English

University of Bergen

The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

June 07, 2019, from 09.00 to 13.00

The exam consists of two parts:

Exercises 1-20 is of type "multiple choice". You have to choose the correct answer and mark it. In exercise 20 the questions can have several correct answers. This part assumes that you give answers on the computer.

The exercises 21-22 require from you an ability to make a proof of some statements. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver to the Inspera system as a pdf file.

1.2 The augmented matrix

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 & a \\ -2 & 4 & 5 & -5 & 3 & b \\ 3 & -6 & -6 & 8 & -3 & c \end{bmatrix}$$

corresponds to a consistent system if (choose the correct answer)

- ○ $-a-b+c=0$
- ○ $a+b+c=0$
- ○ $-a+b+c=0$ and $2a+b\neq 0$
- • $-a+b+c=0$
- ○ non of them

2.2 A linear transformation $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by a rotation on some angle in the counterclockwise direction. Let Φ rotate the vector $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ to the vector $\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$. Then the matrix that corresponds to Φ is given by (choose the correct answer)

○ $\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

○ $\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$

○ $\begin{bmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$

● $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$

○ non of them

3.2 Let $\alpha = (\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4)$ where

$$\overrightarrow{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{\alpha}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T(\overrightarrow{\alpha}_1) = \overrightarrow{\beta}_1, \quad T(\overrightarrow{\alpha}_2) = \overrightarrow{\beta}_2 + \overrightarrow{\beta}_3, \quad T(\overrightarrow{\alpha}_3) = \overrightarrow{\beta}_2 - \overrightarrow{\beta}_3, \quad T(\overrightarrow{\alpha}_4) = \overrightarrow{\beta}_4,$$

where the vectors $\overrightarrow{\beta}_j$, $j = 1, 2, 3, 4$ are given by

$$\overrightarrow{\beta}_1 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_2 + \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4, \qquad \overrightarrow{\beta}_2 = \overrightarrow{\alpha}_2 + \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4,$$
$$\overrightarrow{\beta}_3 = \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4, \qquad \overrightarrow{\beta}_4 = \overrightarrow{\alpha}_1.$$

Then the standard matrix of the transformation T is given by (choose the correct answer)

○ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- ○ $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$
- ● $\begin{bmatrix} 3 & -3 & 7 & -2 \\ 2 & -1 & 4 & -2 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 3 & -1 \end{bmatrix}$
- ○ $\left(\begin{bmatrix} 3 & -3 & 7 & -2 \\ 2 & -1 & 4 & -2 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 3 & -1 \end{bmatrix} \right)^{-1}$
- ○ non of them

4.2 Let vectors $\overrightarrow{\gamma}_1, \overrightarrow{\gamma}_2, \overrightarrow{\gamma}_3, \overrightarrow{\gamma}_4$ are given by

$$\overrightarrow{\gamma}_1 = -\overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4, \qquad \overrightarrow{\gamma}_2 = \overrightarrow{\alpha}_2 - \overrightarrow{\alpha}_3 - \overrightarrow{\alpha}_4,$$
$$\overrightarrow{\gamma}_3 = \overrightarrow{\alpha}_1 - \overrightarrow{\alpha}_4, \qquad \overrightarrow{\gamma}_4 = \overrightarrow{\alpha}_1 + \overrightarrow{\alpha}_3 + \overrightarrow{\alpha}_4,$$

where $\overrightarrow{\alpha}_1, \overrightarrow{\alpha}_2, \overrightarrow{\alpha}_3, \overrightarrow{\alpha}_4$ is a basis of a vector space V. The dimension of $W = \text{span}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is equal to (choose the correct answer)

- ○ 1
- ○ 2
- ○ 3
- ● 4
- ○ 0

5.2 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given by

$$T(\overrightarrow{e}_1) = -\overrightarrow{e}_3 + \overrightarrow{e}_4, \quad T(\overrightarrow{e}_2) = \overrightarrow{e}_2 - \overrightarrow{e}_3 - \overrightarrow{e}_4,$$
$$T(\overrightarrow{e}_3) = \overrightarrow{e}_1 - \overrightarrow{e}_4, \quad T(\overrightarrow{e}_4) = \overrightarrow{e}_1 + \overrightarrow{e}_3 + \overrightarrow{e}_4,$$

in standard basis $\overrightarrow{e}_j$, $j = 1, 2, 3, 4$. Then the dimension of the null space is (choose the correct answer)

- ○ 1
- ○ 2
- ○ 3
- ○ 4
- ● 0

6.2 Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation given in the problem 5.2 Then the basis of the null space of the transformation is (choose the correct answer)

○ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

● non of them

7.2 The determinant of the matrix

$$\begin{bmatrix} 3+5\sin x & 3-5\sin x & 2 \\ 1 & 1 & 2 \\ \sin x & -\sin x & 2 \end{bmatrix}$$

is equal to (choose the correct answer)

○ $-28\sin x$

● $28\sin x$

○ $\sin x$

○ $14(\sin x)^2$

○ $-\sin x$

○ non of them

8.2 Let S be the parallelogram determined by the vectors

$$\overrightarrow{b}_1 = \begin{bmatrix} 4 \\ -7 \end{bmatrix}, \qquad \overrightarrow{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 5 & 2 \\ -1 & -1 \end{bmatrix}$. The area of the image of S under the mapping $\overrightarrow{x} \mapsto A\overrightarrow{x}$ is equal to (choose the correct answer)

○ 4
○ 0
○ -12
○ 3
● 12
○ non of them

9.2 Let

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}.$$

The inverse to adjugate matrix $(\text{adj}A)^{-1}$ is given by (choose the correct answer)

○ $\begin{bmatrix} -15 & -20 \\ -10 & -5 \end{bmatrix}$

○ $\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$

○ $\begin{bmatrix} -1 & -4/5 \\ -2/5 & -1/5 \end{bmatrix}$

○ $\begin{bmatrix} -1/5 & 4/5 \\ 2/5 & -1 \end{bmatrix}$

○ $\begin{bmatrix} -3 & -12 \\ 6 & -9 \end{bmatrix}$

● non of them

10.2 Let

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \vec{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$$

be two bases in $\mathbb{R}^2$. The change-of-basis matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{C}}$ is equal to (choose the correct answer)

- ○ $\begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}$
- ● $\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$
- ○ $\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$
- ○ $\begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$
- ○ non of them

11.2 Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$. The eigenvalues of A are given by (choose the correct answer)

- ○ $\lambda = -4, 0, 3$
- ○ $\lambda = -4, 3$
- ○ $\lambda = 4, -3$
- ● $\lambda = 4, 0, -3$
- ○ non of them

12.2 Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$. The eigenvectors of A are given by (choose the correct answer)

- ○ $\begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$
- ○ $\begin{bmatrix} 7 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- ○ $\begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- ● $\begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- ○ non of them

13.2 Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. It is known that $\lambda = 5, 1$ are among the eigen values of A. The diagonalization of the matrix A is given by (choose the correct answer)

- ○ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- ○ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
- ○ $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

○ The matrix A is not diagonalisable

14.2 Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$. The matrix $(A^T A)^{-1}$ is given by (choose the correct answer)

○ $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

○ $\begin{bmatrix} 24 & 0 \\ 0 & 3 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 \\ 0 & 1/8 \end{bmatrix}$

- $\frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1/8 \end{bmatrix}$

○ non of these

15.2 The least square solution to the problem $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

is given by (choose the correct answer)

○ $\begin{bmatrix} 12 \\ 3/2 \end{bmatrix}$

○ $\frac{1}{2} \begin{bmatrix} 24 \\ 3 \end{bmatrix}$

○ $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

• $\frac{1}{2}\begin{bmatrix}6\\1\end{bmatrix}$

○ non of these

16.2. The quadratic form $Q = -x_1^2 - 2x_1x_2 - x_2^2$ is (choose the correct answer)

○ positive definite
○ negative definite
○ positive semidefinite
• negative semidefinite
○ non of them

17.2 The matrix P that orthogonally diagonalises the matrix $A = \begin{bmatrix}-1 & -1\\-1 & -1\end{bmatrix}$ is given by (choose the correct answer)

○ $\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$

○ $\frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}$

○ $\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

• $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

○ The matrix A is not diagonalisable

18.2 The orthogonal complement to $W = \text{span}\left\{\begin{bmatrix}-2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}\right\}$, is given by (choose the correct answer)

○ $\begin{bmatrix}0\\0\\0\end{bmatrix}$

- ○ $\text{span}\left\{\begin{bmatrix}4\\2\\-5\end{bmatrix}\right\}$
- ● $\text{span}\left\{\begin{bmatrix}-4\\2\\-5\end{bmatrix}\right\}$
- ○ $\begin{bmatrix}-4\\2\\-5\end{bmatrix}$
- ○ non of these

19.2 The distance between points $p = \begin{bmatrix}2\\3\\-4\end{bmatrix}$ and $q = \begin{bmatrix}2\\5\\-4\end{bmatrix}$ is equal to (choose the correct answer)

- ○ -2
- ● 2
- ○ $\sqrt{2}$
- ○ $-\sqrt{2}$
- ○ $\sqrt{20}$
- ○ non of these

20.2 A square $(n \times n)$-matrix A is diagonalizable, if (choose the correct answer. It can be several correct answers.)

- ○ the matrix A is a low triangular matrix
- ○ $\det A \neq 0$
- ○ The rows of the matrix A are linearly independent
- ● The eigen values of the matrix A are all different
- ● $A^T = A^{-1}$
- ● The matrix A is symmetric
- ○ The matrix has n eigen vectors
- ● There is an orthogonal matrix P such that PAP^T is diagonal
- ○ The dimension of the null space is 0
- ○ There is a matrix P such that $PAP^T = A$
- ○ The rank of the matrix A is equal to n

- ○ The eigen values of the matrix A are all positive
- • There is an invertible matrix P such that $P^{-1}AP$ is diagonal

21.2. Let W be a subspace of $\mathbb{R}^n$. Show that the orthogonal complement $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Let $\overrightarrow{x}_1$ and $\overrightarrow{x}_2 \in W^\perp$, that is

$$\overrightarrow{x}_1 \cdot \overrightarrow{v} = 0, \quad \overrightarrow{x}_2 \cdot \overrightarrow{v} = 0$$

for any $\overrightarrow{v} \in W$. We have

$$(a_1\overrightarrow{x}_1 + a_2\overrightarrow{x}_2) \cdot \overrightarrow{v} = a_1(\overrightarrow{x}_1 \cdot \overrightarrow{v}) + a_2(\overrightarrow{x}_2 \cdot \overrightarrow{v}) = 0$$

for any $\overrightarrow{v} \in W$ by the properties of the inner product. So we conclude that the linear combination $a_1\overrightarrow{x}_1 + a_2\overrightarrow{x}_2 \in W^\perp$.

22.2. Find a formula for least square solution of $A\overrightarrow{x} = \overrightarrow{b}$ when the columns of A are orthonormal.

If the columns of A are orthonormal, they are linearly independent and $A^TA = I$. Then we obtain that the normal equation take the form

$$A^TA\overrightarrow{x} = \overrightarrow{x} = A^T\overrightarrow{b}$$

So the least square solution $\overrightarrow{\hat{x}}$ is given by

$$\overrightarrow{\hat{x}} = A^T\overrightarrow{b}.$$

Professor Irina Markina

Good luck!