ENGLISH

# UNIVERSITY OF BERGEN

The Faculty of Mathematics and Natural Sciences

**Solutions to Examination in MAT111 - Calculus 1**

Tuesday May 8, 2018, Kl. 09.00-14.00.

## Question 1

Express the complex number $\left(\frac{2}{1-i}\right)^3$ in the form $a + ib$, where $a, b \in \mathbb{R}$.

## Solution

Let $z = \frac{2}{1-i}$. Then

$$z = \frac{2}{1-i} * \frac{1+i}{1+i} = \frac{2+2i}{1-i^2} = \frac{2+2i}{2} = 1+i$$

Find the modulus and the Principal Argument of $z$:

$$\mid z \mid = \sqrt{1^2+1^2} = \sqrt{2} \qquad \text{and} \qquad \mathrm{Arg}(z) = \frac{\pi}{4}$$

Write $z$ in polar form: $z = \sqrt{2}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$

$$\therefore \quad \left(\frac{2}{1-i}\right)^3 = z^3 = (\sqrt{2})^3\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)^3 = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}\right)$$

Notice the use of de Moivre's theorem in the last equality. Use the addition formulas for sine and cosine to get:

$$\cos\frac{3\pi}{4} = \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \cos\frac{\pi}{2}\cos\frac{\pi}{4} - \sin\frac{\pi}{2}\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}},$$

$$\sin\frac{3\pi}{4} = \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \sin\frac{\pi}{2}\cos\frac{\pi}{4} + \cos\frac{\pi}{2}\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

$$\therefore \quad \left(\frac{2}{1-i}\right)^3 = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}i}\right) = -2+2i.$$

## Question 2

Evaluate the limit $\lim_{x\to-\infty}\left(x + \sqrt{x^2-4x+1}\right)$.

## Solution

$$\begin{aligned}\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x+1}\right) &= \lim_{x\to-\infty}\frac{\left(x+\sqrt{x^2-4x+1}\right)\left(x-\sqrt{x^2-4x+1}\right)}{x-\sqrt{x^2-4x+1}}\\ &= \lim_{x\to-\infty}\frac{x^2-(x^2-4x+1)}{x-|x|\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}}\\ &= \lim_{x\to-\infty}\frac{4x-1}{x+x\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}}\\ &= \lim_{x\to-\infty}\frac{4-\frac{1}{x}}{1+\sqrt{1-\frac{4}{x}+\frac{1}{x^2}}}\\ &= \lim_{x\to-\infty}\frac{4-0}{1+\sqrt{1-0+0}}\\ &= 2\end{aligned}$$

## Question 3

Decide whether or not the real function

$$f(x)=\begin{cases}\frac{\ln(\cos 2x)}{2x^2}, & x\neq 0,\\ -1, & x=0,\end{cases}$$

is continuous at $x=0$.

## Solution

$f(0)=-1$.

$$\lim_{x\to 0}\frac{\ln(\cos 2x)}{2x^2}=\frac{0}{0}$$

The limit is indeterminate so use l'Hopital rule

$$\lim_{x\to 0}\frac{-2\tan 2x}{4x}=\frac{0}{0}.$$

The limit is indeterminate so we use l'Hopital rule again

$$\lim_{x\to 0}\frac{-4\sec^2 2x}{4}=-1$$

Hence, the function is continuous since

$$\lim_{x\to 0}f(x)=f(0).$$

# Question 4

(a) Compute the Taylor polynomial of order 3, $P_3(x)$, about $x = 0$ for the function

$$f(x) = x\cos x, \quad \text{for} \quad x \in \mathbb{R}.$$

(b) Show that the real function $g(x) = 4\sin x + x\cos x$ is strictly increasing on the interval $[0, \pi/4]$.

(c) If $0 < x < \frac{\pi}{4}$, find constants $G$ and $H$ such that

$$P_3(x) + Gx^4 < f(x) < P_3(x) + Hx^4.$$

***Hint:*** *You may find (b) useful*

# Solution

(a) The Taylor polynomial of order 3, $P_3(x)$, for the given function $f(x) = x\cos x$ about $x = 0$ is

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

Next is to compute the derivatives and substitute into the above expression

$$\begin{aligned}
f(x) &= x\cos x & \Rightarrow \quad & f(0) = 0, \\
f'(x) &= \cos x - x\sin x & \Rightarrow \quad & f'(0) = 1, \\
f''(x) &= -2\sin x - x\cos x & \Rightarrow \quad & f''(0) = 0, \\
f'''(x) &= -3\cos x + x\sin x & \Rightarrow \quad & f'''(0) = -3.
\end{aligned}$$

$$\therefore \quad P_3(x) = x - \frac{1}{2}x^3.$$

(b) To show that $g(x) = 4\sin x + x\cos x$ is strictly increasing on $[0, \pi/4]$, use the first derivative test to check that

$$g'(x) > 0 \ \ \forall x \in [0, \pi/4].$$

$$g'(x) = 5\cos x - x\sin x > 0 \quad \forall x \in [0, \pi/4]$$

Hence, the function is strictly increasing on $[0, \pi/4]$.

(c) The error of approximation is

$$E_3(x) = \frac{f^{(4)}(c)}{4!}x^4, \quad 0 < c < x < \pi/4.$$

$$f^{(4)}(c) = 4\sin c + x\cos c \quad \Rightarrow \quad E_3(x) = \frac{4\sin c + x\cos c}{24}x^4$$

By (b) $f^{(4)}(c) = g(c)$ is strictly increasing on $[0, \pi/4]$ which implies that $E_3(x)$ is strictly increasing for $0 < x < \pi/4$. Consequently,

$$\begin{aligned}
\frac{4\sin 0 + 0}{24}x^4 < \frac{4\sin c + c\cos c}{24}x^4 &< \frac{4\sin(\pi/4) + (\pi/4)\cos(\pi/4)}{24}x^4 \\
0 < E_3(x) &< \frac{1}{24}\left(\frac{4}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right)x^4 \\
P_3(x) < E_3(x) + P_3(x) &< \left(\frac{1}{6\sqrt{2}} + \frac{\pi}{96\sqrt{2}}\right)x^4 + P_3(x) \\
P_3(x) < f(x) &< \left(\frac{1}{6\sqrt{2}} + \frac{\pi}{96\sqrt{2}}\right)x^4 + P_3(x)
\end{aligned}$$

$$\therefore \quad G = 0 \quad \text{and} \quad H = \frac{1}{6\sqrt{2}} + \frac{\pi}{96\sqrt{2}}$$

# Question 5

Without solving for $y(x)$, find the value of $y''(0)$ in the initial value problem

$$(x+2)y' + y = 1, \quad x > 0, \quad y(0) = 2.$$

# Solution

$$\begin{aligned}
(x+2)y' + y = 1 \quad &\Rightarrow \quad (x+2)y'' + y' + y' = 0 \\
&\Rightarrow \quad y'' = -\frac{2y'}{x+2}
\end{aligned}$$

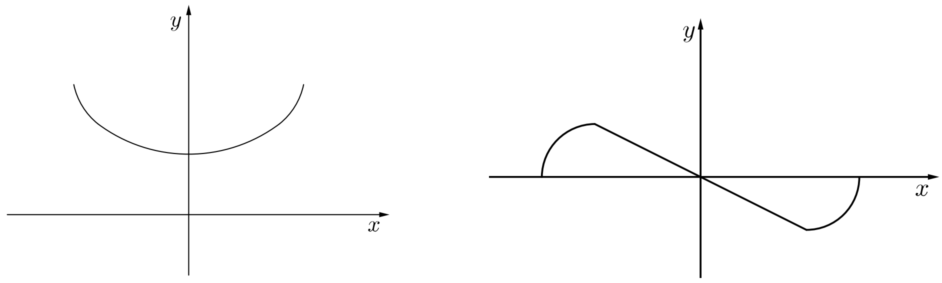
Now find the initial value for $y'(0)$

$$y(0) = 2 \quad \Rightarrow \quad (0+2)y'(0) + 2 = 1 \quad \Rightarrow \quad y'(0) = -\frac{1}{2}$$

$$\therefore \quad y''(0) = -\frac{2y'(0)}{0+2} = \frac{1}{2}$$

# Question 6

The graphs of two functions $y = f(x)$ are shown in the Figure 1 below. For each of them, sketch a qualitatively accurate graphs of the first derivative $f'(x)$ and the second derivative $f''(x)$ from the graphs of $f(x)$.

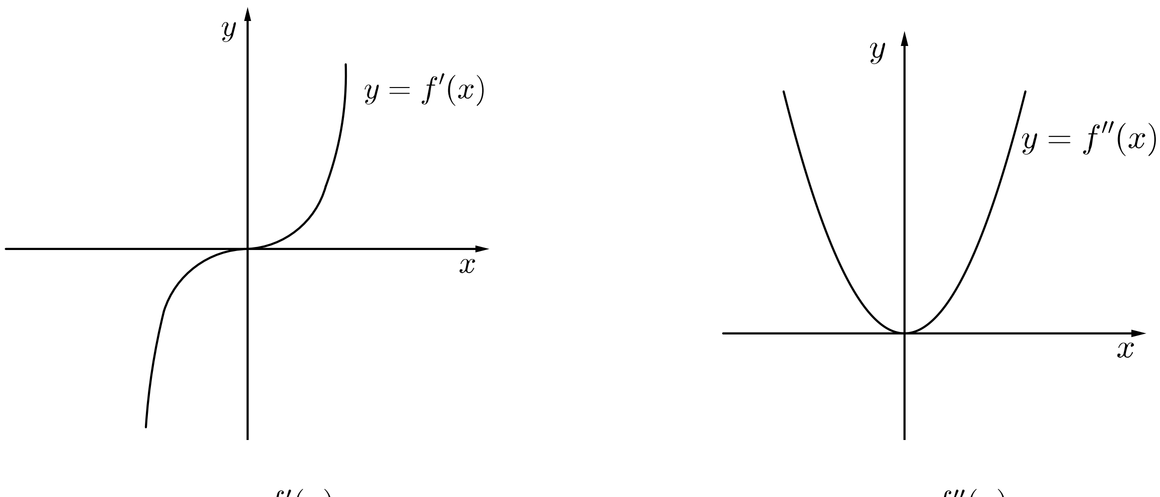



(a) (b)

Figure 1: The graph of $y = f(x)$.

# Solution

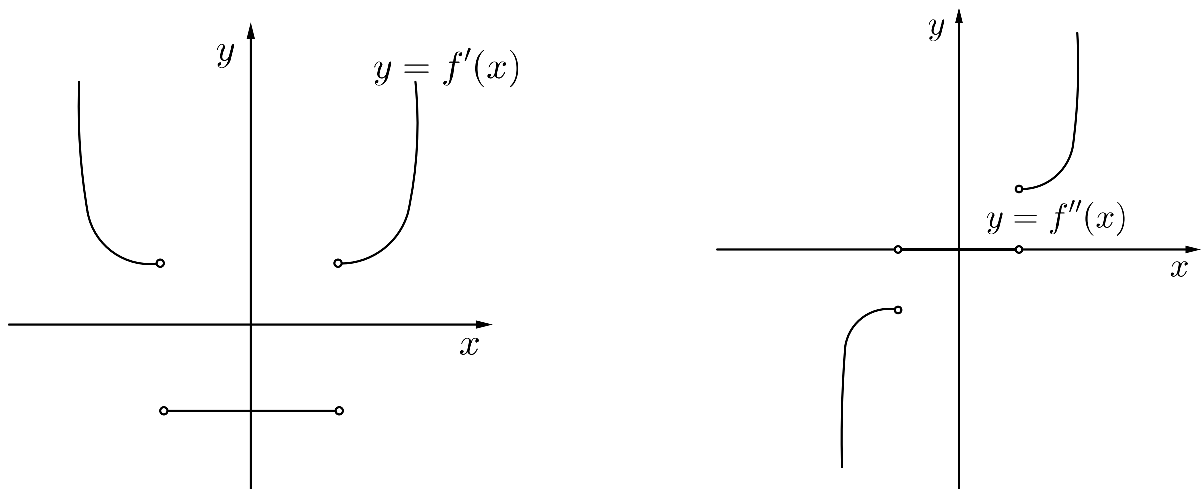
(a)




$y = f'(x)$ $y = f''(x)$

Figure 2: Question 6($a$).

(b)




$y = f'(x)$ $y = f''(x)$

Figure 3: Question 6($b$).

# Question 7

Compute the following indefinite integrals

(a) $\int x^3 \sin(x^2)\; dx$

(b) $\int \sqrt{4 - x^2}\; dx$

# Solution

(a) Use substitution by letting $u = x^2$. Then

$$u = x^2 \quad \Rightarrow \quad du = 2x dx$$
$$\therefore \quad \int x^3 \sin(x^2)\; dx = \frac{1}{2}\int u \sin u\; du.$$

Proceed by integration by parts

$$f = u \quad \Rightarrow \quad df = du$$
$$dg = \sin u \quad \Rightarrow \quad g = -\cos u$$

$$\therefore \quad \frac{1}{2}\int u \sin u\; du = -\frac{1}{2}u \cos u + \frac{1}{2}\int \cos u\; du$$
$$= -\frac{1}{2}u \cos u + \frac{1}{2}\sin u + C.$$

Substitute $u = x^2$ back into the result of the last integral to get

$$\int x^3 \sin(x^2)\; dx = -\frac{1}{2}x^2 \cos(x^2) + \frac{1}{2}\sin(x^2) + C.$$

(b) Use trigonometric substitution:

$$u = 2\sin u \quad \Rightarrow \quad dx = 2\cos u \; du$$

$$\begin{aligned}
\Rightarrow \quad \int \sqrt{4-x^2}\; dx &= 2\int \sqrt{4-\sin^2 u}\; \cos u \; du \\
&= 4\int \sqrt{1-\sin^2 u}\; \cos u \; du \\
&= 4\int \sqrt{\cos^2 u}\; \cos u \; du \\
&= 4\int \cos^2 u \; du \\
&= 2\int (1+\cos 2u)\; du \\
&= 2u + \sin 2u + C \\
&= 2u + 2\sin u \cos u + C
\end{aligned}$$

$$x = 2\sin u \quad \Rightarrow \quad u = \sin^{-1}\left(\frac{x}{2}\right).$$

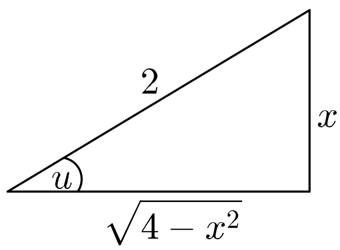



Figure 4: $\sin u = \frac{x}{2}$.

From Figure 4, we have

$$\sin u = \frac{x}{2}, \qquad \cos u = \frac{\sqrt{4-x^2}}{2}.$$

$$\therefore \quad \int \sqrt{4-x^2}\; dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

# Question 8

The (real) function $f$ defined on some interval has the property that all normals to the graph of the function pass through the point $(2,1)$. (The normal is the line which is perpendicular to the tangent line to the graph of $f$ at a point).

(a) Show that $y = f(x)$ satisfies the differential equation

$$y' = \frac{2-x}{y-1}.$$

***Hint:*** *Use the given information to calculate the slope of the graph of the function at an arbitrary point* $(x, y)$.

(b) Find all solutions to the differential equation in (a). Leave your answer in implicit form. (You may solve this problem without having solved (a)).

(c) What kind of curves are described by the implicit solutions in (b)? (Specify the values of the integration constant which give such curves.)

# Solution

(a) The equation of the normal through the point $(2, 1)$ is

$$\begin{aligned} y - 1 &= -\frac{1}{y'}(x-2) \\ \Rightarrow \quad y' &= \frac{2-x}{y-1} \end{aligned}$$

(b) This is a separable equation so

$$\frac{dy}{dx} = \frac{2-x}{y-1} \quad \Rightarrow \quad (y-1)dy = (2-x)dx.$$

$$\begin{aligned} \Rightarrow \quad & \int (y-1)\, dy = \int (2-x)\, dx \\ \Rightarrow \quad & \frac{y^2}{2} - y = 2x - \frac{x^2}{2} + C \\ \Rightarrow \quad & x^2 - 4x + y^2 - 2y = C \\ \Rightarrow \quad & (x-2)^2 + (y-1)^2 = C + 5 \end{aligned}$$

(c)

$$\begin{aligned} C > -5 \quad &\to \quad \text{a circle centred at } (2,1) \\ C = -5 \quad &\to \quad \text{a single point} \\ C < -5 \quad &\to \quad \text{null set} \end{aligned}$$

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