English

University of Bergen

The Faculty of Mathematics and Natural Sciences

Exam in MAT121 - Linear algebra

June, 02, 2020, from 09.00 to 15.00

- Allowed help resources: all, except for communication between students

The exam consists of two parts:

The first set of exercises is of type "multiple choice". You have to choose the correct answer and mark it. This part assumes that you give answers on the computer.

The second set of exercises requires from you an ability to make a proof of some statement. If you have difficulty to write it on the computer, just write it by hand on the additional ark and deliver.

Solutions of exercises.

1.1 Consider the vectors:

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{x}_2 = \begin{bmatrix} h \\ 1 \\ -h \end{bmatrix}, \qquad \vec{x}_3 = \begin{bmatrix} 1 \\ 2h \\ 3h+1 \end{bmatrix}.$$

The set of all values of h for which $\{\vec{x}_1, \vec{x}_2, \vec{x}_2\}$ are linearly independent is given by: (choose the correct answer)

3 points

Solution to 1.1

We construct the matrix $A = [\vec{x}_1 \vec{x}_2 \vec{x}_3]$. By the invertible matrix theorem, we know that the columns of A are linearly independent if and only if the determinant of A is nonzero.

We therefore compute the determinant of A as a function of h.

$$A = \begin{bmatrix} 1 & h & 1 \\ 0 & 1 & 2h \\ 0 & -h & 3h+1 \end{bmatrix}$$

$$det(A) = 1 \cdot \begin{vmatrix} 1 & 2h \\ -h & 3h+1 \end{vmatrix} = 1[1 \cdot (3h+1) - (-h) \cdot 2h] = 3h + 1 + 2h^2 = 2h^2 + 3h + 1$$

By using the quadratic formula, we find that $det(A) = 0$ for $h = -1$ and $h = -1/2$. Hence, the vectors $\overrightarrow{x}_1, \overrightarrow{x}_2, \overrightarrow{x}_3$ are linearly independent for $h \neq -1$ and $h \neq -1/2$.

Correct answer:

- $h \neq -1,\ h \neq -\frac{1}{2}$

2.1 The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

is not invertible if: (choose the correct answer)

3 points

Solution to 2.1

By the invertible matrix theorem, we know that in order for a matrix to be invertible, the determinant has to be nonzero. Hence, the matrix A is not invertible if the determinant is 0.

We compute the determinant of A as a function of k, using cofactor expansion along the first row.

$$det(A) = 1 \cdot \begin{vmatrix} 2 & k \\ 4 & k^2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & k \\ 1 & k^2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = (2k^2 - 4k) - (k^2 - k) + (4 - 2)$$
$$= 2k^2 - 4k - k^2 + k + 2 = k^2 - 3k + 2$$

By using the quadratic formula, we find that the determinant of A is 0 for $k = 1$ and $k = 2$. By the IMT, the matrix A is not invertible for these values of k.

Correct answer:

- $k = 1$, $k = 2$

3.1 Suppose the following information is known about a (3×3) matrix A:

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

Then the matrix A has the following eigenvalues: (choose the correct answer)

4 points

Solution to 3.1

From the first equation, we see that $A\vec{v} = \lambda\vec{v}$ for $\lambda = 6$. Hence 6 is an eigenvalue of A.

From the second equation, we see that $A\vec{v} = \lambda\vec{v}$ for $\lambda = 3$. Hence 3 is also an eigenvalue of A.

The third equation is not on the form $A\vec{v} = \lambda\vec{v}$, so we can not conclude that 3 is an eigenvalue for more than one eigenvector. However, since both the second and the third equation has the same right side, we can use them to look for a third eigenvalue.

We let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

So we have the following equations:

$$A\vec{v}_1 = 3\vec{v}_1$$
$$A\vec{v}_2 = 3\vec{v}_1$$

Equating them gives us:

$$A\vec{v}_1 = A\vec{v}_2$$
$$A\vec{v}_1 - A\vec{v}_2 = \vec{0}$$
$$A(\vec{v}_1 - \vec{v}_2) = \vec{0}$$

We know that if we multiply a vector by 0, we get the zero vector. Hence, we have the third eigenvalue-eigenvector relation

$$A(\vec{v}_1 - \vec{v}_2) = 0(\vec{v}_1 - \vec{v}_2)$$

So $\lambda = 0$ is also an eigenvalue of A.

Correct answer:

- $\lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 0$

4.1 Suppose the following information is known about a (3×3) matrix A:

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

Then the matrix A has the following eigenvectors: (choose the correct answer)

4 points

Solution to 4.1

These are the same equations that we studied in the last problem.

From the first equation, we have that $A\vec{u}_1 = \lambda\vec{u}_1$ for $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Hence $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of A.

From the second equation, we have that $A\vec{u}_2 = \lambda\vec{u}_2$ for $\vec{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is also an eigenvector of A.

In the last problem, we discovered the relation $A(\vec{v}_1 - \vec{v}_2) = 0(\vec{v}_1 - \vec{v}_2)$, where $\vec{v}_1$ and $\vec{v}_2$ were defined in that problem. Hence, $\vec{v}_1 - \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is also an eigenvector of A.

Correct answer:

- $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

5.1 Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & -3 & 4 & 5 \\ -3 & 10 & -6 & -7 \end{bmatrix}.$$

We denote $\mathcal{B}_C$ the basis of the column space Col (A), $\mathcal{B}_R$ the basis of the row space Row (A), and $\mathcal{B}_{ON}$ an orthonormal basis of the null space Null (A) of A. The mentioned above bases are given by: (choose the correct answer)

4 points

Solution to 5.1

We start by row reducing the matrix A to reduced echelon form.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & -3 & 4 & 5 \\ -3 & 10 & -6 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 5/4 \end{bmatrix} = B$$

The first three columns of A are pivot columns, and by theorem 6 chapter 4 they form a basis for $Col(A)$.

A and B are row equivalent, and B is in echelon form. Thus, by theorem 13 chapter 4, the rows of B, which are all nonzero, form a basis for $Row(A)$.

Finally, we find an orthonormal basis for the null space of A.

$$[A|\overrightarrow{0}] \sim [B|\overrightarrow{0}] = \begin{bmatrix} 1 & 0 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 \end{bmatrix}$$

Solving the equation $A\overrightarrow{x} = \overrightarrow{0}$, we get:

x_4 is free

$x_3 = (-5/4)x_4$

$x_2 = (-1/2)x_4$

$x_1 = (-3/2)x_4$

$$\overrightarrow{x} = \begin{bmatrix} -3/2 \\ -1/2 \\ -5/4 \\ 1 \end{bmatrix} x_3 = \overrightarrow{v} x_4$$

A basis for $Null(A)$ is just $\overrightarrow{v}$. Normalizing this vector, we get

$$\frac{\overrightarrow{v}}{|\overrightarrow{v}|} = \frac{1}{\sqrt{(-3/2)^2 + (-1/2)^2 + (-5/4)^2 + 1^2}} \begin{bmatrix} -3/2 \\ -1/2 \\ -5/4 \\ 1 \end{bmatrix}$$

$$= \frac{4}{9} \begin{bmatrix} -3/2 \\ -1/2 \\ -5/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/9 \\ -5/9 \\ 4/9 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -6 \\ -2 \\ -5 \\ 4 \end{bmatrix}$$

Correct answer:

- $$\mathcal{B}_C = \left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} \right\},$$

$$\mathcal{B}_R = \left\{ \begin{bmatrix} 1, & 0, & 0, & \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 0, & 1, & 0, & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0, & 0, & 1, & \frac{5}{4} \end{bmatrix} \right\},$$

$$\mathcal{B}_{ON} = \left\{ \frac{1}{9} \begin{bmatrix} -6 \\ -2 \\ -5 \\ 4 \end{bmatrix}, \right\},$$

6.1 Suppose the matrix A is diagonalizable and has the characteristic polynomial

$$\det(A - I(\lambda)) = \lambda^2(\lambda - 3)(\lambda + 2)^3(\lambda - 4)^3.$$

Let $(m \times n)$ be the size of the matrix A, d is the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 4$ and $p = \dim\big(\text{Null}\,(A)\big)$. Which of the following numbers correspond to the matrix A? (choose the correct answer)

4 points

Solution to 6.1

If A is diagonalizable, it has to be a square matrix. By theorem 7b chapter 9, the sum of the eigenspaces has to equal n. We also know that the simension of the eigenspace for each eigenvalue has to equal the multiplicity of that eigenvalue. Counting the multiplicities in the characteristic polynomial, we see that A has to be (9×9).

Since the dimension of the eigenspace of an eigenvalue is equal to the multiplicity of that eigenvalue, we see that d has to be 3.

By looking at the characteristic polynomial, we can also find the dimension of the null space of A. For the eigenvalue λ, the dimension of the eigenspace is equal to the number of free variables in the equation $[A - \lambda I]\overrightarrow{x} = \overrightarrow{0}$. For $\lambda = 0$, this is just the equation $A\overrightarrow{x} = \overrightarrow{0}$. We see that $\lambda = 0$ has multiplicity 2, which means that $A\overrightarrow{x} = \overrightarrow{0}$ has 2 free variables. Hence $p = dim(Null(A)) = 2$.

Correct answer:

- $m \times n = 9 \times 9, \quad d = 3, \quad p = 2$

7.1 Let A, B, C be $(n \times n)$ invertible matrices. When you simplify the expression

$$C^{-1}(AB^{-1})^{-1}(CA^{-1})^{-1}C^2$$

which matrix do you get? (choose the correct answer)

3 points

Solution to 7.1

We simplify the expression using the properties of matrix multiplication from theorem 2 chapter 2, the properties of invertible matrices from theorem 6 chapter 2, as well as the fact that $M^k = M \cdot ... \cdot M$ (k times).

$$\begin{gathered} C^{-1}(AB^{-1})^{-1}(CA^{-1})^{-1}C^2 \\ = C^{-1}(B^{-1})^{-1}A^{-1}(A^{-1})^{-1}C^{-1}CC \\ = C^{-1}BA^{-1}AC^{-1}CC \\ = C^{-1}B(A^{-1}A)(C^{-1}C)C \\ = C^{-1}BIIC \\ = C^{-1}BC \end{gathered}$$

Correct answer:

- $C^{-1}BC$

8.1 Let $\mathbb{P}_3$ be the vector space of all polynomials of degree 3 or less. Let

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\}, \qquad Q = \text{span}\{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$p_1(t) = 1 + 3x + 2x^2 - x^3, \quad p_2(t) = x + x^3,$$

$$p_3(t) = x + x^2 - x^3, \quad p_4(t) = 3 + 8x + 8x^3.$$

The coordinates of the polynomials $\{p_1(t), p_2(t), p_3(t), p_4(t)\}$ in the standard basis $\mathcal{E} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ are: (choose the correct answer)

3 points

Solution to 8.1

By the first definition in chapter 2.9, when a vector is written as a linear combination of a set of basis vectors, the coordinates are given by the coefficients in front of the basis vectors. If one or more of the basis vectors are not included in the linear combination (i.e. for $p_2(t)$), the coefficient corresponding to that basis vector is just 0.

Correct answer:

- $$[p_1(t)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 8 \\ 0 \\ 8 \end{bmatrix}$$

9.1 Let $\mathbb{P}_3$ be the vector space of all polynomials of degree 3 or less. Let

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\}, \qquad Q = \text{span}\{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$p_1(t) = 1 + 3x + 2x^2 - x^3, \quad p_2(t) = x + x^3,$$

$$p_3(t) = x + x^2 - x^3, \quad p_4(t) = 3 + 8x + 8x^3.$$

The basis $\mathcal{B}$ of Q chosen from the set S is given by: (choose the correct answer)

3 points

Solution to 9.1

The coordinate mapping is a one-to-one correspondence between Q and $\mathbb{R}^4$. Q is isomorphic to $\mathbb{R}^4$, which means that linear dependence is conserved when we apply the coordinate mapping $p(t) \mapsto [p(t)]_{\mathcal{E}}$.

We construct the matrix A where the columns are the coordinate vectors we found in the last problem. Then we row reduce A to reduced echelon form. We only need the echelon form here, but the reduced echelon form will be useful in the next problem.

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 1 & 1 & 8 \\ 2 & 0 & 1 & 0 \\ -1 & 1 & -1 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns are pivot columns, and therefore linearly independent. Since the coordinate mapping conserves linear dependence, this means that $p_1(t), p_2(t)$ and $p_3(t)$ are linearly independent.

In order for a set to be a basis for a given subspace, it needs to be linearly independent and span that subspace. The four polynomials in S span Q, but only $p_1(t), p_2(t)$ and $p_3(t)$ are linearly independent in Q. Hence, $\{p_1(t), p_2(t), p_3(t)\}$ form a basis for Q.

Correct answer:

- $$p_1(t), \quad p_2(t), \quad p_3(t)$$

10.1 Let $\mathbb{P}_3$ be the vector space of all polynomials of degree 3 or less. Let

$$S = \{p_1(t), p_2(t), p_3(t), p_4(t)\}, \qquad Q = \text{span}\{p_1(t), p_2(t), p_3(t), p_4(t)\},$$

where

$$p_1(t) = 1 + 3x + 2x^2 - x^3, \quad p_2(t) = x + x^3,$$

$$p_3(t) = x + x^2 - x^3, \quad p_4(t) = 3 + 8x + 8x^3.$$

The polynomials in S has the following coordinates in the basis $\mathcal{B}$ of Q: (choose the correct answer). Remember that the basis $\mathcal{B}$ is chosen from the polynomial belonging to the set S.

4 points

Solution to 10.1

Recall that the basis $\mathcal{B}$ of Q was $\{p_1(t), p_2(t), p_3(t)\}$. We need to find $[p_i(t)]_{\mathcal{B}}$ for $i = 1, 2, 3, 4$, that is, finding the coefficients such that

$$p_i(t) = a_i p_1(t) + b_i p_2(t) + c_i p_3(t)$$

Then $[p_i(t)]_{\mathcal{B}} = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$

This is easy for the first three polynomials in S, as they are basis vectors themselves. For example,

$$p_1(t) = 1 \cdot p_1(t) + 0 \cdot p_2(t) + 0 \cdot p_3(t)$$

Thus,

$$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $p_4(t)$, we return to the matrix of coordinate vectors that we constructed in the last problem. From the reduced echelon form, we see that $p_4(t)$ is linearly dependent of the other vectors, in the sense that

$$p_4(t) = 3p_1(t) + 5p_2(t) + (-6)p_3(t).$$

Thus,

$$[p_4(t)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}$$

Correct answer:

- $$[p_1(t)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [p_2(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [p_3(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [p_4(t)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}$$

11.1 Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$(1) \qquad T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}, \qquad T\left(\begin{bmatrix}4\\3\end{bmatrix}\right) = \begin{bmatrix}0\\-5\\1\end{bmatrix}.$$

The standard matrix A of T, rank r of T, and $d = \dim(\ker(T))$ are given by: (choose the correct answer)

4 points

Solution to 11.1

Since $T\colon \mathbb{R}^2 \to \mathbb{R}^3$, we know that the standard matrix A of T has to be a (3×2) matrix.

$$A = \begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}$$

However since T is linear, we also know that the standard matrix is given by

$$A = \begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2)\end{bmatrix}$$

So instead of computing each element of A, we can compute each column directly using the given information.

Let $\vec{a} = \begin{bmatrix}3\\2\end{bmatrix}$ and $\vec{b} = \begin{bmatrix}4\\3\end{bmatrix}$. We want to express the unit vectors in terms of $\vec{a}$ and $\vec{b}$.

$$x \cdot \vec{a} + y \cdot \vec{b} = \vec{e}_1$$

$$\begin{bmatrix}3 & 4 & 1\\2 & 3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 3\\0 & 1 & -2\end{bmatrix}$$

So we know that $\vec{e}_1 = 3\vec{a} - 2\vec{b}$.

$$x \cdot \vec{a} + y \cdot \vec{b} = \vec{e}_2$$

$$\begin{bmatrix}3 & 4 & 0\\2 & 3 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -4\\0 & 1 & 3\end{bmatrix}$$

Thus, we find that $\overrightarrow{e}_2 = -4\overrightarrow{a} + 3\overrightarrow{b}$.

By using these relations and the fact that T is linear, we can find the columns of the standard matrix.

$$T(\overrightarrow{e}_1) = T(3\overrightarrow{a} - 2\overrightarrow{b}) = 3T(\overrightarrow{a}) - 2T(\overrightarrow{b}) = 3\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}0\\-5\\1\end{bmatrix} = \begin{bmatrix}3\\16\\7\end{bmatrix}$$

$$T(\overrightarrow{e}_2) = T(-4\overrightarrow{a} + 3\overrightarrow{b}) = -4T(\overrightarrow{a}) + 3T(\overrightarrow{b}) = -4\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}0\\-5\\1\end{bmatrix} = \begin{bmatrix}-4\\-23\\-9\end{bmatrix}$$

Hence the standard matrix A of T is given by

$$A = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix}$$

The rank r of T is the dimension of the range of T, which is the same as the dimension of the column space of A, or the rank of A. The dimension d of the kernel of T is just the same as the dimension of the null space of A. See chapter 4.2.

We row reduce A:

$$A = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix} \sim \begin{bmatrix}1 & -4/3\\0 & 1\\0 & 0\end{bmatrix}$$

A has two pivot columns. Hence the rank of A is 2, which means that the rank r of T is 2.

Finally, by the rank theorem $dim(Null(A)) = n - rank(A) = 2 - 2 = 0$. So $d = dim(Ker(T)) = 0$. This just means that T is one-to-one.

Correct answer:

- $$A = \begin{bmatrix}3 & -4\\16 & -23\\7 & -9\end{bmatrix}, \quad r = 2, \quad d = 0.$$

12.1 Let

$$A = \begin{bmatrix}1 & -14 & 4\\-1 & 6 & -2\\-2 & 24 & -7\end{bmatrix} \quad \text{and} \quad \overrightarrow{v} = \begin{bmatrix}4\\-1\\-7\end{bmatrix}.$$

Then $A^{10}\overrightarrow{v}$ equals to the vector: (choose the correct answer).

You can use the following information without proving it: the eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$, and the corresponding eigenspaces are

$$\text{span}\left\{\begin{bmatrix}3\\-1\\-5\end{bmatrix}\right\}, \quad \text{span}\left\{\begin{bmatrix}-2\\1\\4\end{bmatrix}\right\}, \quad \text{span}\left\{\begin{bmatrix}-4\\2\\7\end{bmatrix}\right\}.$$

4 points

Solution to 12.1

The three given eigenvectors correspond to different eigenvalues, and are therefore linearly independent. Then A is a $(3 \times 3$ matrix with 3 linearly independent eigenvectors, and therefore it is diagonalizable.

We construct P and D from the given information.

$$P = \begin{bmatrix}3 & -2 & -4\\-1 & 1 & 2\\-5 & 4 & 7\end{bmatrix}, \quad D = \begin{bmatrix}-1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

We then find the inverse of P, using the algorithm in chapter 2.2.

$$[P|I] = \begin{bmatrix}3 & -2 & -4 & 1 & 0 & 0\\-1 & 1 & 2 & 0 & 1 & 0\\-5 & 4 & 7 & 0 & 0 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 1 & 2 & 0\\0 & 1 & 0 & 3 & -1 & 2\\0 & 0 & 1 & -1 & 2 & -1\end{bmatrix}$$

This means that $P^{-1} = \begin{bmatrix}1 & 2 & 0\\3 & -1 & 2\\-1 & 2 & -1\end{bmatrix}$.

We know that $A = PDP^{-1}$. Then,

$$\begin{aligned}A^{10} &= (PDP^{-1})^{10} = (PDP^{-1})(PDP^{-1})...(PDP^{-1})(PDP^{-1})\\&= PD(P^{-1}P)D(P^{-1}P)...(P^{-1}P)DP^{-1} = PD^{10}P^{-1}\end{aligned}$$

Since D is a diagonal matrix,

$$D^{10} = \begin{bmatrix}(-1)^{10} & 0 & 0\\0 & 0 & 0\\0 & 0 & 1^{10}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Now we are ready to compute $A^{10}\overrightarrow{v}$.

$$A^{10}\overrightarrow{v} = PD^{10}P^{-1}\overrightarrow{v}$$

$$= PD^{10}(P^{-1}\overrightarrow{v}) = PD^{10}\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ -7 \end{bmatrix} = PD^{10}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = P\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -4 \\ -1 & 1 & 2 \\ -5 & 4 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

Correct answer:

- $$\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

13.1 The quadratic form

$$Q = x^2 - 4xz + 5y^2 + 4z^2$$

is: (choose the correct answer)

3 points

Solution to 13.1

First, we construct the vector $\overrightarrow{x} = \begin{bmatrix} x \\ y \\ x \end{bmatrix}$. Now we can write Q on the form $(\overrightarrow{x})^T A\overrightarrow{x}$, using the method presented in the beginning of chapter 7.2.

$$Q(\overrightarrow{x}) = \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We determine the definiteness of Q by studying the eigenvalues of the matrix A.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -2 \\ 0 & 5-\lambda & 0 \\ -2 & 0 & 4-\lambda \end{bmatrix}$$

Calculating the characteristic polynomial using cofactor expansion along the second row of $A - \lambda I$, we get:

$$\begin{aligned} det(A - \lambda I) &= (5-\lambda)[(1-\lambda)(4-\lambda) - (-2)(-2)] \\ &= (5-\lambda)(4 - \lambda - 4\lambda + \lambda^2 - 4) \\ &= (5-\lambda)(\lambda^2 - 5\lambda) \\ &= 5\lambda^2 - 25\lambda - \lambda^3 + 5\lambda^2 \\ &= -\lambda^3 + 10\lambda^2 - 25\lambda \\ &= -\lambda(\lambda^2 - 10\lambda + 25) \\ &= -\lambda(\lambda - 5)^2 \end{aligned}$$

The roots of the characteristic polynomial, and thus the eigenvalues of A, are $\lambda_1 = 0$ and $\lambda_2 = 5$.

The eigenvalues of A are greater than or equal to zero, which means that $Q(\overrightarrow{x}) \geq 0$ for all $\overrightarrow{x}$. Hence, Q is positive semidefinite.

Correct answer:

- positive semidefinite

14.1 Let

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \qquad \overrightarrow{v} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The value c for which the vectors

$$c\overrightarrow{u} + \overrightarrow{v} \quad \text{and} \quad \overrightarrow{u} + c\overrightarrow{v}$$

are orthogonal is: (choose the correct answer)

3 points

Solution to 14.1

We define the two vectors

$$\overrightarrow{x}_1 = c\overrightarrow{u} + \overrightarrow{v} = c\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c-3 \\ c \\ 0 \\ -2c+1 \end{bmatrix}$$

$$\overrightarrow{x}_2 = \overrightarrow{u} + c\overrightarrow{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-3c \\ 1 \\ 0 \\ -2+c \end{bmatrix}$$

The two vectors are orthogonal to each other if and only if their inner product is 0. We calculate the inner product as a function of c:

$$\begin{aligned} \overrightarrow{x}_1 \cdot \overrightarrow{x}_2 &= (c-3)(1-3c) + c \cdot 1 + 0 \cdot 0 + (-2c+1)(-2+c) \\ &= c - 3c^2 - 3 + 9c + c + 4c - 2c^2 - 2 + c \\ &= -5c^2 + 16c - 5 \end{aligned}$$

Using the quadratic formula, we get that the inner product is 0 for $c = \frac{8 \pm \sqrt{39}}{5}$. Hence the two vectors are orthogonal to each other for these values of c.

Correct answer:

- $c = \frac{8 \pm \sqrt{39}}{5}$

15.1 Let

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

The spectral decomposition of A is given by: (choose the correct answer)

3 points

Solution to 15.1

The spectral decomposition of A is given by its eigenvalues and eigenvectors. We construct $A - \lambda I$ and calculate the eigenvalues from the characteristic polynomial.

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$det(A - \lambda I) = (3-\lambda)(3-\lambda) - 1 \cdot 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8$$

Using the quadratic formula, we get that the roots of the characteristic polynomial are $\lambda_1 = 2$ and $\lambda_2 = 4$, so these are the eigenvalues of A.

Next, we find the corresponding eigenvectors by solving the system $A\overrightarrow{u}_i = \lambda_i \overrightarrow{u}_i$.

$$[A - 2I|\overrightarrow{0}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 is free, $x_1 = -x_2$. We choose $u_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$$[A - 4I|\overrightarrow{0}] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2 is free, $x_1 = x_2$. We choose $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The two eigenvectors are orthogonal since their inner product is one, so to make them orthonormal we only have to normalize them by dividing them with their length.

$$\overrightarrow{v}_1 = \frac{\overrightarrow{u}_1}{|\overrightarrow{u}_1|} = \frac{1}{\sqrt{(-1)^2 + 1^2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\overrightarrow{v}_2 = \frac{\overrightarrow{u}_2}{|\overrightarrow{u}_2|} = \frac{1}{\sqrt{1^2 + 1^2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

Now we can compute the two terms of the spectral decomposition of A, using the representation presented in chapter 7.1.

$$\lambda_1 \overrightarrow{v}_1 \overrightarrow{v}_1^T = 2 \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = 2 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lambda_2 \overrightarrow{v}_2 \overrightarrow{v}_2^T = 4 \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The spectral decomposition of A is given by $A = \lambda_1 \overrightarrow{v}_1 \overrightarrow{v}_1^T + \lambda_2 \overrightarrow{v}_2 \overrightarrow{v}_2^T$.

Correct answer:

- $$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

16.1. The distance from $\vec{b} = \begin{bmatrix} 3 \\ 5 \\ -9 \end{bmatrix} \in \mathbb{R}^3$ to a subspace

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 + x_2 + x_3 = 0\}$$

is equal to: (choose the correct answer)

4 points

Solution to 16.1

The Best Approximation Theorem (theorem 9 chapter 6) tells us that the shortest distance from a vector to a subspace is along a vector that is perpendicular to that subspace. We therefore want to find a vector that is perpendicular to W.

Any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in W has the property that $x_1+x_2+x_3 = 0 \Rightarrow x_3 = -x_1-x_2$, and can thus be written as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = x_1 \vec{v}_1 + x_2 \vec{v}_2$$

Hence, $W = span\{\vec{v}_1, \vec{v}_2\}$. A vector that is perpendicular to W therefore has to be orthogonal to $\vec{v}_1$ and $\vec{v}_2$.

Let us denote this vector as $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then we know that:

$$\vec{u} \cdot \vec{v}_1 = 0 \Rightarrow a - c = 0 \Rightarrow a = c$$
$$\vec{u} \cdot \vec{v}_2 = 0 \Rightarrow b - c = 0 \Rightarrow b = c$$

So any vector $\vec{u} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ is perpendicular to W, and we choose $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Thus, the distance from $\vec{b}$ to W is the length of the component of $\vec{b}$ along $span\,\vec{u}$, which is $proj_{span\{\vec{u}\}}\vec{b}$.

$$d = ||proj_{span\{\vec{u}\}}\vec{b}|| = ||\frac{\vec{b} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}|| = ||\frac{3 \cdot 1 + 5 \cdot 1 + (-9) \cdot 1}{1^2 + 1^2 + 1^2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ||$$

$$= || -\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} || = \sqrt{(-1/3)^2+(-1/3)^2+(-1/3)^2} = \frac{1}{\sqrt{3}}$$

Correct answer:

- $\frac{1}{\sqrt{3}}$

17.1 The certain experiment produces the data

$$\{(0,2),(-1,-1),(2,0),(1,1)\}.$$

The least square curve $y = \beta_1 x + \beta_2 x^2$ is given by: (choose the correct answer)

3 points

Solution to 17.1

Here we use the method presented in chapter 6.6. We want to find the least squares solution of the equation $X\vec{\beta} = \vec{y}$. We will do this by solving the normal equation $X^T X\vec{\beta} = X^T\vec{y}$.

First we construct the design matrix X, the observation vector $\vec{y}$ and the parameter vector $\vec{\beta}$ using the given data. Keep in mind that since we do not want the β_0 term in the expression for our curve, our design matrix will only contain the x and x^2 columns, and not the first columns of 1s that we usually include when solving problems of this type.

$$X = \begin{bmatrix}0 & 0\\-1 & 1\\2 & 4\\1 & 1\end{bmatrix}, \qquad \vec{y} = \begin{bmatrix}2\\-1\\0\\1\end{bmatrix}, \qquad \vec{\beta} = \begin{bmatrix}\beta_1\\\beta_2\end{bmatrix}$$

Next, we compute $X^T X$ and $X^T\vec{y}$.

$$X^T X = \begin{bmatrix}0 & -1 & 2 & 1\\0 & 1 & 4 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\-1 & 1\\2 & 4\\1 & 1\end{bmatrix} = \begin{bmatrix}6 & 8\\8 & 18\end{bmatrix}$$

$$X^T\overrightarrow{y} = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Now we can solve the normal equation for $\overrightarrow{\beta}$ and find the coefficients for our curve. X^TX is a square matrix with linearly independent columns and thus invertible, so we can write:

$$\overrightarrow{\beta} = (X^TX)^{-1}X^T\overrightarrow{y} = \frac{1}{6 \cdot 18 - 8 \cdot 8} \begin{bmatrix} 18 & -8 \\ -8 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{1}{44} \begin{bmatrix} 36 \\ -16 \end{bmatrix} = \begin{bmatrix} 9/11 \\ -4/11 \end{bmatrix}$$

This means that $\beta_1 = 9/11$ and $\beta_2 = -4/11$.

Correct answer:

- $y = \frac{9}{11}x - \frac{4}{11}x^2$

18.1 The following equation

$$2x_1^2 - 6x_1x_2 - 6x_2^2 = \pi$$

defines: (choose the correct answer)

3 points

Solution to 18.1

We can write this quadratic form as $\overrightarrow{x}^TA\overrightarrow{x}$ in the following way:

$$2x_1^2 - 6x_1x_2 - 6x_2^2 = \pi = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The geometric shape of the quadratic form will not change if we use a change of variable. We find the eigenvalues of A.

$$det(A - \lambda I) = det\left(\begin{bmatrix} 2-\lambda & -3 \\ -3 & -6-\lambda \end{bmatrix}\right) = (2-\lambda)(-6-\lambda) - (-3)(-3)$$

$$= -12 - 2\lambda + 6\lambda + \lambda^2 - 9 = \lambda^2 + 4\lambda - 21$$

Using the quadratic formula, we find that A has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -7$. Since A is symmetric, it is orthogonally diagonalizable, and can be written as $A = PDP^T$, where $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$.

Thus by finding two orthonormal eigenvectors of A and constructing P, we could write the quadratic form as $\overrightarrow{y}^T D \overrightarrow{y}$, using the change of variable $\overrightarrow{x} = P\overrightarrow{y} \Rightarrow \overrightarrow{y} = P^{-1}\overrightarrow{x}$.

The quadratic form can then be written as:

$$\overrightarrow{y}^T D \overrightarrow{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 3y_1^2 - 7y_2^2 = \pi$$

We recognize this as the equation of a hyperbola.

Correct answer:

- a hyperbola

19.1 Let

$$\overrightarrow{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \overrightarrow{x}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \quad \overrightarrow{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}.$$

The orthogonal basis of $V = \text{span}\{\overrightarrow{x}_1, \overrightarrow{x}_2, \overrightarrow{x}_3\}$ is given by the following vectors: (choose the correct answer)

3 points

Solution to 19.1

Here we will use the Gram-Schmidt algorithm to produce an orthogonal basis for V. Note that there are multiple orthogonal bases for V, depending on which vector we use as our $\overrightarrow{v}_1$. However since all of our options contain $\overrightarrow{x}_1$, we choose this as our $\overrightarrow{v}_1$.

Then we use the Gram-Schmidt algorithm to find $\overrightarrow{v}_2$ and $\overrightarrow{v}_3$ as described in theorem 11 chapter 6.

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \frac{(-5)\cdot 3 + 1 \cdot 1 + 5 \cdot (-1) + (-7) \cdot 3}{3^2 + 1^2 + (-1)^2 + 3^2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{1 \cdot 3 + 1 \cdot 1 + (-2) \cdot (-1) + 8 \cdot 3}{3^2 + 1^2 + (-1)^2 + 3^2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{1 \cdot 1 + 1 \cdot 3 + (-2) \cdot 3 + 8 \cdot (-1)}{1^2 + 3^2 + 3^2 + (-1)^2} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Correct answer:

- $$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

20.1. Let

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

The matrix R in the QR factorisation of A is given by: (choose the correct answer)

4 points

Solution to 20.1

By theorem 12 chapter 6, the QR-factorisation of A is given by $A = QR$. Q is a matrix whose columns form an orthonormal basis for $Col(A)$, and hence is an orthogonal matrix. We then know that $Q^TQ = I$.

$$A = QR \Leftrightarrow Q^TA = Q^TQR \Leftrightarrow Q^TA = IR \Leftrightarrow R = Q^TA$$

We need to construct Q. In the previous problem we found an orthogonal basis for $V = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$, which we recognize as $Col(A)$. In order to find the columns of Q, we just need to normalize $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

$$\vec{q}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{\sqrt{3^2 + 1^2 + (-1)^2 + 3^2}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{1}{\sqrt{1^2 + 3^2 + 3^2 + (-1)^2}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{|\vec{v}_3|} = \frac{1}{\sqrt{(-3)^2 + 1^2 + 1^2 + 3^2}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}$$

Now we can calculate R using Q and A.

$$R = Q^TA = \frac{1}{2\sqrt{5}} \begin{bmatrix} 3 & 1 & -1 & 3 \\ 1 & 3 & 3 & -1 \\ -3 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{5}} \begin{bmatrix} 20 & -40 & 30 \\ 0 & 20 & -10 \\ 0 & 0 & 20 \end{bmatrix} = \frac{1}{2\sqrt{5}} \begin{bmatrix} 10 & -20 & 15 \\ 0 & 10 & -5 \\ 0 & 0 & 10 \end{bmatrix}$$

Correct answer:

- $$R = \frac{1}{\sqrt{5}} \begin{bmatrix} 10 & -20 & 15 \\ 0 & 10 & -5 \\ 0 & 0 & 10 \end{bmatrix}$$

21.1. Let

$$\mathcal{A} = \{\overrightarrow{a}_1, \overrightarrow{a}_2, \overrightarrow{a}_3\}, \qquad \mathcal{B} = \left\{\overrightarrow{b}_1, \overrightarrow{b}_2, \overrightarrow{b}_3\right\},$$

be two bases of a vector space V. Suppose that

$$\overrightarrow{a}_1 = 4\overrightarrow{b}_1 - \overrightarrow{b}_2, \quad \overrightarrow{a}_2 = -\overrightarrow{b}_1 + \overrightarrow{b}_2 - \overrightarrow{b}_3, \quad \overrightarrow{a}_3 = \overrightarrow{b}_2 - 2b_3.$$

Let also

$$\overrightarrow{x} = 3\overrightarrow{a}_1 + 4\overrightarrow{a}_2 + \overrightarrow{a}_3.$$

Then the matrix $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$ of the change of coordinates from the basis $\mathcal{A}$ to the basis $\mathcal{B}$ and $[\overrightarrow{x}]_{\mathcal{B}}$ are given by: (choose the correct answer)

3 points

Solution to 21.1

The change of coordinates matrix $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$ is given by

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \left[[\overrightarrow{a}_1]_{\mathcal{B}} \quad [\overrightarrow{a}_2]_{\mathcal{B}} \quad [\overrightarrow{a}_3]_{\mathcal{B}}\right]$$

We find the $\mathcal{B}$-coordinate vectors of the basis $\mathcal{A}$ using the definition in chapter 2.8. Later we also find the $\mathcal{A}$-coordinate vector of $\overrightarrow{x}$ in the same way.

$$[\overrightarrow{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \qquad [\overrightarrow{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \qquad [\overrightarrow{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

These vectors are the columns of $\mathcal{P}_{\mathcal{A}\to\mathcal{B}}$, so

$$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}$$

By theorem 15 in chapter 4.7, the coordinate vector of $\overrightarrow{x}$ relative to $\mathcal{B}$ is

$$[\overrightarrow{x}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{A}\to\mathcal{B}}[\overrightarrow{x}]_{\mathcal{A}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$$

Correct answer:

- $$\mathcal{P}_{\mathcal{A}\to\mathcal{B}} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix}, \qquad [\overrightarrow{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$$

22.1. Let

$$\mathcal{B} = \left\{ \overrightarrow{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \overrightarrow{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

be a basis in $\mathbb{R}^2$ and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by $\overrightarrow{x} \mapsto A\overrightarrow{x}$, where

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

Then the $\mathcal{B}$-matrix $M_{\mathcal{B}}$ for the linear transformation T in the basis $\mathcal{B}$ is given by: (choose the correct answer)

4 points

Solution to 22.1

The $\mathcal{B}$-matrix $M_{\mathcal{B}}$ is the matrix that takes us from $[\overrightarrow{x}]_{\mathcal{B}}$ to $[T(\overrightarrow{x})]_{\mathcal{B}}$. By the section "Linear transformations from V into V" in chapter 5.4, it is given by:

$$M_{\mathcal{B}} = \begin{bmatrix} [T(\overrightarrow{b}_1)]_{\mathcal{B}} & [T(\overrightarrow{b}_2)]_{\mathcal{B}} \end{bmatrix}$$

$$T(\overrightarrow{b}_1) = A\overrightarrow{b}_1 = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

We easily see that $[T(\overrightarrow{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, since $T(\overrightarrow{b}_1) = 1\overrightarrow{b}_1 + 0\overrightarrow{b}_2$.

$$T(\overrightarrow{b}_2) = A\overrightarrow{b}_2 = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -3 \end{bmatrix}$$

We need to find the coefficients c_1 and c_2 such that $T(\overrightarrow{b}_2) = c_1\overrightarrow{b}_1 + c_2\overrightarrow{b}_2$.

$$\begin{bmatrix} \overrightarrow{b}_1 & \overrightarrow{b}_2 & T(\overrightarrow{b}_2) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 11 \\ -1 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

Hence $[T(\overrightarrow{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, and the matrix $M_{\mathcal{B}}$ is given by:

$$M_{\mathcal{B}} = \begin{bmatrix} [T(\overrightarrow{b}_1)]_{\mathcal{B}} & [T(\overrightarrow{b}_2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

Correct answer:

- $$M_{\mathcal{B}} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

23.1. Suppose that $\{\overrightarrow{v}_1, \overrightarrow{v}_2\}$ are linear independent in 5-dimensional vector space V. Prove that

$$\{\overrightarrow{u}_1 = \overrightarrow{v}_1 + \overrightarrow{v}_2,\ \overrightarrow{u}_2 = \overrightarrow{v}_1 - \overrightarrow{v}_2\}$$

are also linear independent in V.

8 points

Solution to 23.1

By the definition of linear independence, $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ are linearly inependent if and only if

$$c_1\overrightarrow{u}_1 + c_2\overrightarrow{u}_2 = \overrightarrow{0}$$

only for $c_1 = c_2 = 0$.

$$\begin{aligned} c_1\overrightarrow{u}_1 + c_2\overrightarrow{u}_2 &= \overrightarrow{0} \\ c_1(\overrightarrow{v}_1 + \overrightarrow{v}_2) + c_2(\overrightarrow{v}_1 - \overrightarrow{v}_2) &= \overrightarrow{0} \\ c_1\overrightarrow{v}_1 + c_1\overrightarrow{v}_2 + c_2\overrightarrow{v}_1 - c_2\overrightarrow{v}_2 &= \overrightarrow{0} \\ (c_1 + c_2)\overrightarrow{v}_1 + (c_1 - c_2)\overrightarrow{v}_2 &= \overrightarrow{0} \end{aligned}$$

We know that $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are linearly independent in V. So a linear combination of them can only equal the zero vector if the coefficients in front of the vectors are all 0.

Hence, the above equation is true only for $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. These equations are fulfilled only for $c_1 = c_2 = 0$.

Thus, we have proved that

$$c_1 \overrightarrow{u}_1 + c_2 \overrightarrow{u}_2 = \overrightarrow{0}$$

only for $c_1 = c_2 = 0$. Then, $\overrightarrow{u}_1$ and $\overrightarrow{u}_2$ must also be linearly independent in V.

24.1. Let A be a (5×3) matrix. Suppose that there is a (3×5) matrix B such that

$$BA = I_3.$$

Suppose further that the system $A\overrightarrow{x} = \overrightarrow{b}$ has at least one solution. Prove that this solution is actually unique solution.

8 points

Solution to 24.1

In order to show that the solution of $A\overrightarrow{x} = \overrightarrow{b}$ is unique, we assume by contrary, that there are two different solutions

$$A\overrightarrow{x}_1 = \overrightarrow{b} \quad \text{and} \quad A\overrightarrow{x}_2 = \overrightarrow{b}, \quad \text{and} \quad \overrightarrow{x}_1 \neq \overrightarrow{x}_2.$$

Subtracting from the first equation the second one we obtain

$$A(\overrightarrow{x}_1 - \overrightarrow{x}_2) = \overrightarrow{0}$$

We multiply both sides by B from the left, and use the given relation between A and B.

$$B\big(A(\overrightarrow{x}_1 - \overrightarrow{x}_2)\big) = B\overrightarrow{0}$$
$$(BA)(\overrightarrow{x}_1 - \overrightarrow{x}_2) = \overrightarrow{0}$$
$$I_3(\overrightarrow{x}_1 - \overrightarrow{x}_2) = \overrightarrow{0}$$
$$\overrightarrow{x}_1 - \overrightarrow{x}_2 = \overrightarrow{0}$$
$$\overrightarrow{x}_1 = \overrightarrow{x}_2$$

Hence, the vectors $\overrightarrow{x}_1$ and $\overrightarrow{x}_2$ are equal that contradict to the assumption that they were different.

We have thus proved that the solution of the equation is unique.

25.1. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The sum of the diagonal entries $\operatorname{tr} A = a + d$ in A is called the trace of the matrix A. Show that the characteristic polynomial of A can be written as

$$\det(A - \lambda I) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A.$$

Hence give the condition for A to have real eigenvalues.

8 points

Solution to 25.1

We first construct the matrix $A - \lambda I$ and find the characteristic polynomial in the usual way.

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}$$

$$det(A - \lambda I) = (a - \lambda)(d - \lambda) - c \cdot b = ad - a\lambda - d\lambda + \lambda^2 - bc = \lambda^2 - (a + d)\lambda + (ad - bc)$$

We recognize $a + d$ as the trace of A, and $ad - bc$ as the determinant of A. Hence, the characteristic polynomial is given by

$$det(A - \lambda I) = \lambda^2 - (trA)\lambda + det(A)$$

The eigenvalues of A are the roots of the characteristic polynomial. Using the quadratic formula, we get

$$\lambda_{1,2} = \frac{-(-trA) \pm \sqrt{(-trA)^2 - 4 \cdot 1 \cdot det(A)}}{2 \cdot 1}$$

In order for the eigenvalues to be real, the discriminant has to be greater than or equal to 0. We then have the condition:

$$(-trA)^2 - 4 \cdot 1 \cdot det(A) \geq 0$$
$$(trA)^2 - 4det(A) \geq 0$$

Hence, A has real eigenvalues when $(trA)^2 - 4det(A) \geq 0$.