ENGLISH MAT111 - Spring 2015

UNIVERSITY OF BERGEN

Department of Mathematics and Natural Sciences

Final Exam, MAT 111: Calculus I-Solutions

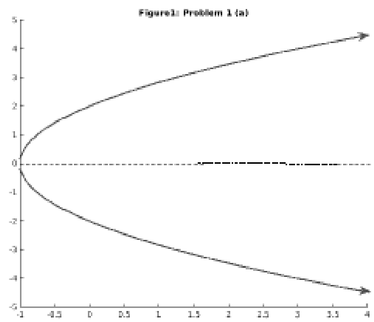
Wednesday 13. May 2015, 9 a.m. to 2 p.m.

Allowed aids: Textbook (Adams & Essex: Calculus - a complete course) and an approved calculator.

Problem 1.

(a) (10 points) Describe geometrically and sketch the curve given by $2|z| = z + \bar{z} + 4$ where z is a complex number.

Solution:
Let $z = x + iy$, thus $|z| = \sqrt{x^2 + y^2}$ and $z + \bar{z} = 2x$. By substituting in the given equation we have $2\sqrt{x^2+y^2} = 2(x+2)$. Take square of both side and simplify the equation we will find $y^2 = 4x + 4$ which is a sideway parabola at vertex$(-1, 0)$.■



(b) (10 points) Find x and y so that $\frac{x}{1+i}+\frac{y}{2-i}=2+4i$.

Solution:
$\frac{x(1-i)}{2}+\frac{y(2+i)}{5}=2+4i \rightarrow (5x+4y)+(2y-5x)i=20+40i$

$$\begin{cases} 5x+4y=20 \\ 2y-5x=40 \end{cases} \rightarrow y=10, \quad x=-4. \blacksquare$$

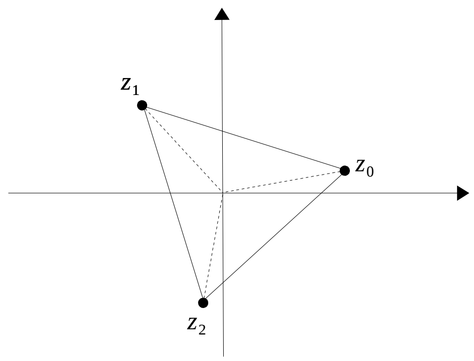
(c) (10 points) Find the three cube roots of $1+i$, and sketch the roots in the complex plane.

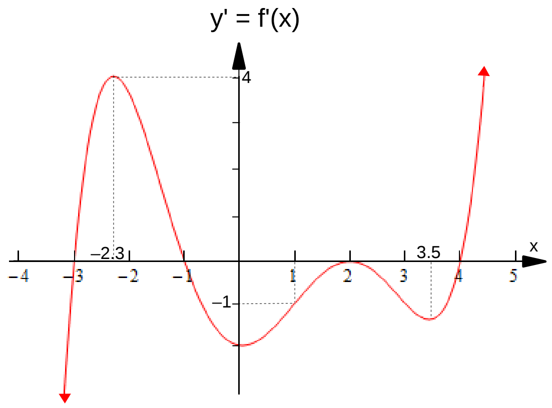
Solution:
Let $w=1+i$. Since $r_w=\sqrt{2}$ and $arg(1+i)=\frac{\pi}{4}$, the polar representation of $1+i$ is $\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$. Now we are looking for roots z_k which satisfy in equation $z_k^3=1+i$. Let consider $z_k=re^{i\theta}$ then
$r^3e^{3i\theta}=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}) \rightarrow r^3=\sqrt{2} \rightarrow r=\sqrt[6]{2}$
and $\quad 3\theta=\frac{\pi}{4}+2k\pi \quad$ for $\quad k=0,1,2$. Then the solutions are:

$$z_0=\sqrt[6]{2}(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}),$$

$$z_1=\sqrt[6]{2}(\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}),$$

$$z_2=\sqrt[6]{2}(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}).\blacksquare$$





Problem 2.

(15 points) Find a and b so that $f(x) = \begin{cases} \dfrac{a(1-\cos x)}{x^2} & x > 0 \\ a(x+1)+b & x = 0 \\ \dfrac{|x|}{x}\cos x & x < 0 \end{cases}$

is continuous at $x = 0$.

Solution:

Continuity: $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{|x|}{x}\cos x = \lim_{x\to 0^-} \dfrac{-x}{x}\cos x = -1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{a(1-\cos x)}{x^2} = \dfrac{0}{0} \xRightarrow{\text{L'Hopital Rule}} \lim_{x\to 0^+} \dfrac{a\sin x}{2x} = \dfrac{0}{0}$

$\xRightarrow{\text{L'Hopital Rule}} \lim_{x\to 0^+} \dfrac{a\cos x}{2} = \dfrac{a}{2}$.

$f(0) = a(0+1)+b = a+b$.

$$\Rightarrow \begin{cases} \dfrac{a}{2} = a+b \to \dfrac{a}{2}+b = 0 \to b = 1 \\ \dfrac{a}{2} = -1 \to a = -2. \blacksquare \end{cases}$$

Problem 3.

Above is the graph of $f'(x)$, the derivative of $f(x)$, use any suitable information you can obtain from the graph and answer the following questions.

(a) (15 points) Find the x-value of the critical points of $f(x)$. Determine whether the critical points are local maximum, local minimum, or neither.

Solution:
Critical points: $f'(x) = 0 \Rightarrow x = -3, -1, 2, 4$.

x	$-\infty$	-3		-1		2		4	$+\infty$
f'	-	0	+	0	-	0	-	0	+
f	↘		↗		↘		↘		↗

According to the above table
x-coordinates of the local minimum: x= -3, 4
x-coordinates of the local maximum: x=-1
neither minimum nor maximum: x= 2. ■

(b) (10 points) Find the equation of the tangent line to $f(x)$ at point $(1, -2)$.
Solution:
Tangent line equation: $y - f(x_0) = f'(x_0)(x - x_0)$.
Since $(x_0, f(x_0)) = (1, -2)$ and from the graph of f' we find $f'(1) = -1$.
Therefore, the tangent line at point $(1, -2)$ is

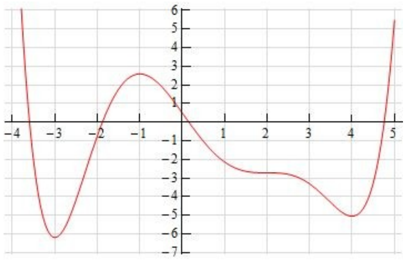
$$y + 2 = -1(x - 1) \to y = -x - 1. \blacksquare$$

(c) (15 points) Determine the x-value of the inflection points of the curve $y = f(x)$ and the intervals of concavity.
Solution:
Inflection points: $f''(x) = 0 \to x = -2.3, 0, 2, 3.5$.

x	$-\infty$	-2.3		0		2		3.5	$+\infty$
f''	+	0	-	0	+	0	-	0	+
f	◡		◠		◡		◠		◡

As we can see in the above table:
$f(x)$ is concave up: $(-\infty, -2.3) \cup (0.2) \cup (3.5, +\infty)$
$f(x)$ is concave down: $(-2.3, 0) \cap (2, 3.5)$.■

(d) (10 points) Use the information obtained from the previous parts and sketch a possible graph of $f(x)$, given that $f(-3) = -6, f(-1) = 2.5, f(2) = -2.5$ and $f(4) = -5$.
Solution:



(e) (10 points) Calculate $\int_2^{-2.3} f''(x)\,dx$.

Solution:

$\int_2^{-2.3} f''(x)\,dx = -\int_{-2.3}^{2} f''(x)\,dx = -f'(x)\Big|_{-2.3}^{2} = -(f'(2) - f'(-2.3)) = -(0-4) = 4.$■

Problem 4.

(15 points) The equation $e^{-2x} = 3x^2$ has a positive root near $x = 0$. By finding a suitable polynomial approximation to e^{-2x} find an approximation to this root.

Solution:

Maclaurin polynomial for e^{-2x} is:

$P_n(x) = 1 - \frac{2x}{1!} + \frac{4x^2}{2!} - \frac{(2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!}$. So we need to use the first three terms of this polynomial to find the solution of given equation. Thus we have $1 - 2x + 2x^2 = 3x^2 \Rightarrow x^2 + 2x - 1 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{8}}{2} \to x \approx -2.4, 0.4$. Since the given equation has a positive root near $x = 0$, the approximated solution is $x \approx 0.4$.■

Problem 5.

Determine whether each integral is convergent or divergent and justify your answer.

(a) (15 points) $\displaystyle\int_0^{\infty} (1-x)e^{-x}\,dx$ **(b)** (10 points) $\displaystyle\int_1^{\infty} \frac{2+\cos x}{x-1}\,dx$

Solution (a):

$\int_0^{\infty} (1-x)e^{-x}\,dx = \lim_{t\to\infty} \int_0^t (1-x)e^{-x}\,dx$ (integrate by parts)

$\lim_{t\to\infty} \int_0^t (1-x)e^{-x}\,dx = \lim_{t\to\infty}(-(1-t)e^{-t} + e^{-t}) - (-1+1)$

$= lim_{t\to\infty} te^{-t} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hopital Rule}} lim_{\to\infty} \frac{1}{e^t} = 0 \Rightarrow$ Improper integral converges.■

Solution (b):

First rewrite the improper integral as:

$\int_1^\infty \frac{2+\cos x}{x-1} dx = 2\int_1^\infty \frac{1}{x-1} dx + \int_1^\infty \frac{\cos x}{x-1} dx$. The first integral in right-hand side is an improper integral of both types, therefore, we need to rewrite it as: $2\int_1^\infty \frac{1}{x-1} dx = 2\int_1^2 \frac{1}{x-1} dx + 2\int_2^\infty \frac{1}{x-1} dx$

$\xrightarrow{\text{P-integrals theorem}} \int_2^\infty \frac{1}{x-1} dx \to$ diverges, thus $\int_1^\infty \frac{2+\cos x}{x-1} dx$ also diverges.■

Problem 6.

(a) (5 points) Find the exact value of the definite integral $\int_0^1 \frac{1}{1+x^2} dx$.

Solution:

$\int_0^1 \frac{1}{1+x^2} dx = \arctan x \Big|_0^1 = \frac{\pi}{4}$.■

(b) (15 points) How large should we take n, the number of subintervals, in order to guarantee that the Trapezoid Rule approximation for the value of π is accurate to within 10^{-2}.

Solution:

There are many ways to compute π. A nice one is:

$$\pi = 4\arctan(1) = 4\int_0^1 \frac{1}{1+x^2} dx.$$

Thus we need to evaluate $\int_0^1 \frac{1}{1+x^2} dx$ to within an error of $\frac{1}{4}.10^{-2}$. To estimate the error in the Trapezoid rule, we need an upper bound on the second derivative of $f(x) = \frac{1}{1+x^2}$. Compute:

$$f'(x) = \frac{-2x}{(1+x^2)^2} \qquad \text{and} \qquad f''(x) = \frac{(6x^2-2)}{(1+x^2)^3}.$$

A rough estimate is $f''(x) \le 4$ for x in $[0,1]$. Thus

$$|E_T| \le \frac{4.1^3}{12n^2} \le \frac{1}{4}10^{-2}.$$

Then $\frac{400}{3} \le n^2 \to n \approx 11.54$. Thus $n = 12$ will do.■

(c) (10 points) Show that Simpson's Rule gives an exact value for

$$\int_a^b (Ax^3+Bx^2+Cx+D)\,dx.$$

Solution:

To estimate the error in Simpson rule, we need an upper bound on the fourth derivative of $f(x)=Ax^3+Bx^2+Cx+D$. Compute:

$$f'(x)=3Ax^2+2Bx+C,\ f''(x)=6Ax+2B,\ f'''(x)=6A,\ f^{(4)}(x)=0.$$

Thus

$$|E_S|\le \frac{0.(b-a)^5}{180n^4}=0\to E_S=0,$$

therefore, Simpson rule is exact.■

Problem 7.

(a) (10 points) Find the general solution of $xy'-2y=x^2$.

Solution:

We first need to put this equation into standard form. If we divide the differential equation on both sides by x then

$$y'-\frac{2}{x}y=x.$$

Hence $p(x)=\dfrac{-2}{x}$. An integrating factor is

$$\mu(x)=\exp\left(\int -\frac{2}{x}dx\right)=\exp(-2\ln|x|)=x^{-2}.$$

Multiplying the differential equation, in standard form, through by $\mu(x)$,

$$x^{-2}-\frac{2}{x^3}y=\frac{1}{x}\to (x^{-2}y)'=\frac{1}{x}.$$

Now integrating $x^{-2}y=\ln|x|+c\to y=x^2\ln|x|+cx^2$.■

(b) (15 points) Solve the initial value problem

$$y'\sin x+y\ln y=0,\qquad y(\frac{\pi}{2})=e.$$

Solution:
Rewrite it as $\dfrac{dy}{dx}=-y\dfrac{\ln y}{\sin x}\xrightarrow{\text{separate variable}}\dfrac{dy}{y\ln y}=-\dfrac{dx}{\sin x}\xrightarrow{\text{integrate}}$
$\ln(\ln y)=\ln|\csc x-\cot x|+c\xrightarrow{\text{impose initial value}} c=0\Rightarrow y=e^{\ln|\csc x-\cot x|}$.■

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