UNIVERSITY OF BERGEN
Faculty of Mathematics and Natural Sciences

Exam in MAT220 - Algebra

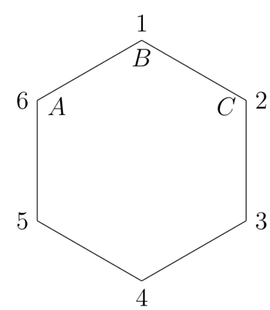
Exam available Friday May 21. Deadline for handing in written solution Wednesday May 26 at 09:00
Oral exam in the period May 26 to Friday May 28.

Exercise 1

1. Decide if the quotient ring
$$\mathbb{Q}[x]/(x^6 + 250x^4 + 35x + 15)$$
is an integral domain.
2. Decide if the quotient ring
$$\mathbb{Q}[x]/(x^6 + x^3 + x^2 + x)$$
is an integral domain.
3. Find a prime ideal in $\mathbb{Z}_{120}$.
4. Give an example of an integral domain R and a unit u in R such that $3u = 0$.

Exercise 2

We consider the dihedral group D_6 consisting of the symmetries by rotation and reflection of the regular hexagon.



Write s for clockwise rotation by 60 degrees. We consider s as the 6-cycle $s = (1, 2, 3, 4, 5, 6)$ rotating the corner marked B from the position 1 to the position 2. Here the letters are fixed to the corners of the hexagon while the numbers from 1 to 6 are painted on a fixed wall.

1. Let $\sigma = (2, 6)(3, 4) \in S_6$. Is σ an element of D_6? Justify your answer.
2. List all the elements of D_6 written as products of transpositions.
3. Let $\tau = (2, 6)(3, 5)$ considered as an element of D_6 and let
$$\lambda_\tau : S_6 \to S_6$$
be the function given by $\lambda_\tau(\sigma) = \tau\sigma$. Explain how the function λ_τ gives a bijection $f : D_6 \to D_6$.

4. Let $f\colon D_6 \to D_6$ be the bijection described in question 3. Let R be the set of rotations in D_6 and let S be the set of reflections in D_6 (speilinger in Norwegian). Explain how and why f gives a bijection between R and S. Is S a subgroup of D_6? Justify your answer.
5. Find a subgroup H of D_6 of order 6. Is this subgroup normal?

Exercise 3

1. List all abelean groups of order 72 up to isomorphism. In each case give both the elementary divisors and the invariant factors.
2. Find an element of every possible order in the group $\mathbb{Z}_{72}$.
3. Give an example of an injective group homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{72}$. Does there exist an injective group homomorphism from $\mathbb{Z}_{16}$ to $\mathbb{Z}_{72}$?
4. Find a non-abelian group of order 72.

Exercise 4

Let u be the complex number $u = (1 + i\sqrt{3})/2$ and let F be the field $F = \mathbb{Q}(u)$.

1. Describe an injective group homomorphism from $\mathbb{Z}_6$ to the group of units in F.
2. Find the minimal polynomial of u over $\mathbb{Q}$.
3. What is the degree of the field extension $\mathbb{Q} \subseteq \mathbb{Q}(u, \sqrt{2})$?

Exercise 5

Let M_3 be the ring of all real 3×3-matrices and let G_3 be the group of units in M_3.

1. Show that the function

$$\varphi\colon S_3 \to G_3$$

taking $\sigma \in S_3$ to the matrix $\varphi(\sigma) = (a_{ij})$ where a_{ij} is the number in the i-th row and j-th columnn and

$$a_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

is a group homomorphism.

2. Use that the determinant $\det\colon G_3 \to \mathbb{R}^*$ is a group homomorphism and that

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

to show that if σ is an even permutation then $\det(\varphi(\sigma)) = 1$ and that if σ is an odd permutation, then $\det(\varphi(\sigma)) = -1$.

3. Explain why the subgroup of G_3 consisting of matrices with positive determinant is a normal subgroup.

Exercise 6

Let $F \subseteq K$ be a field extension.

1. Explain why $[K : F] = 1$ if and only if $K = F$.
2. Let $u \in K$ be a transcendental element. Describe an injective ring homomorphism from $F[x]$ to K.
3. Given $z \in \mathbb{C}$, what can you say about the degree of the field extension $\mathbb{R} \subseteq \mathbb{R}(z)$?
4. Given a field extension $\mathbb{C} \subseteq K$, what can you say about K?
5. Does there exist a field extension $\mathbb{R} \subseteq K$ of degree 4?