SOLUTION TO EXAM IN MAT111 – SPRING 2019

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Problem 0.1. *Find all cubic roots (3-th order roots) of the complex number*

$$z = \frac{2+6i}{1-2i}.$$

Solution. First one finds

$$z = -2 + 2i$$

and then represents this number in the trigonometric form $z = r(\cos\varphi + i\sin\varphi)$ as

$$r = 2\sqrt{2},$$

$$\cos\varphi = -\frac{1}{\sqrt{2}}, \quad \sin\varphi = \frac{1}{\sqrt{2}}$$

and so the angle

$$\varphi = \frac{3\pi}{4}.$$

All cubic roots have the form

$$w_k = \sqrt{2}(\cos\psi_k + i\sin\psi_k)$$

where

$$\psi_0 = \frac{\pi}{4},$$

$$\psi_1 = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12},$$

$$\psi_2 = \frac{\pi}{4} + \frac{4\pi}{3} = \frac{19\pi}{12}.$$

Thus the cubic roots are

$$w_0 = 1 + i,$$

$$w_1 = (1+i)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2} + i\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right),$$

$$w_2 = (1+i)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2} - i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right).$$

□

Problem 0.2. *Consider the function*

$$f(x) = \frac{\cos(\pi x)}{x} - 1$$

with domain of definition $D(f) = (0, +\infty)$. Using the formal limit definition, prove that the function f has a limit as x approaches $+\infty$.

Solution. Using the squeeze theorem one can see that $f \to -1$ at infinity. Taking $\varepsilon > 0$ one needs to find $R > 0$ such that if $x > R$ then $|f(x) + 1| < \varepsilon$.

Note that for any positive R and $x > R$ we have

$$|f(x) + 1| = \left|\frac{\cos(\pi x)}{x}\right| \leqslant \frac{1}{x} < \frac{1}{R}.$$

Thus for any $\varepsilon > 0$ number $R = 1/\varepsilon$, for example, suits the problem, i. e. if $x > R$ then $|f(x) + 1| < \varepsilon$. □

Problem 0.3. *Prove that the inequalities*

$$x \geqslant \arctan x \geqslant \frac{x}{1 + \frac{2}{\pi}x}$$

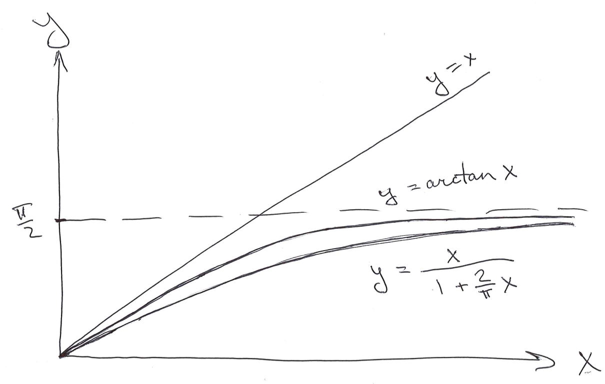
hold for any $x \geqslant 0$. Draw the graphs of the three functions.

Hint: *Minimize the functions $\varphi(x) = x - \arctan x$ and $\psi(x) = \arctan x - \frac{x}{1+2x/\pi}$ on the interval $[0, +\infty)$.*

Solution. **The proof of the first inequality:** One has to show that $\varphi(x) \geqslant 0$ for any $x \geqslant 0$. Firstly, one can notice that $\varphi(0) = 0$. It is enough to show that $\varphi(x)$ is an increasing function on $[0, +\infty)$. Indeed, the derivative

$$\varphi'(x) = 1 - \frac{1}{1 + x^2} = \frac{x^2}{1 + x^2} \geqslant 0$$

for any $x \geqslant 0$. Hence $\varphi(x)$ is increasing. Thus for any $x \geqslant 0$ we have $\varphi(x) \geqslant \varphi(0) = 0$ giving the first inequality.



The proof of the second inequality: One has to show that $\psi(x) \geqslant 0$ for any $x \geqslant 0$. Its derivative

$$\psi'(x) = \frac{1}{1 + x^2} - \frac{(\pi/2)^2}{(x + \pi/2)^2} = \frac{(1 - (\pi/2)^2)\, x^2 + \pi x}{(1 + x^2)(x + \pi/2)^2}.$$

Equation $\psi'(x) = 0$ is equivalent to the quadratic equation $(1 - (\pi/2)^2)\, x^2 + \pi x = 0$ which has two solutions. The first one is $x_1 = 0$. The second one is $x_2 = \frac{4\pi}{\pi^2 - 4} > 0$. Note that

$$\psi'(x) > 0 \quad \text{for} \quad x \in (0, x_2),$$
$$\psi'(x) < 0 \quad \text{for} \quad x \in (x_2, +\infty).$$

In other words, $\psi(x)$ increases on $(0, x_2)$, reaches maximum at x_2 and decreases on $(x_2, +\infty)$. Moreover, $\psi(0) = 0$ and

$$\lim_{x\to+\infty} \psi(x) = \lim_{x\to+\infty} \left(\arctan x - \frac{x}{1 + 2x/\pi} \right) = \lim_{x\to+\infty} \arctan x - \lim_{x\to+\infty} \frac{1}{\frac{1}{x} + \frac{2}{\pi}} = \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

Hence for any $x \in [0, x_2]$ we have $\psi(x) \geqslant \psi(0) = 0$ due to the increase. For any $x \in [x_2, +\infty)$ we have $\psi(x) \geqslant \lim_{x\to+\infty} \psi(x) = 0$ due to the decrease. Thus $\psi(x) \geqslant 0$ for all $x \geqslant 0$, which concludes the proof of the second inequality. □

Problem 0.4. *Find the limit*

$$\lim_{x\to 0} \frac{e^{-2x} + 2\sin(x) - \cos(2x)}{x^2}.$$

Solution. Note that

$$\exp(-2x) = 1 - 2x + 2x^2 + O(x^3),$$
$$\sin(x) = x + O(x^3),$$
$$\cos(2x) = 1 - 2x^2 + O(x^4)$$

as $x \to 0$. Thus

$$\lim_{x\to 0} \frac{e^{-2x} + 2\sin(x) - \cos(2x)}{x^2} = \lim_{x\to 0} \frac{4x^2 + O(x^3)}{x^2} = \lim_{x\to 0} (4 + O(x)) = 4.$$

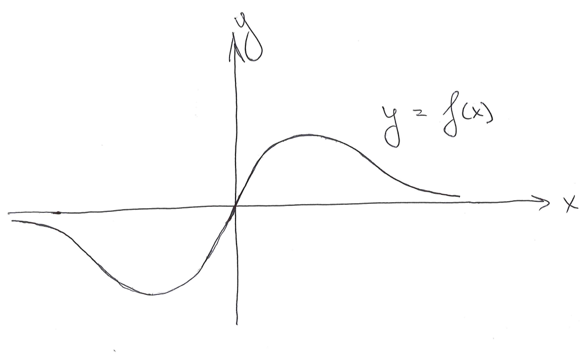
□

Problem 0.5. *Consider the function*

$$f(x) = xe^{-x^2/2}$$

with the domain $D(f) = \mathbb{R}$. Find extreme values for this function, and determine where $f(x)$ is convex or concave. Find the limits at $+\infty$ and $-\infty$. Does the function have any asymptotes? Sketch the graph of $f(x)$.

Solution. Firstly, one can notice that f is symmetric with respect to the origin, i.e. $f(-x) = -f(x)$. Secondly, $f(0) = 0$, $f(x) > 0$ for $x > 0$ and $f(x) \to 0$ as $x \to +\infty$. At this point one can already depict an approximate graph of this function.



To make the graph more precise, calculate the derivatives

$$f'(x) = (1 - x^2)e^{-x^2/2},$$

$$f''(x) = x(x^2-3)e^{-x^2/2}.$$

Function $f(x)$ has a global minimum at $x=-1$ and a global maximum at $x=1$. It is concave on the intervals $(-\infty,-\sqrt{3}]$, $[0,\sqrt{3}]$. It is convex on the intervals $[-\sqrt{3},0]$, $[\sqrt{3},+\infty)$. □

Problem 0.6. *Calculate the integral*

$$\int_0^{\pi/2} \sin(2x)\cos^{2019}(x)dx.$$

Solution. Note that $\sin(2x) = 2\sin x\cos x$ and so

$$\int_0^{\pi/2} \sin(2x)\cos^{2019}(x)dx = 2\int_0^{\pi/2} \sin(x)\cos^{2020}(x)dx = -2\int_1^0 y^{2020}dy = \frac{2}{2021}$$

where the change $\cos x = y$ of variables was used. □

Problem 0.7. *Calculate the improper integral*

$$\int_0^{+\infty} x^3e^{-x^2}dx.$$

Solution. Changing variables ($y=x^2$) and then integrating by parts obtain

$$\int_0^{+\infty} x^3e^{-x^2}dx = \frac{1}{2}\int_0^{+\infty} ye^{-y}dy = \frac{1}{2}.$$

To justify these calculations one has to check convergence of the given integral. Consider

$$J(\beta) = \int_0^{\beta} x^3e^{-x^2}dx$$

as a function of $\beta \in [0,+\infty)$. The integrand is obviously non-negative and so J is an increasing function. By the theorem about limit of monotone function one concludes that there exists the limit of $J(\beta)$ at infinity. It is left to check that this limit is finite. Indeed, we have

$$-x^2 \leqslant -2x+1$$

since $(x-1)^2 \geqslant 0$, and

$$\frac{x^3}{3!} \leqslant e^x$$

following from the Taylor expansion for $x \geqslant 0$ (or one can instead integrate $1 \leqslant e^x$ from 0 to x three times, to get the same inequality). Thus

$$J(\beta) \leqslant \int_0^{\beta} 6e^{-x+1}dx \leqslant 6e$$

and so $\lim_{\beta\to+\infty} J(\beta)$ is finite. □

Problem 0.8. *Let Γ be the graph of the function $f(x)=\sin(2x)$ with domain of definition $D(f)=[0,\pi]$. Find the volume of the solid of revolution generated by rotating the curve Γ about the x-axis.*

Hint: *The formula $\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$ might be of help.*

Solution. The volume is $\pi\int_0^{\pi} f^2(x)dx = \pi/2\int_0^{\pi}(1-\cos(4x))dx = \frac{\pi^2}{2}$. □

Problem 0.9. *Find all solutions of the differential equation*

$$y' \sin x - y \cos x = 0.$$

Then find the solution that satisfies the initial-value problem $y(\pi/2) = -1$.

Solution. Rewrite the equation in the form

$$\frac{y'}{y} = \frac{\cos x}{\sin x}$$

and then integrate

$$\int \frac{dy}{y} = \int \frac{y'(x)dx}{y(x)} = \int \frac{\cos x}{\sin x} dx$$

and so

$$\ln |y| + C_1 = \ln |\sin x| + C_2.$$

Thus the general solution has the form

$$y = C \sin x, \text{ with } C \in \mathbb{R}.$$

The initial-value problem associated with $y(\pi/2) = -1$ has the following solution

$$y = -\sin x.$$

□