English

University of Bergen

The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

September, 28, 2020, from 09.00 to 15.00

- Allowed help resources: all, except of communication between students

The exam consists of two parts:

The first set of exercises is of type "multiple choice". You have to choose the correct answer and mark it. This part assumes that you give answers on the computer.

The second set of exercises requires from you an ability to make a proof of some statement. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver.

1.1 Consider the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{bmatrix}.$$

The set of all values of $\{a, b, c \in \mathbb{R}\}$ for which the matrix A is diagonalizable is given by: (choose the correct answer)

3 points

- ○ all possible $a, b, c \in \mathbb{R}$.
- ○ $a = b = c = 0$
- ○ those values of $a, b, c \in \mathbb{R}$ for which the columns in A are orthogonal
- ○ there are no such $a, b, c \in \mathbb{R}$
- ○ none of them

2.1 The set of all values of x for which the matrix

$$A = \begin{bmatrix} 2 & 0 & 10 \\ 0 & 7+x & -3 \\ 0 & 4 & x \end{bmatrix}$$

is not invertible is given by: (choose the correct answer)

3 points

- ○ $x = -3$, $x = -4$
- ○ $x \neq -3$, $x \neq -4$
- ○ $x = 3$, $x = 4$
- ○ $x = 0$, $x = -7$
- ○ none of them

3.1 Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of some (2×2) matrix A corresponding to the eigenvalue 3 and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue -2. Then $A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ is equal to: (choose the correct answer)

4 points

- ○ $\begin{bmatrix} 26 \\ 22 \end{bmatrix}$
- ○ $\begin{bmatrix} 16 \\ 9 \end{bmatrix}$
- ○ $\begin{bmatrix} 24 \\ 17 \end{bmatrix}$
- ○ $\begin{bmatrix} 9 \\ 16 \end{bmatrix}$
- ○ none of them

4.1 Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

be a basis of $\mathbb{R}^3$. Then the vector

$$\overrightarrow{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

has the following coordinates in the basis $\mathcal{B}$: (choose the correct answer)

4 points

- ○ $$[\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} a+b+c \\ b+c \\ c \end{bmatrix}$$

- ○ $$[\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- ○ $$[\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} a-b \\ b-c \\ c \end{bmatrix}$$

- ○ $$[\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} a \\ -a+b \\ -b+c \end{bmatrix}$$

- ○ none of them

5.1 Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

defines the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^5$ by the rule $T(\overrightarrow{x}) = A\overrightarrow{x}$. Denote by $\mathcal{B}_K$ a basis of Ker (T), by $\mathcal{B}_R$ a basis of range of T, and by $\mathcal{B}_{Rw}$ a basis of Row (A). Then the bases are given by the following sets: (choose the correct answer)

4 points

- ○
$$\mathcal{B}_K = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{B}_R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\},$$
$$\mathcal{B}_{Rw} = \left\{ \begin{bmatrix} 1, & -1, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0, & 1, & 1, & 1 \end{bmatrix} \right\}.$$

- ○
$$\mathcal{B}_K = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{B}_R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\},$$
$$\mathcal{B}_{Rw} = \left\{ \begin{bmatrix} 1, & -1, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0, & 1, & 1, & 1 \end{bmatrix} \right\}.$$

- ○
$$\mathcal{B}_K = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{B}_R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$
$$\mathcal{B}_{Rw} = \left\{ \begin{bmatrix} 1, & -1, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0, & 1, & 1, & 1 \end{bmatrix} \right\}.$$

- ○
$$\mathcal{B}_K = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{B}_R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\},$$
$$\mathcal{B}_{Rw} = \left\{ \begin{bmatrix} 1, & -1, & 0, & 0 \end{bmatrix}, \begin{bmatrix} 0, & 1, & 1, & 1 \end{bmatrix}, \begin{bmatrix} 0, & 2, & 2, & 2 \end{bmatrix} \right\}.$$

- ○ none of them

6.1 Let A be a square matrix that has the characteristic polynomial

$$\det(A - \lambda I) = (\lambda - 1)^2(\lambda - 2)^3(\lambda - 3)^4(\lambda - 4)^5.$$

The rank of the matrix A is equal to: (choose the correct answer)

4 points

- ○ 14
- ○ 4
- ○ 10
- ○ 0
- ○ none of them

7.1 Let $\overrightarrow{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by

$$T(\overrightarrow{x}) = \mathrm{proj}_{\mathrm{span}\{\overrightarrow{u}\}}(\overrightarrow{x}) = \frac{\overrightarrow{x} \cdot \overrightarrow{u}}{\overrightarrow{u} \cdot \overrightarrow{u}}\overrightarrow{u}.$$

The standard matrix A_T of the transformation T and the rank r of T are given by: (choose the correct answer)

3 points

- ○
$$A_T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad r = 1$$

- ○
$$A_T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad r = 1$$

○

$$A_T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad r = 2$$

○

$$A_T = \frac{1}{2}\begin{bmatrix} x_1 + x_2 & 0 & 0 \\ 0 & x_1 + x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad r = 1$$

○ none of them

8.1 Let $\mathbb{M}_{2\times 2}$ be the vector space of (2×2) matrices. Let

$$V = \left\{ A \in \mathbb{M}_{2\times 2} |\ A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, a, b, c \in \mathbb{R} \right\}.$$

We denote $\mathcal{B}$ the basis of V and $d = \dim(V)$. Then the basis and the dimension of V are given by: (choose the correct answer)

3 points

○

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad d = 3$$

○

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad d = 4$$

○

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad d = 3$$

○

$$\mathcal{B} = \left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \right\}, \quad d = 4$$

○ none of them

9.1 Let $\mathbb{P}_2$ be the vector space of all polynomials of degree 2 or less. Consider the set of polynomials

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$\begin{array}{rclrcl} p_1(t) & = & -1+t-2t^2, & p_2(t) & = & t+3t^2, \\ p_3(t) & = & 1+2t+8t^2, & p_4(t) & = & 1+t+t^2. \end{array}$$

The following polynomials from the set S form the basis $\mathcal{B}$ of $\mathbb{P}_2$: (choose the correct answer)

3 points

○

$$\mathcal{B} = \{p_1(t),\ p_2(t), p_3(t)\}$$

○

$$\mathcal{B} = \{p_1(t),\ p_2(t), p_3(t), p_4(t)\}$$

○

$$\mathcal{B} = \{p_1(t),\ p_2(t)\}$$

○

$$\mathcal{B} = \{p_3(t),\ p_4(t)\}$$

○ none of them

10.1 Let $\mathbb{P}_2$ be the vector space of all polynomials of degree 2 or less. Consider the set of polynomials

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$\begin{array}{rclrcl} p_1(t) & = & -1+t-2t^2, & p_2(t) & = & t+3t^2, \\ p_3(t) & = & 1+2t+8t^2, & p_4(t) & = & 1+t+t^2. \end{array}$$

The coordinates of the given polynomials in the basis $\mathcal{B}$ chosen from the set S are given by: (choose the correct answer)

3 points

- ○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix}-\frac{4}{3}\\5\\-\frac{7}{3}\end{bmatrix},$$

- ○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix}2\\3\\-3\end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix}1\\2\\-7\end{bmatrix},$$

- ○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix}-1\\1\\-2\end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix}0\\1\\3\end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix}1\\2\\8\end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix}1\\1\\1\end{bmatrix},$$

- ○

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix}\frac{4}{3}\\-5\\\frac{7}{3}\end{bmatrix},$$

- ○ none of them

11.1 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation given by

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1 - x_2\\x_2\\x_1 + x_2\end{bmatrix}.$$

The orthonormal basis of the range of T is given by

3 points

- ○

$$\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$$

- ○
$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

- ○
$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

- ○
$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- ○ none of them

12.1 Let
$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 3 & 3 & 0 \end{bmatrix}.$$
Then
$$\det\left(A^2(C^3B)^{-1}A^{-2}(B^T)^2C^4\right)$$
is equal to: (choose the correct answer)

4 points

- ○ -24^2
- ○ 24^2
- ○ $364 \cdot 12$
- ○ 0
- ○ none of them

13.1 The quadratic form
$$Q = 4x^2 + 4xy + 4y^2 + 4xz + 4z^2 + 4yz$$
is: (choose the correct answer)

3 points

- ○ positive definite
- ○ degenerate and positive semidefinite
- ○ negative definite
- ○ degenerate and negative semidefinite
- ○ none of them

14.1 Let

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \qquad \overrightarrow{v} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The value c for which the vector

$$\frac{1}{2}\overrightarrow{u} - c\overrightarrow{v}$$

is unit: (choose the correct answer)

3 points

- ○ $c = \frac{-5\pm\sqrt{5}}{20}$
- ○ $c = \frac{2}{5}$
- ○ $c = -\frac{2}{5}$
- ○ c does not exist
- ○ none of them

15.1 Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. The spectral decomposition of A is given by: (choose the correct answer)

3 points

- ○

$$A = \frac{1}{5}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{11}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + 11 \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○

$$A = 1 \overrightarrow{u}_1^T \overrightarrow{u}_1 + 11 \overrightarrow{u}_2^T \overrightarrow{u}_2$$

○

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} + \frac{11}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

○ none of these

16.1. The distance from $\overrightarrow{b} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix} \in \mathbb{R}^3$ to a subspace

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 - x_2 - x_3 = 0\}$$

is equal to: (choose the correct answer)

3 points

○ $\frac{2}{\sqrt{3}}$

○ $-\frac{2}{\sqrt{3}}$

○ 2

○ 6

○ none of them

17.1 The least square line $y = \beta_0 + \beta_1 x$ that best fits the data points

$$\{(2, 3), \ (3, 2), \ (5, 1), \ (6, 0)\}$$

is given by: (choose the correct answer)

4 points

○ $y = \frac{43}{10} - \frac{7}{10}x$

- ○ $y = 86 - 14x$
- ○ $y = -1 + 5x$
- ○ the line does not exist
- ○ none of them

18.1 The following equation

$$5x_1^2 - 4x_1x_2 + 5x_2^2 = 122$$

defines: (choose the correct answer)

3 points

- ○ an ellipse
- ○ a plain
- ○ a hyperbola
- ○ a stright line
- ○ none of them

19.1 Let

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

The orthogonal basis of $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is given by: (choose the correct answer)

4 points

- ○

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

○

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -4 \end{bmatrix}$$

○

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 7 \\ -2 \end{bmatrix}$$

○

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 15 \\ -25 \end{bmatrix}$$

○ none of them

20.1. Let

$$A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

The matrix R in the QR factorisation of A is given by: (choose the correct answer)

4 points

○

$$R = \frac{1}{2\sqrt{3}} \begin{bmatrix} 12 & -36 & 6 \\ 0 & 12 & 30 \\ 0 & 0 & 12 \end{bmatrix}$$

○

$$R = \begin{bmatrix} 12 & -36 & 6 \\ 0 & 12 & 30 \\ 0 & 0 & 12 \end{bmatrix}$$

○

$$R = 2\sqrt{3}\begin{bmatrix} 12 & -36 & 6 \\ 0 & 12 & 30 \\ 0 & 0 & 12 \end{bmatrix}$$

○

$$R = \begin{bmatrix} 12 & 0 & 0 \\ -36 & 12 & 0 \\ 6 & 30 & 12 \end{bmatrix}$$

○ none of them

21.1. Let

$$\mathcal{A} = \left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}.$$

Then the matrix $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$ of change of coordinates from the basis $\mathcal{A}$ to the basis $\mathcal{B}$ and the matrix $\mathcal{P}_{\mathcal{B}\to\mathcal{A}}$ of change of coordinates from the basis $\mathcal{B}$ to the basis $\mathcal{A}$ are given by

4 points

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}, \qquad \mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}, \qquad \mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \frac{1}{8}\begin{bmatrix} 1 & -11 \\ -5 & -9 \end{bmatrix}, \qquad \mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \frac{1}{8}\begin{bmatrix} -9 & 11 \\ 5 & 1 \end{bmatrix},$$

○

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 24 & -8 \\ 40 & -16 \end{bmatrix}, \qquad \mathcal{P}_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} -16 & 8 \\ -40 & 24 \end{bmatrix}$$

○ none of them

22.1. Let

$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

be a basis in $\mathbb{R}^2$ and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by $\vec{x} \mapsto A\vec{x}$, where

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}.$$

Then the $\mathcal{B}$-matrix $M_{\mathcal{B}}$ for the linear transformation T in the basis $\mathcal{B}$ is given by: (choose the correct answer)

4 points

- ○

$$M_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

- ○

$$M_{\mathcal{B}} = \begin{bmatrix} 5 & 10 \\ -10 & 5 \end{bmatrix}$$

- ○

$$M_{\mathcal{B}} = \frac{1}{5} \begin{bmatrix} -19 & 26 \\ -26 & 29 \end{bmatrix}$$

- ○

$$M_{\mathcal{B}} = \begin{bmatrix} -19 & 26 \\ -26 & 29 \end{bmatrix}$$

- ○ none of them

23.1. Let A be a (6×4) matrix and B a (4×6) matrix. Show that (6×6) matrix AB can not be invertible.

8 points

24.1. Let A be an $(n \times n)$ real matrix and suppose that A can be written in the form

$$A = URU^T,$$

where U is an orthogonal matrix and R is an upper triangular matrix. Show that A and R have the same characteristic polynomial. Deduce that A has n real eigenvalues, counting with multiplicity.

8 points

25.1. Suppose that

$$\{\overrightarrow{v}_1, \overrightarrow{v}_2\}$$

are linear independent in 5-dimensional vector space V. Prove that

$$\{\overrightarrow{u}_1 = \overrightarrow{v}_1 + 2\overrightarrow{v}_2,\ \overrightarrow{u}_2 = 2\overrightarrow{v}_1 - \overrightarrow{v}_2\}$$

are also linear independent in V.

8 points

Professor Irina Markina

Good luck!