{30}$
$+ 1732513659516\, z^{31} + 2132875765648\, z^{32}$
$- 5504624223076\, z^{33} + 2839601081928\, z^{34}$
$+ 13003095944716\, z^{35} - 46935551014128\, z^{36}$

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$- 5259493058416108\, z^{43} - 18696243912842660\, z^{44}$
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$-25226152492560735802066201576697940480\,z^{185}+3157046251089350517441204120237613792\,z^{186}$

$-11767976471893844910786987740079895360\,z^{187}-11227643569687922512840420440594394880\,z^{188}$

$+43926811203820688275463732585264227520\,z^{189}-26327361341277561929175319590648343680\,z^{190}$

$+64938852889666173083499216117019798720\,z^{191}-67771942300006536749986374696815224320\,z^{192}$

$+22242464508319899707575164127684440320\,z^{193}+27290457991847083665856556002413746240\,z^{194}$

$-74868221157724010850682734036160608320\,z^{195}+51284532876874789310008171159436808320\,z^{196}$

$-120926273925459547102724835880270980160\,z^{197}+112412607752498441836329722464778214400\,z^{198}$

$-34586399082901044092856913405695766720\,z^{199}-42682876198715219760362426601672381760\,z^{200}$

$+94726180500628223530557055885647660160\,z^{201}-65214315296246583862297261468793850560\,z^{202}$

$+169715109942102008694327555211563328000\,z^{203}-141625284109851230877035945548638548800\,z^{204}$

$+40493976205584930938624526217752505600\,z^{205}+47491999597043685684195820062906731520\,z^{206}$

$-86575685977488584142921717497460055360\,z^{207}+50087865439360513166218151190861942060\,z^{208}$

$-180114635085719387242559778874926788400\,z^{209}+128967367282699630738728925063720121960\,z^{210}$

$-29387426200444545137828813451456722960\,z^{211}-37889737381621736145792599572927364040\,z^{212}$

$+51555625782768917320426981313611951320\,z^{213}-77464246190709866964670497096249018400\,z^{214}$

$+140114546314469913091404028074806568680\,z^{215}-71605859857837219190633276062241419120\,z^{216}$

$+33774635595274280397718154086992604\,z^{217}+20124225384399774327493563314061844880\,z^{218}$

$-129708245042376736367055869710483185600\,z^{219}-36076935239165780828768859338843569400\,z^{220}$

$-71729342529717441280287643740559471720\,z^{221}+57595169224518213955692264511034472\,z^{222}$

$+32111061649794845234484958377276056760\,z^{223}-34624690241915825095194265038585719200\,z^{224}$

$-49051259010564039768498826216224308400\,z^{225}+53602784005260400722201292119991557400\,z^{226}$

$+138236128333930139971435523600092995200\,z^{227}+43895298002558954050255506819350397400\,z^{228}$

$-49079855662837760492009222243794794880\,z^{229}-71195426754424109674956730521972690000\,z^{230}$

$+17747933659405386272608187552298482400\,z^{231}-39765850687852102863060483950192567520\,z^{232}$

$+102601184047549289952598329777563452400\,z^{233}-47673236232550425912570305366454311020\,z^{234}$

$+44376051672339206985574803529728347040\,z^{235}+102376158047314914377762372053856412000\,z^{236}$

$+34917930056562752678005045397897027200\,z^{237}+138396559249165647539949137604838560800\,z^{238}$

$-63066826088077122290657805167480053600\,z^{239}+27975856770350102388029474981185370560\,z^{240}$

$-27204275905106602346992199747297320560\,z^{241}-67552644280772742058284500148609710400\,z^{242}$

$-44868342218940985795926574448849592\,z^{243}+24613561959261046291614117898900246400\,z^{244}$

$-43548673990362951732795547666885990400\,z^{245}-85400321617440072435771106821466695200\,z^{246}$

$+97784597691307122189823402105133524000\,z^{247}+101148435680475771086235587661574608\,z^{248}$

$-38696192158291037308189701605610591200\,z^{249}-45090255279239595731180663678671753600\,z^{250}$

$+69131185472315523795723071533335973600\,z^{251}-69989962299523131847698031893402200\,z^{252}$

$$+1328252996772558856534458547872435 04\,z^{253}+151200608610615277445796871304127384\,z^{254}$$
$$+243385858623557782945962987264575712\,z^{255}+117186526387359087817059629564937288\,z^{256}$$
$$-274618646765452043897088696206864768\,z^{257}+198904494053814342601687125791875424\,z^{258}$$
$$-450816357226852857884502151068959704\,z^{259}+53709176818889032590299989976988 82\,z^{260}$$
$$+116862439601503569707622321090855144\,z^{261}+46192004838979470424624295163444516\,z^{262}$$
$$-32418889592115401278436101997441528\,z^{263}-44714299520782179257953308912514196\,z^{264}$$
$$+239888071071740910117714318670078524\,z^{265}-162175016724419035310227432723211992\,z^{266}$$
$$-165609680545774655229741015759462572\,z^{267}-19275920316525736580293385119691736\,z^{268}$$
$$+36198355586384865946651577884397756\,z^{269}-41659395929504794746490981993538440\,z^{270}$$
$$+7786635174902685950275444785399344\,z^{271}+9497157965611486680979765187627026 0\,z^{272}$$
$$+8125575087941870640198677306749372\,z^{273}-21115462568779320989948693805632504\,z^{274}$$
$$+5365380589796405892149456579874732\,z^{275}+18003741046784896808167000230261960\,z^{276}$$
$$-50653375983847174894217253011548428\,z^{277}+9701903666070990461694963648223708\,z^{278}$$
$$+44648759889598205956359694317070960\,z^{279}+16840616289852388319583240400983180\,z^{280}$$
$$+3300057843058874145092084204269600\,z^{281}+4468985881345379762016907693527372\,z^{282}$$
$$-12532230568343024084083040459947 44\,z^{283}-22085063201717797234586187350235120\,z^{284}$$
$$-1691145355218836670345011316703908\,z^{285}+3528238667143186380371432340026654\,z^{286}$$
$$-4877752179207543874198547262504384\,z^{287}-1011609870477855920479729752224480\,z^{288}$$
$$+11361590901637179370595420338232320\,z^{289}+4434629790592698073172547918648032\,z^{290}$$
$$-11265863361740061971630225515300384\,z^{291}-7118188346158268956279483820073344\,z^{292}$$
$$-2553050354756279678096327572545568\,z^{293}-1028217317580504590360873322780736\,z^{294}$$
$$+2861103054808843323899918087561952\,z^{295}+1898905892175465377551309420490752\,z^{296}$$
$$-2388373313788629207138613329363 20\,z^{297}-218683471927095789652291446935392\,z^{298}$$
$$+811729457714811792627571093511008\,z^{299}-211336951281524421522671463391680\,z^{300}$$
$$-1618351449981618976381379178557088\,z^{301}-1489978721836904262999060970593024\,z^{302}$$
$$+1383910899535720905402471945644320\,z^{303}+1569756158434357294403513646353632\,z^{304}$$
$$+1045067113666712355236949126986560\,z^{305}+349371289668880476264986919557408\,z^{306}$$
$$-1166492374572650886663164146885632\,z^{307}-432844739811189869167008317871392\,z^{308}$$
$$+387517342718255997795950022322048\,z^{309}+174648957308004456211813735526144\,z^{310}$$
$$+121920503285494927104967047333088\,z^{311}+158368418248995632043454982574772\,z^{312}$$
$$-213009046418301927919898063988240\,z^{313}+196823708648087971657713607014520\,z^{314}$$
$$+119192152730199109511752666759056\,z^{315}-207491527040680486122662536979736\,z^{316}$$
$$-172722517242956860845200131053240\,z^{317}-13115229217306837561608136468016\,z^{318}$$
$$+67444217240000055247355057024696\,z^{319}+11930977339869840122458014832043 2\,z^{320}$$
$$-14403519598134839937967526782840\,z^{321}-4070900616812144405159673539022 4\,z^{322}$$
$$-68617803119193694507456817251104\,z^{323}-38590445416628780177455522587816\,z^{324}$$
$$+69401782699412569722823106924808\,z^{325}+14558259184748750877283563632816\,z^{326}$$
$$-40546159622572003011009356468760\,z^{327}+23262353675884554892080377520912\,z^{328}$$
$$-10423898506964415701269562265768\,z^{329}-24145624310005954641042916838392\,z^{330}$$
$$+35242034928488774847563281927424\,z^{331}-4432787429154281268553697265272\,z^{332}$$
$$-15304077109024079397734282641024\,z^{333}+14891999109068865706142102018088\,z^{334}$$
$$-4609836016854062446106206999728\,z^{335}-1160034041805195490294551204380 8\,z^{336}$$
$$+12698570688737967053324354676872\,z^{337}-1542233878204334559453102803188\,z^{338}$$
$$-2392607591192748692132957551072\,z^{339}+5061617853464290042597143201776\,z^{340}$$
$$-2574945805542116976538327385152\,z^{341}-4311903078592444220491926235056\,z^{342}$$
$$+3354804785558101282227562072080\,z^{343}-639298198826019719555437747 84\,z^{344}$$
$$-305560291762388890873884709232\,z^{345}+1171542529538729739654990635104\,z^{346}$$
$$-496553404601482314235313346992\,z^{347}-111185370788928692562188342547 2\,z^{348}$$
$$+642905103357300590709530086016\,z^{349}+101102277159609312603209540912\,z^{350}$$
$$-174310889155475679147934630960\,z^{351}+165896880374294286843386749856\,z^{352}$$
$$+70280806264775255805313106384\,z^{353}-21445861460002132679415031123 2\,z^{354}$$
$$+76928304066612024379027144752\,z^{355}+54961620708687385025581294736\,z^{356}$$
$$-77561175641372505113493828768\,z^{357}+27747668052994332903302443760\,z^{358}$$
$$+57315214733438900939698829120\,z^{359}-47408214938155507656742837168\,z^{360}$$

$$
\begin{aligned}
&- 7241063973381732812403994888\,z^{361} + 16322069073607639580872447808\,z^{362} \\
&- 18744779067479718941528135792\,z^{363} + 11786209291153390681516254588\,z^{364} \\
&+ 14855990768738517632323786224\,z^{365} - 12888047115367947543839766376\,z^{366} \\
&- 2985048462831425518073257744\,z^{367} + 3014703798662258244830032968\,z^{368} \\
&- 2598659223244701870355917912\,z^{369} + 3388230459927774658487267376\,z^{370} \\
&+ 2159477015990999853671006328\,z^{371} - 2584105912442167312260922128\,z^{372} \\
&- 676978064269298626812699992\,z^{373} + 367729915941670120479440656\,z^{374} \\
&- 139906719397136219345346784\,z^{375} + 450613651681263257778146552\,z^{376} \\
&+ 74587362356229015169797160\,z^{377} - 260745355623099724386600528\,z^{378} \\
&- 35699934697632415524435128\,z^{379} + 25830616608272620930346096\,z^{380} \\
&+ 25646504196752645402976504\,z^{381} + 4126835833231748841461608\,z^{382} \\
&- 3812091295850044421001244 8\,z^{383} + 5202275392250133113479368\,z^{384} \\
&+ 12397236179629298338233024\,z^{385} - 811271168931299657826424\,z^{386} \\
&+ 6147445358372433630627440\,z^{387} - 711207763606158788323536 0\,z^{388} \\
&- 6637970054283164558812824\,z^{389} + 491925182614888846212794 0\,z^{390} \\
&+ 2290984924817251989340864\,z^{391} - 494337206175796703745312\,z^{392} \\
&+ 2608645163868503512266 24\,z^{393} - 864965640505963456073184\,z^{394} \\
&- 1094575801998141041469 12\,z^{395} + 532071619828609482857856\,z^{396} \\
&+ 338137740320917257328 96\,z^{397} - 70596433240561360068288\,z^{398} \\
&- 566394112529528192957 12\,z^{399} + 29276932901571227527 68\,z^{400} \\
&+ 72680792922606043441024\,z^{401} - 73914283943085865913 92\,z^{402} \\
&- 25864856193908841460576\,z^{403} - 2474824024479995966272\,z^{404} \\
&- 6435674875408797248864\,z^{405} + 9926494346662828583936\,z^{406} \\
&+ 5991969468726814278624\,z^{407} - 7264071840615763508960\,z^{408} \\
&- 2067517711153646844736\,z^{409} + 1005102040707397735136\,z^{410} \\
&+ 658639556679874306 56\,z^{411} + 678612833552585931296\,z^{412} \\
&- 306057440106686303872\,z^{413} - 373430203302901342208\,z^{414} \\
&+ 128449033981334977824\,z^{415} + 80226483486528144361\,z^{416} \\
&+ 62607855172065547420\,z^{417} - 64501787363681159602\,z^{418} \\
&- 59978044282799988716\,z^{419} + 61423246473483866730\,z^{420} \\
&+ 21054530566877345826\,z^{421} - 11211276190003271612\,z^{422} \\
&+ 3521044003172938638\,z^{423} - 8194857780641646036\,z^{424} \\
&+ 317823218190130290\,z^{425} + 4209683907587474828\,z^{426} \\
&- 530533781319829704\,z^{427} - 511157559390335034\,z^{428} \\
&- 545869565676150798\,z^{429} + 452256494275210124\,z^{430} \\
&+ 362053334096140298\,z^{431} - 472711549194613068\,z^{432} \\
&- 110526414340531994\,z^{433} + 96316032729547858\,z^{434} \\
&- 30016273801734064\,z^{435} + 48415387537143554\,z^{436} \\
&- 2121778914181536\,z^{437} - 16986529588384838\,z^{438} \\
&+ 4258375769295588\,z^{439} - 1086171754958056\,z^{440} \\
&+ 4495149168458162\,z^{441} - 3598511483708441\,z^{442} \\
&- 1402373323475496\,z^{443} + 2771178771678532\,z^{444} \\
&+ 163264327263664\,z^{445} - 360829106137204\,z^{446} \\
&- 19919203842564\,z^{447} - 64238514095728\,z^{448} \\
&+ 43262817392412\,z^{449} - 36875855092024\,z^{450} \\
&- 15962421645812\,z^{451} + 20983978483856\,z^{452} \\
&- 16446086996544\,z^{453} + 13685124009588\,z^{454} \\
&+ 1769933132236\,z^{455} - 6472600542024\,z^{456} \\
&+ 875568934860\,z^{457} - 257282329552\,z^{458} \\
&+ 948497834260\,z^{459} - 544631286308\,z^{460} \\
&- 146587938520\,z^{461} + 391875164548\,z^{462} \\
&- 42196557680\,z^{463} - 15806752628\,z^{464} \\
&- 23381389952\,z^{465} + 9579441616\,z^{466} \\
&+ 4165098652\,z^{467} - 10403175665\,z^{468}
\end{aligned}
$$

$+ 1267376812\, z^{469} + 775761214\, z^{470}$
$+ 253898940\, z^{471} - 70089094\, z^{472}$
$- 56792942\, z^{473} + 152254060\, z^{474}$
$- 20515130\, z^{475} - 16092724\, z^{476}$
$- 586798\, z^{477} + 162436\, z^{478}$
$+ 275624\, z^{479} - 1263434\, z^{480}$
$+ 173970\, z^{481} + 173596\, z^{482}$
$- 3942\, z^{483} - 1828\, z^{484}$
$+ 342\, z^{485} + 5442\, z^{486}$
$- 728\, z^{487} - 822\, z^{488}$
$- 16\, z^{489} + 10\, z^{490}$
$- 4\, z^{491} - 8\, z^{492}$
$+ 2\, z^{493} + z^{494}.$

$d = 7$

$V(7, n)$: [1, 2, 6, 24, 120, 720, 5040, 40320, 287280, 1865520, 11345160, 66349464, 381523758, 2193664790, 12764590275, 75796724309, 455383613924, 2750869551868, 16635586999056, 100439873614656, 604666567043712, 3629299734118656, 21736009354060800, 130082373922081536, 778592461165543296, 4662622196995104768, 27939363678933973680, 167502671965264890484, 1004517822465067933604, 6024645837183028067461].

$d = 8$

$V(8, n)$: [1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 2943360, 21898800, 152622000, 1017952680, 6623303544, 42700751022, 276054834902, 1805409270031, 12020754177001, 80930279045116, 548117873866228, 3720269813727312, 25239622338694272, 170893063638209664, 1154033453027033856, 7773411193506720000, 52264587965895740160, 351151367251083920640, 2359282339736877136896, 15857353030657628048256, 106637166825837550488000].

$d = 9$

$V(9, n)$: [1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 33022080, 277280640, 2184341040, 16427628720, 119892387720, 861175365144, 6157828055310, 44222780245622, 321113303226243, 2369364111428885, 17667206334000068, 132553643382927196, 997400200347756816, 7508509530032233152, 56460003482650313088, 423674310844459593216, 3171704573559141123840, 23692355860934470118400, 176700827837593631232000, 1316959907015491216634880].

$d = 10$

$V(10, n)$: [1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, 402796800, 3770686080, 33187593600, 278598101040, 2261952938160, 17986137205800, 141564484858104, 1112444773251726, 8787513806478134, 70146437009397871, 568128719132038153, 4647312969412825372, 38254281070116978580, 315956466319092418512, 2612682260760752048640, 21595706234798803330176, 178245404794110657897216, 1468252111498173531429120, 12068682678159535562019840].