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Planar Subgraph Isomorphism Revisited

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Abstract

Ten years after Eppstein's results on planar subgraph isomorphism for k-sized patterns, we improve the exponential term of the running time $2^{O(k \log k)} \cdot n$ of Eppstein's algorithm to $2^{O(k)}$ (keeping the term in n linear!) Next to deciding subgraph isomorphism, we can construct a solution and enumerate all solutions in the same asymptotic running time. We may list ω subgraphs with an additive term $O(\omega n)$ in the running time of our algorithm. For exact algorithms, this means we obtain a truly subexponential algorithm for patterns of size $O(\sqrt{n})$ of running time $2^{O(\sqrt{n})}$ imrpoving the former bound of $2^{O(\sqrt{n} \log n)}$.

1 Introduction

The GRAPH ISOMORPHISM problem is one crux of modern theoretic computer science; it is not known whether it belongs to P or NP[19]. The fastest exact algorithm for solving this problem with guaranteed solution is of running time $2^{O(\sqrt{n})}$ [2]. However, there exists an $O(n)$ algorithm [22] to decide if two planar graphs are isomorphic. For several graph classes GRAPH ISOMORPHISM has been proven to possess polynomial time algorithms, such as graphs embedded in surfaces of bounded genus ([27]), graphs of bounded treewidth k ([4] gives an $n^{O(k)}$ algorithm), and graphs with polyhedral embeddings in arbitrary surfaces [24].

While the realm of SUBGRAPH ISOMORPHISM is more clearly understood, its instances are harder to tackle. The problem is defined as follows. Given a *pattern* H and a graph G on n vertices, does G contain a subgraph that is isomorphic to H? SUBGRAPH ISOMORPHISM is a generalization of many important graph problems, such as HAMILTONICITY, LONGEST PATH, and CLIQUE.Contrary to GRAPH ISOMORPHISM, this problem is known to be NP-complete, even when restricted to planar graphs (since it is a generalization of PLANAR HAMILTONICITY, which is NP-complete [19]). Until now, the best known algorithm to solve SUBGRAPH ISOMORPHISM is the naïve approach with running time $O(n^{|H|})$. For pattern H of treewidth at most t, [1] give an algorithm of running time $2^{O(|H|)}n^{O(t)}$. [20] suggest a polynomial time algorithm for input graphs of bounded treewidth and patterns of $\log$-bounded fragmentation. For PLANAR SUBGRAPH ISOMORPHISM, given planar pattern and input graph, some considerable improvements have been made mostly during the 90's. The first improvement was provided by Plehn and Voigt [29], with running time $2^{O(|H| \log |H|)} n^{O(\sqrt{|H|})}$. As an application of the infamous Color-coding technique of Alon et al. [1], one can devise an algorithm of running time $2^{O(|H|)} n^{O(\sqrt{|H|})}$. The actual benchmark has been set by Eppstein [16] to $2^{O(|H| \log |H|)} n$, by employing graph decomposition methods, similar to the Baker-approach [3] for approximating NP-complete problems on planar graphs. Eppstein's algorithm is actually the first FPT-algorithm for PLANAR SUBGRAPH ISOMORPHISM with k as parameter ([15]).

For exact algorithms, if $|H| = f(|G|)$ for some sub-linear function f, the fastest algorithm can be obtained by straight-forward dynamic programming by implementing techniques and bounds presented in [18], and it has the running time $2^{O(\sqrt{n}\log n)}$.

Planar subgraph isomorphism problems with restricted pattern have been studied exhaustively: take for example triangles and K_4 in [23, 28], or outerplanar graphs [33, 26]. For a survey, please consider [16]. Let us add some more recent results to this list. *Bidimensionality Theory* employs results of the seminal Graph Minors Theory by Robertson and Seymour for planar graphs [31] and other structural graph classes to algorithmic graph theory (entry [7], for a survey [8]). Bidimensionality enables to create algorithms for a range of graph problems, when searching for specific subgraphs of size k, with running time subexponential in k and polynomial in n [13]. For k-LONGEST PATH on planar graphs, [14] give the first truly subexponential algorithm of running time $2^{O(\sqrt{k})}n + O(n^3)$. However, Bidimensionality Theory is heavily based on minor properties related to the graph problems considered. We can recognize paths as subgraphs by minor testing, but such methods do not work for subgraph isomorphism problems in general.

Our results. In this paper, we give an algorithm for PLANAR SUBGRAPH ISOMORPHISM for a fixed sized pattern H of running time $2^{O(|H|)}n$. That improves the formerly best algorithm of [16] in the exponential part of the running time, which stays linear in n and contains relatively small constants. In particular, we obtain the running time $O(2^{c\cdot|H|}n)$, where $c \approx 11$ if H is a planar triangulation, $c \approx 21$ if H is 3-connected or a plane graph, and $c \approx 30$ for any other H. If $|H| \in O(\sqrt{n})$, we obtain the fastest exact algorithm of running time $2^{O(\sqrt{n})}$ with similar constants.

Our algorithm is divided into two parts. First, we employ Baker's approach [3] with similar arguments as in the algorithm of [16] in combination with [34] for computing some structural graph decomposition. Our improvement results from the second part, a more involved dynamic programming design and analysis. We obtain our tool kit from the technique used in [14]. Namely, we employ the structural properties of graphs embedded in the plane or sphere. *Sphere-cut decompositions* are natural extensions of tree-decompositions to plane graphs, where the separators *cut* the sphere into disks. The boundary of the subgraph embedded in such disk has structural properties, that we can exploit for efficient dynamic programming. The idea in this paper can be described as follows: given the input plane graph G and *plane* pattern H. A separator in G cuts out a disk which may contain a plane subgraph of G isomorphic to a part of H. The boundary of that part forms some structural restricted subgraph, so-called *closed plane walks*. In an auxiliary planar structure constructed from H, these closed plane walks determine cycles. We show that the number of partial solutions computed in a dynamic programming step for a separator of size k is asymptotically bounded by the number of cycles in a planar graph of size $O(k)$. If H is not planar embedded, we have to argue in addition on the number and construction of planar triangulations and drawings of H. We obtain the following result:

Theorem 1.1 *Given a planar graph G on n vertices and a pattern H of size k. We can find a subgraph of G isomorphic to H in time $2^{O(k)}n$. We enumerate all subgraphs of G isomorphic to H in time $2^{O(k)}n$ and list ω subgraphs in time $2^{O(k)}n + O(\omega k)$.*

Organization. After giving some definitions, we argue in Section 2 how to obtain a sphere-cut decomposition of small width from the approach in [34]. In Section 3, we show how to find closed plane walks associated with triangulated graphs, and how they relate to cycles in some planar graphs. The set of cycles, on the other hand, gives us a restriction on the number of solutions to our sub-problems in dynamic programming. We extend our results in Section 4 to plane graphs and finally planar graphs using slightly more involved arguments. The entire algorithm together with the description and analysis of the dynamic programming is given in Section 5 with additional results.

2 Preliminaries

Let G, H be two graphs. We call G and H *isomorphic* if there exists a bijection $\nu : V(G) \to V(H)$ with $\{v,w\} \in E(G) \Leftrightarrow \{\nu(v), \nu(w)\} \in E(H)$. We call H *subgraph isomorphic to* G if there is a subgraph H' of G isomorphic to H.

Triangulations. Let G be a planar graph with vertex set $V(G)$, edge set $E(G)$ and face set $F(G)$. A subgraph of G, induced by the vertices and edges incident to a face $f \in F(G)$, is called a *bound* of f. If G is 2-connected, each bound of a face is a cycle. We call this cycle *face-cycle* (for further reading, see e.g. [11]). We call G *planar triangulated* or simply *triangulated* if every face in $F(G)$ is bounded by a triangle (a cycle of length three). Please do not confuse this notion with chordal graphs [21]! If H is a subgraph of triangulation H, we call G a *triangulation of* H.

Let G be planar triangulated. We add an set A of *auxiliary vertex* v into each region Δ of G. Let triangle $\{x, y, z\}$ bound Δ. We then connect v to each of x, y, z, obtaining three new regions bounded by $\{x, v, y\}$, $\{y, v, z\}$ and $\{x, v, z\}$. We name the resulting graph the *super-triangulation* G' of G.

Branch Decompositions. A *branch decomposition* $\langle T, \mu \rangle$ of a graph G consists of an unrooted ternary tree T (i.e., all internal vertices have degree three) and a bijection $\mu : L \to E(G)$ from the set L of leaves of T to the edge set of G. We define for every edge e of T the *middle set* $\mathrm{mid}(e) \subseteq V(G)$ as follows: Let T_1 and T_2 be the two connected components of $T \setminus \{e\}$. Then let G_i be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of G_1 and G_2, i.e., $\mathrm{mid}(e) := V(G_1) \cap V(G_2)$. The *width* bw of $\langle T, \mu \rangle$ is the maximum order of the middle sets over all edges of T, i.e., $\mathrm{bw}(\langle T, \mu \rangle) := \max\{|\mathrm{mid}(e)| : e \in T\}$. An optimal branch decomposition of G is defined by a tree T and a bijection μ which together provide the minimum width, the *branchwidth* $\mathrm{bw}(G)$.

Plane graphs. Let Σ be a sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. By a Σ-*plane graph*, or simply *plane graph* G we denote a planar graph G with the vertex set $V(G)$, the edge set $E(G)$ drawn (without crossings) in Σ. We may refer to G sometimes also as an *embedding* or *drawing* of a planar graph into sphere Σ. Every face of $F(G)$ is then represented by an open disk in Σ, which we call *region* Δ. By a *closed region* $\overline{\Delta}$, we denote the union of region Δ and its face-cycle bounding the corresponding face. Throughout the paper we denote by n the number of vertices of G. To simplify notations, whenever we talk about plane graphs, we do not distinguish between a vertex of the graph and the point of Σ used in the drawing to represent the vertex or between an edge and the open line segment representing it. For a subgraph H of a plane graph G, we refer to the drawing of G reduced to the vertices and edges of H as a *subdrawing* of G.

Sphere cut Decompositions. A *noose* of a Σ-plane graph G is a subset of Σ homeomorphic to a circle and meets G only in vertices. A *tight* noose N intersects with a closed region $\overline{\Delta}$ only in two vertices bounding a line segment of N in Δ. The length of a noose N is $|N \cap V(G)|$, the number of vertices it meets. Every noose N bounds two open discs Δ_1, Δ_2 in Σ, i.e., $\Delta_1 \cap \Delta_2 = \emptyset$ and $\Delta_1 \cup \Delta_2 \cup N = \Sigma$.

For a Σ-plane graph G, we define a *sphere cut decomposition* or *sc-decomposition* $\langle T, \mu, \pi \rangle$ as a branch decomposition such that for every edge e of T there exists a tight noose N_e bounding the two open discs Δ_1 and Δ_2 such that $G_i \subseteq \Delta_i \cup N_e$, $1 \le i \le 2$. Thus N_e meets G only in $\mathrm{mid}(e)$ and its length is $|\mathrm{mid}(e)|$. A clockwise traversal of N_e in the drawing of G defines a cyclic ordering π of $\mathrm{mid}(e)$. We always assume that the vertices of every middle set $\mathrm{mid}(e) = V(G_1) \cap V(G_2)$ are enumerated according to π.

Radial graphs. For a plane graph G we define a *radial graph* R_G as follows: R_G is a bipartite graph with the bipartition $F(G) \cup V(G)$. A vertex $v \in V(G)$ is adjacent in R_G to a vertex $f \in F(G)$ if and only if the vertex v is incident to the face f in the drawing of G. We note that radial graphs are 3-connected planar graphs and thus uniquely embedded in Σ.

A *plane walk* W of length k ($k \geq 1$) of a plane graph G is an alternating sequence $[v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k]$ of vertices and edges of G such that

- e_i is an edge between v_i, v_{i+1} for all $i < k$;
- duplications of vertices and edges may occur;
- when rotating in clockwise order around vertex $v \in W$, the edges of W incident to v appear in increasing order.

A walk is closed if $v_0 = v_k$. A *closed radial walk* W in radial graph R_G of G with vertex-set $\{V(G), F(G)\}$ is a closed plane walk with the restriction vertices of $\{W \cap V(G)\}$ appear only twice in W.

Observation 2.1 *A cycle in radial graph R_G forms a tight noose in plane graph G. A closed radial walk in R_G forms a noose in G. In particular, there is a bijection between the set of cycles in R_G and the set of tight nooses in G, and between the set of closed radial walks in R_G and the set of nooses in G, respectively.*

2.1 Sphere-cut decompositions of bounded width

Assume H is subgraph of G, $|V(H)| = k$ and let vertex $v \in V(H)$. Then H is a subgraph of the induced subgraph G^v of G, where $G^v = G[S]$ with $S \subseteq V(G)$ and $\forall w \in S$: dist$(v, w) \leq k$ (dist(v, w) denotes the length of a shortest path between v and w in G.) This observation will help us to shrink the search space of our algorithm by cutting out chunks of G of bounded width and solve subgraph isomorphism separately. With a similar approach to [3], we obtain a sc-decomposition of bounded width.

Lemma 2.2 *([34]) Let plane graph G have a rooted spanning tree of longest root-leaf-path of length at most ℓ. Then a sphere-cut decomposition of width $2\ell + 1$ can be found in time $O(\ell n)$.*

Proof. Determine a face f adjacent to root r as outer face of G. Let T be a spanning tree rooted at r that is determined by breadth first search. If the longest path from r to a leaf of T is ℓ then the distance d_f in the radial graph R_G from vertex f to any other (face)vertex x is at most $2\ell + 1$. This is due to the fact that in R_G there exists edge $\{f, r\}$, and a detour of at most two edges for every edge in T. [34] shows how to obtain a branch decomposition of width d_f out of a BFS spanning tree rooted at r of the radial graph. With exception of the leafs, which are naturally enclosed by a cycle of length four in R_G, the vertices of each non-leaf middle set form a cycle in the radial graph including outer face-vertex f and having length at most $2d_f$. This cycle hence forms a noose in G using d_f vertices. Thus, the branch decomposition computed by [34] and our notion of sphere-cut decomposition coincide. The algorithm presented in [34] runs in time linear in n for fixed ℓ. □

3 Triangulated subgraphs

For the ease of understanding, we first study the special case that our pattern H is a planar triangulation. We follow the idea that H has an unique embedding and thus we are equipped with extra topological tools for searching H in plane graph G. The fact that the faces are bounded by triangles, restricts furthermore the way nooses may intersect with H.

3.1 Nooses in planar triangulations

We will argue that each connected component of $G \backslash H$ is embedded in a region of H. Thus, any noose N of G intersects with a region of H at most once. For applying some counting arguments, we will relate the radial walks in R_H formed by N to cycles in an auxiliary graph structure. The short proofs of the two next lemmata are moved to Appendix 7.1.

Lemma 3.1 *Let triangulated graph H be a subgraph of plane graph G. For every region Δ of H, the triangle bounding Δ is bounding either a region of G too, or a connected component of G and is hence a separating triangle of G.*

Lemma 3.2 *Let H be a subgraph of G with radial graphs R_G and R_H. Every cycle C in R_G determines a closed radial walk W in R_H.*

In the remaining section, we want to give an upper bound on the number of closed radial walks. We will observe that any walk can be interpreted as a cycle of the super-triangulation of H.

Lemma 3.3 *Every closed radial walk W in R_H can be represented by a cycle in the radial graph $R_{H'}$ of the super-triangulation H' of H.*

Proof. As the radial walk W visits a vertex in $V(H) \cap V(R_H)$ once or none, it may also pass a region Δ only once. Let Δ be bounded by triangle $\{x, y, z\}$. If W pass Δ, we have to consider two case: W intersects with $\overline{\Delta}$ in three vertices or in two vertices. For the first case, assume without loss of generality that W enters Δ in x, touches y and leaves through z. We consider the super-triangulation H' and its radial graph $R_{H'}$. For region Δ bounded by $\{x, y, z\}$, let v be the inserted auxiliary vertex. This construction helps "imitating" the behavior of a walk W in R_H. Above intersection of walk W with region Δ results in the path in $R_{H'}$ from x to the face-vertex of $\{x, v, y\}$ then y, from y to face-vertex $\{y, v, z\}$ and then to vertex z. In the second case, W enters and leaves Δ without touching the third vertex. This situation is well represented in $R_{H'}$ by a path, e.g., from x to the face-vertex of $\{x, v, y\}$ then y. All in all, we obtain a cycle in $R_{H'}$. For an illustration, consider Figure 2 in the Appendix. □

Applying this result together with the following lemma, we obtain an upper bound on the number of closed radial walks in R_H by enumerating the cycles in $R_{H'}$.

Lemma 3.4 *([5]) There is no planar graph with more than $2^{1.53n}$ simple cycle. Let $\alpha \in [0, 1]$. For any planar graph G with f faces, the number of cycles of length αn is bounded by $(\frac{f}{\alpha n} + 1)^{\alpha n}$.*

3.2 Enumerating cycles: planar triangulations

In the heart of our algorithm, at every step of dynamic programming, we will compute all possibilities of how a tight noose N corresponding to a middle set of the sc-decomposition $\langle T, \mu, \pi \rangle$ of G intersects a plane subgraph H. If H is a subgraph of G, N forms a unique noose in H. Therefor, we want to find an upper bound on the number of nooses in H, that is, on the number of closed radial walks in R_H.

Lemma 3.5 *For every planar triangulation H on k vertices, the number of closed radial walks in radial graph R_H is $2^{O(k)}$.*

Proof. As we deduce from Lemma 3.3, this number is bounded by the number of cycles in $R_{H'}$. We give a rough bound that results from the number of cycles given in Lemma 3.4. Since H is a planar triangulation on k vertices, it has $2k-4$ faces. Recall from Lemma 3.1, that $G \setminus H$ is a collection of connected components each embedded in a different region of H. For a given planar triangulation H, let H' be its super-triangulation.We have $|V(H')| = 3k-4$, and thus, $|F(H')| = 6k-12$. Now, we are interested in how many cycles there can be in the radial graph $R_{H'}$. We observe that auxiliary vertices A in H' can be removed without harm, since they do not contribute to nooses in H. We give an upper bound on the total number of cycles in the residual graph $R_{H'} \setminus V_A$. With $7k-12$ vertices, we get with Lemma 3.4 an upper bound of $2^{10.71k}$. A tighter upper bound $2^{3.61k}$ we obtain in Appendix 7.2. □

4 Subdrawings and planar subgraphs

As long as pattern H is plane, we may use our arguments relating nooses in G with radial walks in R_H. However, a region in H may be bounded by a large face-cycle and contain several components in G. In order to reduce our counting problem to cycles, we will show that there exists a triangulation of H that does the trick. Applying fundamental results from enumerative combinatorics, we will also find here single exponential bounds. So far, we studied the problem when H being plane and thus having an unique radial graph. On the contrary, if H is planar but not embedded, we show that the number of radial graphs of H is bounded, too. Thus we count the total number of radial walks in every radial graph.

4.1 Nooses in plane graphs

We extend our studies from planar triangulations to (2-connected) plane graphs. We observe similar results to Lemma 3.1 and Lemma 3.2 :

Lemma 4.1 *Let plane graph H be a subdrawing of plane graph G. For every region Δ of H, the face-cycle C bounding Δ is bounding either a region of G too, or some connected components of G and is hence separating G.*

Lemma 4.2 *Given plane graphs G and H such that H is a subdrawing of G. Then every cycle C in R_G determines a unique closed radial walk in radial graph R_H.*

Proof. Let Δ be a region of H and be bounded by face-cycle $C = \{c_1, .., c_\ell\}$. Let v be the face-vertex for Δ in R_H. Any cycle C in R_G corresponds to a noose N. Suppose N intersects with Δ as one connected segment $S = [c_{i_1}, c_{i_2}, \ldots, c_{i_j}]$ with $i_1 < i_2 \ldots$ composed of a collection of line segments $]c_{i_h}, c_{i_{h+1}}[$ in Δ. Then S can be represented as part of radial walk $[c_{i_1}, v, c_{i_2}, v, \ldots, v, c_{i_j}]$. N might intersect with Δ as several segments $S_1, S_2, \ldots, S_g$. However, due to planarity, N is not self-intersecting and each S_i forms a part of the same radial walk. □

Lemma 4.3 *Given a plane graph H and a closed radial walk W in R_H. Then there exists a planar triangulation H' of H, such that W can be represented by a cycle in radial graph $R_{H''}$ of super-triangulation H'' of H'.*

Proof. Let a region Δ of H be bounded by face-cycle $C = \{c_1, .., c_\ell\}$. Let v be the face-vertex for Δ in R_H. Let $[c_{i_1}, v, c_{i_2}, v, \ldots, v, c_{i_j}]$ with $i_1 < i_2 \ldots$ be a connected part of $W \cap \Delta$. For every pair $\{c_{i_n}, c_{i_{n+1}}\}$ consecutively visited in W add an edge to H, if non-existing, and draw it through region Δ. Proceed for the entire W and obtain the new graph H^S. In the radial graph R_{H_S}, W corresponds to yet another closed radial walk W^S. Any pair $\{c_i, c_j\}$ of vertices that is consecutively visited by W_S, induces an edge in H_S. Thus we may triangulate H^S to H' preserving W^S. For the remaining arguments, we find the situation of Lemma 3.3 with H' triangulated. □

4.2 Enumerating cycles: polyhedral graphs

For two planar graphs G and H where $H \subset G$, if H is 3-connected, we know that H is uniquely embedded in every drawing of G. Thus, we simply assume G and H to be plane graphs. With Lemma 4.3, we will argue that the number of closed radial walks in R_H is bounded by the total amount of cycles in $R_{H''}$ for all possible triangulations H' of H. In Appendix 7.3, we will give an upper bound 2^{4k-6} on the cardinality of the set of planar triangulations of a 3-connected planar pattern H. However, we obtain an exact bound by [35]:

Lemma 4.4 *([35]) The number Φ of non-isomorphic maximal planar graphs on n vertices is approximately $2^{3.24n}$.*

Together with Lemma 4.2 and Lemma 4.3, we get analogously to Lemma 3.5:

Lemma 4.5 *For every plane graph H on k vertices, the number of closed radial walks in radial graph R_H is $2^{O(k)}$.*

Proof. For every of possibly $2^{3.24k}$ planar triangulations H' of H, we count the number of cycles in $R_{H''} \setminus A$ like in the proof of Lemma 3.5. According to Lemma 4.3, this gives the upper bound $2^{6.85k}$ on the number of closed radial walks in R_H. □

4.3 Enumerating cycles: planar graphs

The idea that helps for non-embedded graphs is that any planar graph is subgraph of some triangulation.

Tutte's triangulations. We are given the most general case, that pattern H is planar but not plane. We are confronted by the problem, that our arguments so far only work for H being plane and thus a subdrawing of plane graph G. We circumvent this situation by listing all radial graphs R_H of H. Considering R_H rather than H has two main advantages: R_H is 3-connected (for H 2-connected with $|V(H)| > 2$) and thus uniquely embedded. And R_H determines a unique drawing of H.

Lemma 4.6 *The number Φ_R of radial graphs R_G of a planar graph G on n vertices is bounded by $2^{6.24n}$.*

Proof. Given the set $\mathfrak{G}$ of non-isomorphic maximal planar graphs on n vertices of size approximately $2^{3.24n}$(Lemma 4.4). We check for each such graph which of its subgraphs G' with edge set of cardinality $E(G)$ is isomorphic to G. G' gives an embedding of G. Since $\mathfrak{G}$ contains any triangulation of any planar graph, and we may obtain a planar triangulation of a plane graph by triangulating its faces (see Section 7.3), we get *every* drawing of G. □

Together with Lemma 4.5 we get:

Lemma 4.7 *For every planar graph H on k vertices, the total number of closed radial walks in every radial graph R_H is $2^{O(k)}$.*

5 Dynamic programming

In Algorithm 5.1, we follow the arguments of the previous two sections and particularly the constructive proof of Lemma 4.6. Then we call the dynamic programming routine in Algorithm 5.2, that we describe now.

Algorithm 5.1: Main routine.

Input : Plane graph G, Pattern H of size k.

if *H triangulated or 3-connected* **then** Compute radial graph R_H.
  Return DP$(G, \{R_H\})$.
else if *H planar* **then** Compute set $\mathfrak{H}$ of maximal planar graphs on k vertices.
for *every $H' \in \mathfrak{H}$* **do**
  for *every subgraph I of H' on $|E(H)|$ edges* **do**
    if *I isomorphic to H* **then**
    Add radial graph R_I of I to set $\mathfrak{I}$ of all radial graphs of H.

Return DP$(G, \mathfrak{I})$.
else Return(NO).

Algorithm 5.2: Dynamic programming step: DP.

Input : Plane graph G; Set $\mathfrak{I}$ of plane radial graphs R_H of H .

1 Choose an arbitrary vertex v in G.
2 Partition $V(G)$ into $S_0 \cup S_1 \cup \ldots \cup S_\ell$ with $S_i = \{w \in V(G) : \text{dist}(v, w) = i\}$
3 **for** *every $G_i = G[S_i \cup \ldots \cup S_{i+k}]$ with $0 \leq i \leq \ell - k$* **do**
4   Compute sc-decomposition $\langle T, \mu, \pi \rangle$ of G_i.
5   Do dynamic programming on $\langle T, \mu, \pi \rangle$ to find subgraph H of G_i with $H \cap S_i \neq \emptyset$.

Dynamic programming on plane graphs. Partitioning the vertex set in Line 2 of Algorithm 5.2, is a similar approach to the well-known Baker-approach [3]. Every vertex set S_i contains the vertices of distance i to the chosen vertex v. $S_0 = \{v\}$ and ℓ is the maximum distance in G from v. The graphs G_i in Line 3 are induced by the sets $S_i, \ldots, S_{i+k}$. As in [16], we may argue that every vertex in G appears in at most k subgraphs G_i. We also may observe that for every G_i, the diameter of its radial graph R_{G_i} is bounded by k. We can apply Lemma 2.2 to each G_i to compute sc-decompositions $\langle T, \mu, \pi \rangle$, each of width $\leq 2k$.

Given the plane graphs $G := G_i$ and radial graph R_H of some plane graph H (or the collection $\mathfrak{I}$ of radial graphs of planar graph H). For dynamic programming, we root $\langle T, \mu, \pi \rangle$ at some node r. For each middle set $\text{mid}(e)$, the subgraph G_e of G is induced by the $\mu(L)$, the set of leaves L below e. The vertices of $\text{mid}(e)$ form a tight noose N that separates G_e from the residual graph. An orientation $\vec{N}$ of N is determined by order π. I.e., G_e lies on the right hand side of $\vec{N}$. In each step of dynamic programming, all solutions for a sub-problem in G_e are computed, namely subgraphs $H_{sub} \subseteq H$ isomorphic to subgraphs of G_e, subject to all possibilities of how $\text{mid}(e)$ contributes to an overall solution in G. In other words, every way the pattern H may lie in G relative to $\vec{N}$—how H might intersect with $V(N)$ or if entirely contained in one side of $\vec{N}$.

Algorithm 5.2 can easily be turned into an algorithm counting subgraph isomorphisms (similar to [16]), by using a counter in the dynamic programming. Using an inductive argument, for every subgraph G_i in Line 3 we only compute subgraphs H intersecting with vertices in S_i and thus omit double-counting. We can also adopt Algorithm 5.2 to list the subgraphs of G isomorphic to H.

Lemma 5.1 *The algorithm (5.1 and 5.2) for searching and enumerating planar patterns H of size k in a plane graph G takes time $2^{c \cdot k}n$. We can list ω patterns in time $2^{c \cdot k}n + \omega k$. Here, c is some small constant.*

Proof. First, we consider the problem of searching an isomorphic subgraph. Let us assume H is a plane graph with radial graph R_H with bipartition $V(H) \cup F(H)$. We define a coloring $c : V(G) \to \{0,1\}$ of the vertices of G, where $c(v) = 1$ means we guess $v \in V(H)$ else $c(v) = 0$ means we guess $v \in F(H)$. In the sc-decomposition, we first initialize the leaves of T with all possible colorings. We then map the vertices of $V(H)$ to the vertices in the leaf with color 1 and $F(H)$ to vertices of color 0. If two vertices v, w of $V(H)$ are mapped to both vertices in leaf ℓ, and $\{v, w\} \in E(H)$, then $\mu(\ell)$ determines an edge H. All in all, we have $O(k^2)$ mappings to each leaf.

Let N be a tight noose corresponding to a middle set of $\langle T, \mu, \pi \rangle$ and let us guess a coloring c for the vertices of $V(N)$. As shown for plane graphs in Lemma 4.2, for every middle set of $\langle T, \mu, \pi \rangle$, N determines a closed radial walk W in R_H (that might well be entirely included in one region of H). For N, we thus have a list of radial walks W of length $\leq 2|N|$ in R_H with $V(W) \in V(H) \cup F(H)$, that map the vertices of $W \cap V(H)$ to the guessed vertices w of $V(N)$ with $c(w) = 1$. Since the order π of both, vertices in $W \cap V(H)$ and $V(N)$ is given, we may choose an arbitrary vertex $v \in (V \cap W)$ and choose a mapping vertex $w \in V(N)$ with $c(w) = 1$. This determines the one-to-one mapping of the residual vertices given by order π. We also know that, if there are two or more in π consecutive vertices $v_1, v_2, \ldots, v_m$ with $c(v_i) = 0\ \forall i$, they must be in one single region of H. Thus, a single vertex v of $W \cap F(H)$ maps to consecutive vertices $v_1, v_2, \ldots, v_m$ with color 0.

When updating two middle sets in a dynamic programming step, we have the sc-decomposition property that the two nooses N_1, N_2 form a new noose N_3 in G, separating the union of the graph separated by N_1, N_2 [14]. Thus, the symmetric difference of two corresponding walks W_1, W_2 in R_H form a new radial walk W_3 determining N_3. If at some step, an entire subgraph H is formed, we exit the algorithm confirming. We are able to output the subgraph in G isomorphic to H by reconstructing the solution top-down in $\langle T, \mu, \pi \rangle$. If at root r no subgraph isomorphic to H has been found, we output 'FALSE'.

If pattern H is not plane, we compute in Algorithm 5.1 the set $\mathfrak{I}$ of non-isomorphic radial graphs of H. [25] show how to compute the set $\mathfrak{G}$ of non-isomorphic maximal planar graphs in time proportional to its size. The proof of Lemma 4.6 gives a construction of the set of radial graph given $\mathfrak{G}$. We bound the number of radial walks in every radial graph in $\mathfrak{I}$ as in Lemma 4.7. Thus in each step of dynamic programming, we map every suitable walk W in every radial graph R_H to N. □

The constant c in Lemma 5.1 for the different patterns, is rather small. We may calculate c for the different patterns with techniques used in [14]. For H being a planar triangulation, we have $c = 11.17$, for plane graph H, $c = 20.59$, for planar pattern H, $c = 29.55$.

Variations. We can apply our methods to other variations of the problem which are considered in [16]. Amongst others, we can find the girth g of any planar graph G on n vertices in time $2^{O(g)}n$ and a shortest separating cycle in $2^{O(|C|)}n$. Also, we solve the problem of finding a subgraph isomorphic to a *disconnected* pattern H of size k in time $2^{O(k)}n$.

6 Conclusion

We have shown how to use topological graph theory to improve the results on the already mentioned variations of PLANAR SUBGRAPH ISOMORPHISM, solving the open problems posed in [16] and [14]. Let us end with some outlook, limit and open problem. With the results of [17], [16] extends the feasible graph class from planar graphs to apex-minor-free graphs. This cannot be done with the tools presented here. However, [12] devise a truly subexponential algorithm for k-LONGEST PATH in H-minor-free graphs and thus apex-minor-free graphs, employing the structural theorem of Robertson and Seymour [32] and the results of [9, 6, 10]. Can the structure, [12] deal with, be exploited for our purposes?

It seems improbably to extend our work to obtain a subexponential algorithm. The first reason, mentioned in the introduction is that Bidimensionality applies to subgraphs with minor properties rather than to general subgraphs. Secondly, our enumerative bounds are either tight or of lower bound $2^{\Omega(k)}$. We want to pose the open problem: Is PLANAR SUBGRAPH ISOMORPHISM for *triangulated* pattern H solvable in time $2^{o(k)}n^{O(1)}$?

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7 Appendix

7.1 Proofs for Lemma 3.1 and Lemma 3.2

Proof of Lemma 3.1. Planar triangulated graphs are 3-connected and therefor uniquely embedded (see for example [11]). In particular, a triangulated subgraph H of G is uniquely embedded in *any* drawing of G. Suppose G is a plane graph. The triangle $\{x, y, z\}$ bounding a region Δ in H is a triangle in G as well. Since G is plane, no edges of G cross edge of $\{x, y, z\}$. If $\{x, y, z\}$ bounds a region in G, so it does in H. On the contrary, if $\{x, y, z\}$ bounds a region in H, it might separate the graph G. That is, after removal of the vertices $V(G) \cap \Delta$ from G, G has two connected components, the inside and the outside of Δ. $\square$

Proof of Lemma 3.2. Because of Observation 2.1, we can argue in terms of (tight) nooses and show that every (tight) noose in G determines a (unique) noose in H (after removing $G \setminus H$). Let a region Δ in H be bounded by triangle $\{x, y, z\}$. If Δ is a region of G too, then the segment of any noose in G crossing Δ, is a path through the face-vertex of Δ bounded by two vertices of x, y, z in both, R_G and R_H. Assume Δ is a region of H bounded by a triangle which is separating in G. Then we might have the following situation: a tight noose N in G crosses Δ bounded by $\{x, y, z\}$. Thereby, without loss of generality it enters Δ in x, touches y and leaves through z. For an illustration, consider Figure 1. In H this noose is not tight, and thus, in R_H part of a closed radial walk, visiting twice vertex v corresponding to Δ. $\square$

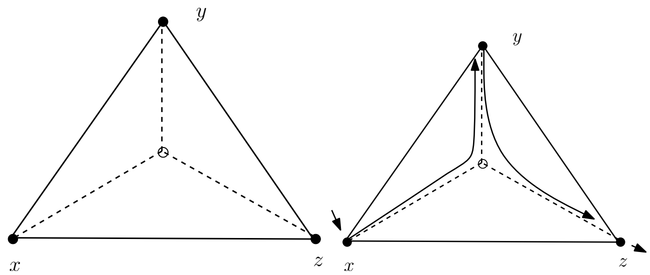

Figure 1: One the left, a region of H bounded by triangle $\{x, y, z\}$ with the part of the radial graph R_H indicated with dashed lines. To the right, the part of the oriented noose indicating the radial walk.

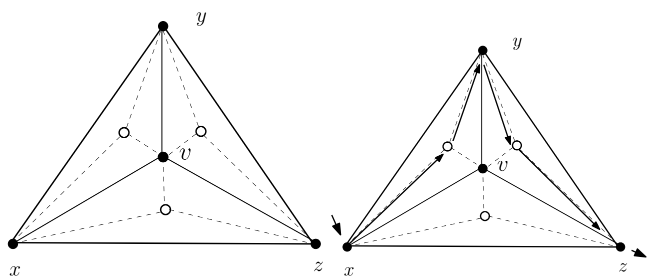

Figure 2: Figure in the proof of Lemma 3.3. On the left, a region of the super-triangulation H' with auxiliary vertex v and the part of the radial graph $R_{H''}$ indicated with dashed lines. On the right, an oriented part of a cycle in $R_{H'}$.

7.2 Tighter bound for Lemma 3.5

A tighter bound is obtained if one observe that H only has k vertices and thus the length of any closed radial walk in R_H is bounded by $2k$ since we visit vertices in H only once. A radial walk results in a cycle in $R_{H'}$ of same length. For applying the second result in Lemma 3.4, we need to bound the number $|F(R_{H'} \setminus A)|$. Observe that every edge in H contributes to one face of $R_{H'}$ and that every vertex v in A contributes to a face of $R_{H'} \setminus A$ after deletion of v. Thus, $|F(R_{H'} \setminus A)| = 5k - 10$. Now the total number Φ_R of cycles in $R_{H'} \setminus A$ is bounded as follows

$$\Phi_R \leq \sum_{i=4}^{2k} (\frac{5k-10}{i} + 1)^i.$$

We further observe

$$\Phi_R \leq 2k \cdot (\frac{5k-10}{2k} + 1)^{2k} \approx 2^{3.61k}.$$

7.3 Number of planar triangulations of 3-connected graphs.

Lemma 7.1 *Given a planar 3-connected graph G on n vertices. Then the number Φ_G of planar triangulations of G is bounded by 2^{4n-6}.*

Proof. Since every 3-connected planar graph is uniquely embedded, we can give an estimation of Φ_G by independently triangulating the faces of G. The resulting set of triangulations might contain a lot of isomorphic copies, and thus our upper bound on Φ_G will be far from optimum in some cases. Let ℓ_j be the length of face f_j. By Euler's formula j is bounded by $2-n+m$. Since every edge is adjacent to two faces, we have that

$$\sum_j \ell_j = 2m.$$

The problem of counting the triangulations of face f_j is well-known as *Euler's polygon division problem* and the solution is the Catalan number $CN(\ell_j - 2)$ [30]. Thus we get for the number of triangulations of G:

$$\Phi_G \leq \prod_j \mathrm{CN}(\ell_j - 2) \approx \prod_j 4^{\ell_j - 2} = 4^{\sum_j \ell_j - 2} \leq 4^{2m-2(2-n+m)} = 2^{4n-6}. \quad (1)$$

□We note that the proof is constructive and can be turned to an algorithm with running time proportional to Φ_G and polynomial per computed graph.