**Department of Mathematics,**
**University of Bergen**
**MAT213, Complex Functions**

# Exam MAT213 spring 2022

The exam will take place on June 08, 2022 from 09.00 to 14.00. The exam consists of two parts:

The first set of exercises is of the type "multiple choice". You have to choose the correct answer and mark it. This part assumes that you give answers on the computer.

The second set of exercises requires from you an ability to make a proof of a statement.

Maximal points 100.

## 1. Real and imaginary parts, 6 points

The real and imaginary parts of the complex number

$$\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}$$

are given by

- • $\operatorname{Re}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = \frac{e^{\frac{\pi}{4}}}{\sqrt{2}}, \quad \operatorname{Im}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = -\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$
- ○ $\operatorname{Re}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = \frac{e^{\frac{\pi}{4}}}{\sqrt{2}}, \quad \operatorname{Im}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = \frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$
- ○ $\operatorname{Re}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = -\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}, \quad \operatorname{Im}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = -\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$
- ○ $\operatorname{Re}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = \frac{e^{\frac{\pi}{4}}}{\sqrt{2}}, \quad \operatorname{Im}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = -i\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$
- ○ $\operatorname{Re}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = -\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}, \quad \operatorname{Im}\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = -i\frac{e^{\frac{\pi}{4}}}{\sqrt{2}}$

Solution. Note that

$$\left|\frac{1-i}{\sqrt{2}}\right| = 1, \quad \arg(\frac{1-i}{\sqrt{2}}) = -\frac{\pi}{4}.$$

By the definition of the power function we obtain

$$\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = e^{(1+i)\log\left(\frac{1-i}{\sqrt{2}}\right)} = e^{(1+i)\left(\ln|\frac{1-i}{\sqrt{2}}|-i\frac{\pi}{4}\right)} = e^{\frac{\pi}{4}}e^{-i\frac{\pi}{4}} = \frac{e^{\frac{\pi}{4}}}{\sqrt{2}}(1-i).$$

## 2. Radius of convergence, 6 points

The region of the convergence for the series

$$\sum_{n=-\infty}^{+\infty} \frac{z^n}{2^{|n|}}$$

is given by

- ● $\frac{1}{2} < |z| < 2$
- ○ $0 < |z| < 2$
- ○ $\frac{1}{2} < |z|$
- ○ $0 < |z|$
- ○ none of them

Solution. The analytic part

$$\sum_{n=0}^{+\infty} \frac{z^n}{2^n}$$

converges in the disc $|z| < 2$ by the ratio test. The principal part

$$\sum_{n=-\infty}^{n=1} \frac{1}{z^n 2^n}$$

converges in the domain $|z| > \frac{1}{2}$ by the ration test.

## 3. Analytic and differentiable functions, 6 points

Consider the function

$$f(z) = x^3 + i(1-y)^3 \quad \text{for} \quad z = x + iy.$$

Choose what is true

- ● The function is not analytic anywhere in $\mathbb{C}$. The function is complex differentiable at $z = i$.
- ○ The function is analytic at $z = i$. The function is complex differentiable at $z = i$.
- ○ The function is complex differentiable at $z = i$ and therefore it is analytic at $z = i$.
- ○ The function is not analytic, but complex differentiable everywhere.
- ○ The function is not differentiable at $z = i$, and therefore is not analytic at $z = i$.

Solution. Verifying the Cauchy - Riemann equations, we find

$$u_x = v_y \quad \text{if} \quad 3x^2 = -3(1-y)^2,$$

which is only satisfied at $x = 0$, $y = 1$. On the other hand,

$$u_y = -v_x \quad \text{if} \quad 0 = 0,$$

which holds everywhere. Note also that the components of $f(z)$, and all its first order partial derivatives exist everywhere. Since the Cauchy-Riemann equations only hold at $z = x+iy = i$, the function $f(z)$ is only differentiable (in the complex sense) at $z = i$. Hence, in particular, it is not differentiable on any neighbourhood of any point, and therefore is nowhere analytic.

## 4. Entire functions, 6 points

Consider the functions of the complex variable $z = x + iy$

$$f_1 = 2xy + i(x^2 + y^2), \quad f_2 = e^y e^{ix}, \quad f_3 = \frac{1}{1 + |z|^2},$$

$$f_4 = \cos z \sinh z, \quad f_5 = \sum_{n=0}^{\infty} \frac{z^n}{(3n)!}.$$

The following functions are entire

- ● $f_4, \ f_5$
- ○ $f_2, \ f_4, \ f_5$
- ○ $f_1, \ f_2, \ f_4$
- ○ $f_1, \ f_2, \ f_3$
- ○ $f_1, \ f_4, \ f_5$

Solution. The functions $f_1, f_2, f_3$ do not satisfy the Cauchy–Riemann equations in whole $\mathbb{C}$, $f_4$ is a combination of entire functions, the radius of convergence of $f_5$ is $\infty$.

## 5. Singularities, 6 points

Choose the correct information about the singularities of the function

$$f(z) = \frac{z^3+1}{z^2(z+1)}.$$

- $z=0$ is a pole of order 2, $z=-1$ is removable singularity
- ○ $z=0$ is a pole of order 2, $z=-1$ is a simple pole
- ○ $z=0$ is a pole of order 2, $z=-1$ is a zero
- ○ $z=\infty$ is a zero of multiplicity 2, $z=-1$ is a simple pole
- ○ $z=0$ is an essential singularity, $z=-1$ is a removable singularity

Solution. Since $z^3+1 = (z+1)(z^2-z+1)$, we can write the function in the form

$$f(z) = \frac{z^3+1}{z^2(z+1)} = \frac{z^2-z+1}{z^2}.$$

## 6. Cauchy integral formula, 6 points

Let $C$ be a positively oriented boundary of the square with the vertices at the points

$$1+i, \quad -1+i, \quad -1-i, \quad 1-i.$$

Let $a \in \mathbb{C}$ be a fixed complex number and $n \in \mathbb{N}$. The value of the integral

$$\frac{1}{2\pi i}\int_C \frac{e^{az}}{z^{n+1}}\,dz$$

is given by

- ● $\frac{a^n}{n!}$
- ○ $\frac{a^n}{n}$
- ○ $a^n$
- ○ $\frac{ae^{az}}{2\pi i n!}$
- ○ $\frac{a^n}{2\pi n!}$

Solution. By the Cauchy integral formula

$$\frac{1}{2\pi i}\int_C \frac{e^{az}}{z^{n+1}}\,dz = \frac{1}{n!}\frac{d^n}{dz^n}(e^{az}) = \frac{a^n}{n!}$$

## 7. Harmonic function, 6 points

Consider the function
$$f(z) = \ln(|z|).$$
Choose the true statement

- ● The function is harmonic and has a harmonic conjugate $\arctan \frac{y}{x}$.
- ○ The function is harmonic and has a harmonic conjugate $2 \arctan \frac{y}{x}$.
- ○ The function is not harmonic, and therefore has no harmonic conjugate.
- ○ The function is not harmonic because it is complex valued.
- ○ The function is harmonic and has a harmonic conjugate $f(z) = \ln(\bar{z})$.

Solution. Let us write $u(x, y) = \frac{1}{2} \ln(x^2 + y^2)$. The function is harmonic because of
$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2},$$
$$u_{xx} = \frac{(y^2 - x^2)}{(x^2 + y^2)^2}, \quad u_{yy} = \frac{(x^2 - y^2)}{(x^2 + y^2)^2}.$$
Let us find the harmonic conjugate. Since $u_x = v_y$ we have
$$v(x, y) = \int \frac{x}{x^2 + y^2} dy = \arctan \frac{y}{x} + g(x).$$
So we have
$$v_x = \frac{-y}{x^2 + y^2} + g'(x) = -u_y = -\frac{y}{x^2 + y^2}.$$
It shows that $g'(x) = 0$ and therefore the function $g(x)$ is constant. We conclude that we can take
$$v(x, y) = \arctan \frac{y}{x}.$$

## 8. Evaluation of the integral, 6 points

Let the parametrisation of the path $C\colon [0,1] \to \mathbb{C}$ be given by $C(t) = 2t + it(t-1)$. The value of the integral

$$\int_C ze^z \, dz$$

is equal to

- $e^2 + 1$
- $e^2 - 1$
- $2e^2$
- $(2t + it(t-1))e^{2t+it(t-1)}$
- $te^t$

Solution. We have $C(0) = 0$ and $C(1) = 2$. The function $ze^z$ is entire function and therefore it has antiderivative. The antiderivative of $ze^z$ can be calculated by the integration by parts

$$\int ze^z dz = ze^z - e^z.$$

Then

$$\int_C ze^z \, dz = \left(ze^z - e^z\right)|_0^2 = e^2 + 1.$$

## 9. Integral, 6 points

The value of the integral

$$\int_0^\pi \frac{d\theta}{2-\cos\theta}$$

is equal to

- ● $\frac{\pi}{\sqrt{3}}$
- ○ $i\frac{\pi}{\sqrt{3}}$
- ○ $-\frac{\pi}{\sqrt{3}}$
- ○ $\frac{1}{2\sqrt{3}}$
- ○ $\frac{i}{2\sqrt{3}}$

Solution. We have

$$\begin{aligned}\int_0^\pi \frac{d\theta}{2-\cos\theta} &= \frac{1}{2}\int_\pi^\pi \frac{d\theta}{2-\cos\theta}\\ &= \int_{|z|=1} \frac{idz}{z^2-4z+1}\\ &= -2\pi \operatorname*{Res}_{z=2-\sqrt{3}} \frac{1}{z^2-4z+1}\\ &= \frac{\pi}{\sqrt{3}}.\end{aligned}$$

## 10. Argument principle, 6 points

Let $f(z) = \frac{(z-2)^2 z^3}{(z+5)^3(z+1)^3(z-1)^4}$. The value of the integral

$$\int\limits_{|z|=3} f'(z)\frac{dz}{f(z)}$$

is equal to

- $\bullet$ $-4\pi i$
- $\circ$ $-2$
- $\circ$ $24\pi i$
- $\circ$ $12$
- $\circ$ $-32\pi i$

Solution. By the argument principle we need to calculate the number of zeros and number of poles, counting multiplicities inside of the disc $|z = 3|$. We have zeros at

$$z = 2 \quad \text{of multiplicity} \quad 2, \qquad z = 0 \quad \text{of multiplicity} \quad 3$$

We have poles at

$$z = -1 \quad \text{of multiplicity} \quad 3, \qquad z = 1 \quad \text{of multiplicity} \quad 4.$$

Thus

$$\int_{|z=3|} \frac{f'(z)}{f(z)}\, dz = 2\pi i(2 + 3 - (3 + 4)) = -4\pi i.$$

## 11. Rouche's theorem, 10 points

Suppose that a function $f$ is analytic in the closed unit disc. Suppose also that $|f(z)| < 1$ for $z$ on the boundary of the unit disc. Show that there is only one point in the unit disc satisfying the equation $f(z) = z$.

Solution Consider the function $h(z) = f(z) - z$. Note that on the boundary of the unit disc we have

$$| - z| = 1 > |f(z)|$$

The function $z$ has only one zero of multiplicity one in the unit disc. Therefore the function $h(z) = -z + f(z)$ has also only one zero inside of the unit disc by the Rouche theorem. Thus the equation $f(z) = z$ has only one solution.

## 12. Liouville's theorem, 10 points

Suppose that $f(z)$ is entire function and $|f(z)| > 1$ for all $z$. Show that $f(z)$ is constant.

Solution. Since $|f(z)| > 1$ for all $z$, the function $f(z)$ is not vanishing. Therefore $g(z) = \frac{1}{f(z)}$ is an entire function. Moreover it is bounded, since $|g(z)| < 1$. By the Liouville theorem, the function $g(z)$ is constant and therefore $f(z)$ is also constant.

## 13. Rational transformations, 10 points

Let $\Omega$ be the third quadrant $\{x < 0,\ y < 0\}$ of the $z$-plane. Find the image of $\Omega$ under the transformation

$$w = \frac{z+i}{z-i}$$

in $w$-plane. Sketch the domain $\Omega$ in the $z$-plane and the image of $\Omega$ in $w$-plane and label the corresponding portions of the boundary.

Solution: A rational transformation maps circles and lines into circles and lines. Circles and lines are determined by three distinct points. We have

$$z = 0 \mapsto w = -1, \quad z = -1 \mapsto w = -i,$$

$$z = -i \mapsto w = 0, \quad z = \infty \mapsto w = 1.$$

As a result the ray $x < 0, y = 0$ is mapped to the semicircle passing through the points $(-1, 0)$, $(0, -1)$, $(1, 0)$. The ray $x = 0, y < 0$ is mapped to the interval $(-1, 1)$ of the $x$-axis. Since $z = i \mapsto w = \infty$

we conclude that the interior of the third quadrant is mapped to the interior of the unit half disc in the lower half $w$-plane.

## 14. Cauchy formula, 10 points

(a) Assume that $f(z)$ is analytic in a simply connected domain $\Omega$ and $C$ is a simple closed contour in $\Omega$. Let $z_0 \in \Omega$ and $z_0 \notin C$. Prove that

$$\int_C \frac{f'(z)}{z - z_0}\, dz = \int_C \frac{f(z)}{(z - z_0)^2}\, dz.$$

(b) Is the formula true for a domain that is not simply connected? Explain why.

Solution: (a) The function $f'(z)$ is analytic inside and on the simple closed contour. We apply the Cauchy integral formula for $f'(z)$ and obtain

$$\int_C \frac{f'(z)}{z - z_0}\, dz = 2\pi i f'(z_0).$$

We also can apply the generalised Cauchy integral formula for $f(z)$ and obtain

$$\int_C \frac{f(z)}{(z - z_0)^2}\, dz = 2\pi i f'(z_0).$$

(b) Yes the formula is true. We denote $g(z) = \frac{f(z)}{z - z_0}$. Then

$$g'(z) = \frac{f'(z)(z - z_0) - f(z)}{(z - z_0)^2} = \frac{f'(z)}{z - z_0} - \frac{f(z)}{(z - z_0)^2}.$$

and $g'(z)$ is analytic around (in a neighbourhood) of the contour $C$. Then for any parametrisation $C(t) = z(t) = (x(t) + iy(t))$, $t \in (a, b)$ we obtain

$$\begin{aligned}
\int_C \Big(\frac{f'(z)}{z - z_0} - \frac{f(z)}{(z - z_0)^2}\Big)\, dz &= \int_C g'(z)\, dz \\
&= \int_a^b g'(z(t)) z'(t)\, dt = \int_a^b \frac{d}{dt} g(t)\, dt \\
&= g(t)\big|_a^b = g(z(a)) - g(z(b)) = 0
\end{aligned}$$

since the contour $C$ is closed.