**Nynorsk**

# UNIVERSITETET I BERGEN

Det matematisk-naturvitskaplege fakultet

## Eksamen i MAT111 - Grunnkurs i Matematikk I

Torsdag 12. Mai 2022, kl 09-14

**Tillatne hjelpemiddel:** Lærebok (Adams & Essex, *Calculus – A Complete Course*) og kalkulator i samsvar med fakultetets reglar. Det er også eit formelark lagt ved eksamen.

Oppgåvesettet er på 2 sider og består av 10 oppgåver som alle tel likt. Les nøye gjennom oppgåvesettet. Alle svar skal grunngjevast. Det blir gitt godt med poeng for riktig framgangsmåte, sjølv om ein ikkje kjem fram til rett svar.

### Oppgåve 1:

Finn tangentlinja og normallinja i punktet $(0, 0)$ på kurva gitt ved likninga

$$y^2 \cos(2x) = \sin(y) - x.$$

### Oppgåve 2:

Vis ved induksjon at $8^n - 3^n$ er deleleg med 5 for alle heiltal $n \geq 1$.

### Oppgåve 3:

Finn konstanten $C$ som gjer funksjonen

$$f(x) = \begin{cases} \dfrac{\int_0^x e^{-t^2}\,dt}{x} & \text{når } x \neq 0 \\ C & \text{når } x = 0 \end{cases}$$

kontinuerleg i 0.

### Oppgåve 4:

Bruk den formelle ("$\epsilon$-$\delta$") definisjonen av grenser til å vise at

$$\lim_{x \to 7} \frac{14}{x} = 2.$$

**Oppgåve 5:**

Skriv det komplekse talet

$$z = \frac{6-2i}{2-4i}$$

på polarform, og finn $z^{16}$.

**Oppgåve 6:**

Bruk taylorpolynomet av grad 2 for funksjonen $f(x) = \frac{1}{\sqrt{x}}$ rundt 9 til å finne ei tilnærming til $\frac{1}{\sqrt{11}}$. Bruk så restleddet $E_2(11)$ til å vise at tilnærminga faktisk er for stor, og finn ei nedre grense for $\frac{1}{\sqrt{11}}$.

**Oppgåve 7:**

Løys initialverdiproblemet

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x^3}{y}, \\ y(1) = -1. \end{cases}$$

**Oppgåve 8:**

Løys det ueigentlege integralet

$$\int_{\ln(2)}^{\infty} \frac{e^x}{(e^x-1)^{11}}\, dx.$$

**Oppgåve 9:**

Forklar korfor funksjonen $f(x) = x^3 - 3x^2 + 4x - 1$ definert i $\mathbb{R}$ har minst eitt nullpunkt i intervallet $I = [0, 1]$? Har $f$ fleire nullpunkt? Kan du si om $f$ har ein inversfunksjon på $\mathbb{R}$?

**Oppgåve 10:**

Er funksjonen

$$f(x) = \begin{cases} x^3 \sin(\frac{1}{x}) & \text{når } x \neq 0, \\ 0 & \text{når } x = 0, \end{cases}$$

deriverbar i 0? Viss ja, kva er $f'(0)$?

**LYKKE TIL!**

Lars M. Salbu

**Bokmål**

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## DIFFERENTIATION RULES

$$\frac{d}{dx}\big(f(x)+g(x)\big)=f'(x)+g'(x) \qquad \frac{d}{dx}\big(cf(x)\big)=cf'(x) \qquad \frac{d}{dx}\big(f(x)g(x)\big)=f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{f'(x)}{\big(f(x)\big)^2} \qquad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)f'(x)-f(x)g'(x)}{\big(g(x)\big)^2} \qquad \frac{d}{dx}f\big(g(x)\big)=f'\big(g(x)\big)g'(x)$$

## ELEMENTARY DERIVATIVES

$$\frac{d}{dx}\frac{1}{x}=-\frac{1}{x^2} \qquad \frac{d}{dx}\sqrt{x}=\frac{1}{2\sqrt{x}} \qquad \frac{d}{dx}x^r=rx^{r-1}$$

$$\frac{d}{dx}e^x=e^x \qquad \frac{d}{dx}a^x=a^x\ln a \quad (a>0) \qquad \frac{d}{dx}\ln x=\frac{1}{x} \quad (x>0)$$

$$\frac{d}{dx}\sin x=\cos x \qquad \frac{d}{dx}\cos x=-\sin x \qquad \frac{d}{dx}\tan x=\sec^2 x$$

$$\frac{d}{dx}\sec x=\sec x\tan x \qquad \frac{d}{dx}\csc x=-\csc x\cot x \qquad \frac{d}{dx}\cot x=-\csc^2 x$$

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2} \qquad \frac{d}{dx}|x|=\operatorname{sgn}x=\frac{x}{|x|}$$

## TRIGONOMETRIC IDENTITIES

$$\sin^2 x+\cos^2 x=1 \qquad \sin(-x)=-\sin x \qquad \cos(-x)=\cos x$$

$$\sec^2 x=1+\tan^2 x \qquad \sin(\pi-x)=\sin x \qquad \cos(\pi-x)=-\cos x$$

$$\csc^2 x=1+\cot^2 x \qquad \sin\left(\frac{\pi}{2}-x\right)=\cos x \qquad \cos\left(\frac{\pi}{2}-x\right)=\sin x$$

$$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y \qquad \cos(x\pm y)=\cos x\cos y\mp\sin x\sin y \qquad \tan(x\pm y)=\frac{\tan x\pm\tan y}{1\mp\tan x\tan y}$$

$$\sin 2x=2\sin x\cos x \qquad \cos 2x=2\cos^2 x-1=1-2\sin^2 x$$

$$\sin^2 x=\frac{1-\cos 2x}{2} \qquad \cos^2 x=\frac{1+\cos 2x}{2}$$

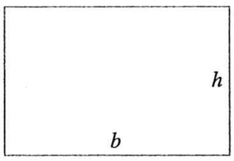
## QUADRATIC FORMULA

If $Ax^2+Bx+C=0$, then $x=\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$.

## GEOMETRIC FORMULAS

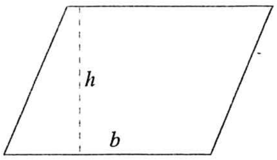
$A$ = area,
$b$ = base,
$h$ = height,
$C$ = circumference,
$V$ = volume,
$S$ = surface area

Rectangle



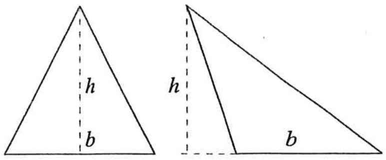

$A=bh$

Parallelogram



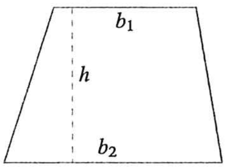

$A=bh$

Triangles



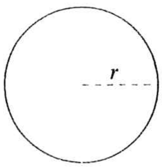

$A=\frac{1}{2}bh$

Trapezoid



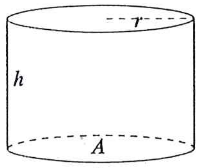

$A=\frac{1}{2}(b_1+b_2)h$

Circle



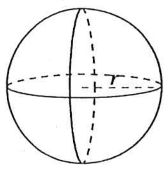

$A=\pi r^2,\ C=2\pi r$

Circular Cylinder



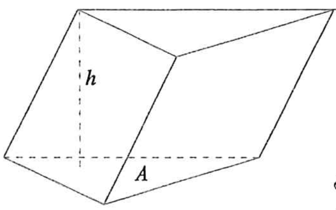

$V=Ah=\pi r^2h,\ S=Ch=2\pi rh$ (cylindrical wall)

Sphere



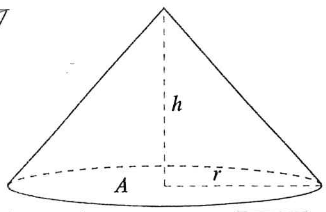

$V=\frac{4}{3}\pi r^3,\ S=4\pi r^2$

Prism



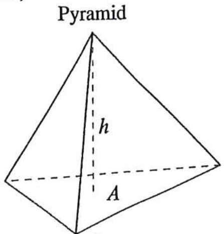

$V=Ah$

Circular Cone




$V=\frac{1}{3}Ah=\frac{1}{3}\pi r^2h,\ S=\pi r\sqrt{r^2+h^2}$ (conical wall)

Pyramid




$V=\frac{1}{3}Ah$

## VECTOR IDENTITIES

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$
$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$
$\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$
then

(dot product) $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

(cross product) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

length of $\mathbf{u} = |\mathbf{u}| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$

angle between $\mathbf{u}$ and $\mathbf{v}$ $= \cos^{-1}\left(\dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$

triple product identities $\quad \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}) \qquad \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$

## IDENTITIES INVOLVING GRADIENT, DIVERGENCE, CURL, AND LAPLACIAN

$\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$ ("del" or "nabla" operator)

$\nabla\phi(x,y,z) = \mathbf{grad}\,\phi(x,y,z) = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$

$\mathbf{F}(x,y,z) = F_1(x,y,z)\,\mathbf{i} + F_2(x,y,z)\,\mathbf{j} + F_3(x,y,z)\,\mathbf{k}$

$\nabla \bullet \mathbf{F}(x,y,z) = div\,\mathbf{F}(x,y,z) = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

$$\nabla \times \mathbf{F}(x,y,z) = curl\,\mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}$$

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$\nabla \bullet (\phi\mathbf{F}) = (\nabla\phi) \bullet \mathbf{F} + \phi\,(\nabla \bullet \mathbf{F})$

$\nabla \times (\phi\mathbf{F}) = (\nabla\phi) \times \mathbf{F} + \phi\,(\nabla \times \mathbf{F})$

$\nabla \times (\nabla\phi) = \mathbf{0}$ (**curl grad** $= \mathbf{0}$)

$\nabla^2\phi(x,y,z) = \nabla \bullet \nabla\phi(x,y,z) = \mathbf{div\,grad}\,\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

$\nabla \bullet (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \bullet \mathbf{G} - \mathbf{F} \bullet (\nabla \times \mathbf{G})$

$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \bullet \mathbf{G}) - \mathbf{G}(\nabla \bullet \mathbf{F}) - (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

$\nabla(\mathbf{F} \bullet \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \bullet \nabla)\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}$

$\nabla \bullet (\nabla \times \mathbf{F}) = 0$ (**div curl** $= 0$)

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \bullet \mathbf{F}) - \nabla^2\mathbf{F}$ (**curl curl** $=$ **grad div** $-$ laplacian)

## VERSIONS OF THE FUNDAMENTAL THEOREM OF CALCULUS

$\displaystyle\int_a^b f'(t)\,dt = f(b) - f(a)$ (the one-dimensional **Fundamental Theorem**)

$\displaystyle\int_C \mathbf{grad}\,\phi \bullet d\mathbf{r} = \phi\big(\mathbf{r}(b)\big) - \phi\big(\mathbf{r}(a)\big)$ if $C$ is the curve $\mathbf{r} = \mathbf{r}(t)$, $(a \le t \le b)$.

$\displaystyle\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C F_1(x,y)\,dx + F_2(x,y)\,dy$ where $C$ is the positively oriented boundary of $R$ (**Green's Theorem**)

$\displaystyle\iint_S \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS = \oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$ where $C$ is the oriented boundary of $S$. (**Stokes's Theorem**)

Three-dimensional versions: $S$ is the closed boundary of $D$, with outward normal $\hat{\mathbf{N}}$

$\displaystyle\iiint_D \mathbf{div\,F}\,dV = \oiint_S \mathbf{F} \bullet \hat{\mathbf{N}}\,dS$ **Divergence Theorem**

$\displaystyle\iiint_D \mathbf{curl\,F}\,dV = -\oiint_S \mathbf{F} \times \hat{\mathbf{N}}\,dS$

$\displaystyle\iiint_D \mathbf{grad}\,\phi\,dV = \oiint_S \phi\hat{\mathbf{N}}\,dS$

## FORMULAS RELATING TO CURVES IN 3-SPACE

Curve: $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

Velocity: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = v\hat{\mathbf{T}}$

Speed: $v = |\mathbf{v}| = \dfrac{ds}{dt}$

Arc length: $s = \displaystyle\int_{t_0}^{t} v\,dt$

Acceleration: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$

Tangential and normal components: $\mathbf{a} = \dfrac{dv}{dt}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}$

Unit tangent: $\hat{\mathbf{T}} = \dfrac{\mathbf{v}}{v}$

Binormal: $\hat{\mathbf{B}} = \dfrac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|}$

Normal: $\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}} = \dfrac{d\hat{\mathbf{T}}/dt}{|d\hat{\mathbf{T}}/dt|}$

Curvature: $\kappa = \dfrac{|\mathbf{v} \times \mathbf{a}|}{v^3}$

Radius of curvature: $\rho = \dfrac{1}{\kappa}$

Torsion: $\tau = \dfrac{(\mathbf{v} \times \mathbf{a}) \bullet (d\mathbf{a}/dt)}{|\mathbf{v} \times \mathbf{a}|^2}$

The Frenet-Serret formulas: $\dfrac{d\hat{\mathbf{T}}}{ds} = \kappa\hat{\mathbf{N}}, \qquad \dfrac{d\hat{\mathbf{N}}}{ds} = -\kappa\hat{\mathbf{T}} + \tau\hat{\mathbf{B}}, \qquad \dfrac{d\hat{\mathbf{B}}}{ds} = -\tau\hat{\mathbf{N}}$

## ORTHOGONAL CURVILINEAR COORDINATES

transformation: $x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w)$

position vector: $\mathbf{r} = x(u,v,w)\mathbf{i} + y(u,v,w)\mathbf{j} + z(u,v,w)\mathbf{k}$

scale factors: $h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right|, \quad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|, \quad h_w = \left|\frac{\partial \mathbf{r}}{\partial w}\right|$

local basis: $\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u}, \quad \hat{\mathbf{v}} = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v}, \quad \hat{\mathbf{w}} = \frac{1}{h_w}\frac{\partial \mathbf{r}}{\partial w}$

volume element: $dV = h_u h_v h_w \, du\, dv\, dw$

scalar field: $f(u,v,w)$

vector field: $\mathbf{F}(u,v,w) = F_u(u,v,w)\hat{\mathbf{u}} + F_v(u,v,w)\hat{\mathbf{v}} + F_w(u,v,w)\hat{\mathbf{w}}$

gradient: $\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\mathbf{w}}$

divergence: $\nabla \bullet \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(h_v h_w F_u\right) + \frac{\partial}{\partial v}\left(h_u h_w F_v\right) + \frac{\partial}{\partial w}\left(h_u h_v F_w\right)\right]$

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

curl: $\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & h_w\hat{\mathbf{w}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ F_u h_u & F_v h_v & F_w h_w \end{vmatrix}$

## PLANE POLAR COORDINATES

transformation: $x = r\cos\theta, \quad y = r\sin\theta$

position vector: $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$

scale factors: $h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r$

local basis: $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$

area element: $dA = r\,dr\,d\theta$

scalar field: $f(r,\theta)$

vector field: $\mathbf{F}(r,\theta) = F_r(r,\theta)\hat{\mathbf{r}} + F_\theta(r,\theta)\hat{\boldsymbol{\theta}}$

gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$

divergence: $\nabla \bullet \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$

laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$

curl: $\nabla \times \mathbf{F} = \left[\frac{\partial F_\theta}{\partial r} + \frac{F_\theta}{r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right]\mathbf{k}$

## CYLINDRICAL COORDINATES

transformation: $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

position vector: $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\mathbf{k}$

scale factors: $h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r, \quad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1$

local basis: $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \quad \hat{\mathbf{z}} = \mathbf{k}$

volume element: $dV = r\,dr\,d\theta\,dz$

surface area element (on $r = a$): $dS = a\,d\theta\,dz$

scalar field: $f(r,\theta,z)$

vector field: $\mathbf{F}(r,\theta,z) = F_r(r,\theta,z)\hat{\mathbf{r}} + F_\theta(r,\theta,z)\hat{\boldsymbol{\theta}} + F_z(r,\theta,z)\mathbf{k}$

gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\mathbf{k}$

divergence: $\nabla \bullet \mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}F_r + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$

laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$

curl: $\nabla \times \mathbf{F} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$

## SPHERICAL COORDINATES

transformation: $x = R\sin\phi\cos\theta, \quad y = R\sin\phi\sin\theta, \quad z = R\cos\phi$

position vector: $\mathbf{r} = R\sin\phi\cos\theta\,\mathbf{i} + R\sin\phi\sin\theta\,\mathbf{j} + R\cos\phi\mathbf{k}$

scale factors: $h_R = \left|\frac{\partial \mathbf{r}}{\partial R}\right| = 1, \quad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = R, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = R\sin\phi$

local basis: $\hat{\mathbf{R}} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}, \quad \hat{\boldsymbol{\phi}} = \cos\phi\cos\theta\,\mathbf{i} + \cos\phi\sin\theta\,\mathbf{j} - \sin\phi\,\mathbf{k}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$

volume element: $dV = R^2\sin\phi\,dR\,d\phi\,d\theta$

surface area element (on $R = a$): $dS = a^2\sin\phi\,d\theta\,d\phi$

scalar field: $f(R,\phi,\theta)$

vector field: $\mathbf{F}(R,\phi,\theta) = F_R(R,\phi,\theta)\hat{\mathbf{R}} + F_\phi(R,\phi,\theta)\hat{\boldsymbol{\phi}} + F_\theta(R,\phi,\theta)\hat{\boldsymbol{\theta}}$

gradient: $\nabla f = \frac{\partial f}{\partial R}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{1}{R\sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$

divergence: $\nabla \bullet \mathbf{F} = \frac{\partial F_R}{\partial R} + \frac{2}{R}F_R + \frac{1}{R}\frac{\partial F_\phi}{\partial \phi} + \frac{\cot\phi}{R}F_\phi + \frac{1}{R\sin\phi}\frac{\partial F_\theta}{\partial \theta}$

laplacian: $\nabla^2 f = \frac{\partial^2 f}{\partial R^2} + \frac{2}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cot\phi}{R^2}\frac{\partial f}{\partial \phi} + \frac{1}{R^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$

curl: $\nabla \times \mathbf{F} = \frac{1}{R^2\sin\phi}\begin{vmatrix} \hat{\mathbf{R}} & R\hat{\boldsymbol{\phi}} & R\sin\phi\hat{\boldsymbol{\theta}} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_R & RF_\phi & R\sin\phi F_\theta \end{vmatrix}$

## INTEGRATION RULES

$$\int (Af(x) + Bg(x))\,dx = A\int f(x)\,dx + B\int g(x)\,dx$$

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

$$\int U(x)\,dV(x) = U(x)\,V(x) - \int V(x)\,dU(x)$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

## ELEMENTARY INTEGRALS

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \text{ if } r \neq -1$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C \quad (a > 0,\ |x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad (a > 0)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C \quad (a > 0)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \quad (a > 0,\ |x| > a)$$

## TRIGONOMETRIC INTEGRALS

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

$$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

$$\int \tan^2 x\,dx = \tan x - x + C$$

$$\int \cot^2 x\,dx = -\cot x - x + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \csc^3 x\,dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$$

$$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C \text{ if } a^2 \neq b^2$$

$$\int \cos ax \cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C \text{ if } a^2 \neq b^2$$

$$\int \sin ax \cos bx\,dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C \text{ if } a^2 \neq b^2$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \cot^n x\,dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \csc^n x\,dx = \frac{-1}{n-1}\csc^{n-2} x \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx \text{ if } n \neq 1$$

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x \cos^m x\,dx \text{ if } n \neq -m$$

$$\int \sin^n x \cos^m x\,dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x \cos^{m-2} x\,dx \text{ if } m \neq -n$$

$$\int x \sin x\,dx = \sin x - x \cos x + C$$

$$\int x \cos x\,dx = \cos x + x \sin x + C$$

$$\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1} \cos x\,dx$$

$$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$$

## INTEGRALS INVOLVING $\sqrt{x^2 \pm a^2}$ $(a > 0)$

(If $\sqrt{x^2 - a^2}$, assume $x > a > 0$.)

$$\int \sqrt{x^2 \pm a^2}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{x^2 + a^2} - a \ln\left|\frac{a + \sqrt{x^2 + a^2}}{x}\right| + C$$

$$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \tan^{-1}\frac{\sqrt{x^2 - a^2}}{a} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\, dx = \frac{x}{8}(2x^2 \pm a^2)\sqrt{x^2 \pm a^2} - \frac{a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}}\, dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{\sqrt{x^2 \pm a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 \pm a^2}}{x} + \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x} + C$$

$$\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} + C$$

$$\int (x^2 \pm a^2)^{3/2}\, dx = \frac{x}{8}(2x^2 \pm 5a^2)\sqrt{x^2 \pm a^2} + \frac{3a^4}{8} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

## INTEGRALS INVOLVING $\sqrt{a^2 - x^2}$ $(a > 0, |x| < a)$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x}\, dx = \sqrt{a^2 - x^2} - a \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$$

$$\int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\int x^2\sqrt{a^2 - x^2}\, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^2}\, dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| + C$$

$$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + C$$

$$\int (a^2 - x^2)^{3/2}\, dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}\sin^{-1}\frac{x}{a} + C$$

## INTEGRALS OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\int \sin^{-1}x\, dx = x\sin^{-1}x + \sqrt{1 - x^2} + C$$

$$\int \tan^{-1}x\, dx = x\tan^{-1}x - \frac{1}{2}\ln(1 + x^2) + C$$

$$\int \sec^{-1}x\, dx = x\sec^{-1}x - \ln|x + \sqrt{x^2 - 1}| + C \quad (x > 1)$$

$$\int x\sin^{-1}x\, dx = \frac{1}{4}(2x^2 - 1)\sin^{-1}x + \frac{x}{4}\sqrt{1 - x^2} + C$$

$$\int x\tan^{-1}x\, dx = \frac{1}{2}(x^2 + 1)\tan^{-1}x - \frac{x}{2} + C$$

$$\int x\sec^{-1}x\, dx = \frac{x^2}{2}\sec^{-1}x - \frac{1}{2}\sqrt{x^2 - 1} + C \quad (x > 1)$$

$$\int x^n\sin^{-1}x\, dx = \frac{x^{n+1}}{n+1}\sin^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{\sqrt{1 - x^2}}\, dx + C \text{ if } n \neq -1$$

$$\int x^n\tan^{-1}x\, dx = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{1 + x^2}\, dx + C \text{ if } n \neq -1$$

$$\int x^n\sec^{-1}x\, dx = \frac{x^{n+1}}{n+1}\sec^{-1}x - \frac{1}{n+1}\int \frac{x^n}{\sqrt{x^2 - 1}}\, dx + C \quad (n \neq -1,\ x > 1)$$

## EXPONENTIAL AND LOGARITHMIC INTEGRALS

$$\int xe^x\, dx = (x - 1)e^x + C$$

$$\int x^n e^x\, dx = x^n e^x - n\int x^{n-1}e^x\, dx$$

$$\int \ln x\, dx = x\ln x - x + C$$

$$\int x^n \ln x\, dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C, \quad (n \neq -1)$$

$$\int x^n(\ln x)^m\, dx = \frac{x^{n+1}}{n+1}(\ln x)^m - \frac{m}{n+1}\int x^n(\ln x)^{m-1}\, dx \quad (n \neq -1)$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$$

## INTEGRALS OF HYPERBOLIC FUNCTIONS

$$\int \sinh x\, dx = \cosh x + C$$

$$\int \cosh x\, dx = \sinh x + C$$

$$\int \tanh x\, dx = \ln(\cosh x) + C$$

$$\int \coth x\, dx = \ln|\sinh x| + C$$

$$\int \operatorname{sech} x\, dx = 2\tan^{-1}(e^x) + C$$

$$\int \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right| + C$$

$$\int \sinh^2 x\, dx = \frac{1}{4}\sinh 2x - \frac{x}{2} + C$$

$$\int \cosh^2 x\, dx = \frac{1}{4}\sinh 2x + \frac{x}{2} + C$$

$$\int \tanh^2 x\, dx = x - \tanh x + C$$

$$\int \coth^2 x\, dx = x - \coth x + C$$

$$\int \operatorname{sech}^2 x\, dx = \tanh x + C$$

$$\int \operatorname{csch}^2 x\, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$$

## MISCELLANEOUS ALGEBRAIC INTEGRALS

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$$

$$\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$$

$$\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left(\frac{ax+b}{n+2} - \frac{b}{n+1}\right) + C \text{ if } n \neq -1, -2$$

$$\int \frac{dx}{(a^2 \pm x^2)^n} = \frac{1}{2a^2(n-1)}\left(\frac{x}{(a^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{dx}{(a^2 \pm x^2)^{n-1}}\right) \text{ if } n \neq 1$$

$$\int x\sqrt{ax+b}\,dx = \frac{2}{15a^2}(3ax-2b)(ax+b)^{3/2} + C$$

$$\int x^n\sqrt{ax+b}\,dx = \frac{2}{a(2n+3)}\left(x^n(ax+b)^{3/2} - nb\int x^{n-1}\sqrt{ax+b}\,dx\right)$$

$$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2}{3a^2}(ax-2b)\sqrt{ax+b} + C$$

$$\int \frac{x^n\,dx}{\sqrt{ax+b}} = \frac{2}{a(2n+1)}\left(x^n\sqrt{ax+b} - nb\int \frac{x^{n-1}}{\sqrt{ax+b}}\,dx\right)$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C \text{ if } b > 0$$

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} + C \text{ if } b < 0$$

$$\int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{b(n-1)x^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{dx}{x^{n-1}\sqrt{ax+b}} \text{ if } n \neq 1$$

$$\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + C \quad (a > 0)$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\frac{x-a}{a} + C \quad (a > 0)$$

$$\int x^n\sqrt{2ax-x^2}\,dx = -\frac{x^{n-1}(2ax-x^2)^{3/2}}{n+2} + \frac{(2n+1)a}{n+2}\int x^{n-1}\sqrt{2ax-x^2}\,dx$$

$$\int \frac{x^n\,dx}{\sqrt{2ax-x^2}} = -\frac{x^{n-1}}{n}\sqrt{2ax-x^2} + \frac{(2n-1)a}{n}\int \frac{x^{n-1}\,dx}{\sqrt{2ax-x^2}}$$

$$\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\frac{x-a}{a} + C \quad (a > 0)$$

$$\int \frac{\sqrt{2ax-x^2}}{x^n}\,dx = \frac{(2ax-x^2)^{3/2}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{2ax-x^2}}{x^{n-1}}\,dx$$

$$\int \frac{dx}{x^n\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{a(1-2n)x^n} + \frac{n-1}{(2n-1)a}\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}}$$

$$\int (\sqrt{2ax-x^2})^n\,dx = \frac{x-a}{n+1}(\sqrt{2ax-x^2})^n + \frac{na^2}{n+1}\int (\sqrt{2ax-x^2})^{n-2}\,dx \text{ if } n \neq -1$$

$$\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{x-a}{(n-2)a^2}(\sqrt{2ax-x^2})^{2-n} + \frac{n-3}{(n-2)a^2}\int \frac{dx}{(\sqrt{2ax-x^2})^{n-2}} \text{ if } n \neq 2$$

## DEFINITE INTEGRALS

$$\int_0^\infty x^n e^{-x}\,dx = n! \ (n \geq 0)$$

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \ a > 0$$

$$\int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a} \text{ if } a > 0$$

$$\int_0^\infty x^n e^{-ax^2}\,dx = \frac{n-1}{2a}\int_0^\infty x^{n-2}e^{-ax^2}\,dx \text{ if } a > 0,\ n \geq 2$$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\,\dfrac{\pi}{2} & \text{if } n \text{ is an even integer and } n \geq 2 \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n} & \text{if } n \text{ is an odd integer and } n \geq 3 \end{cases}$$