NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
MA0003 Mathematical Methods for Computer Science

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Examination time (from–to): 9:00–13:00

Permitted examination support material: C: Specified printed and handwritten materials allowed. Selected, basic calculators allowed.

1. One yellow A4-sized sheet stamped "Institutt for matematiske fag", on which you can write as much as you like by hand.
2. *Matematisk formelsamling*, by Rottmann.
3. Simple calculator: Citizen SR-270X, Citizen SR-270X College or Hewlett Packard HP30S, or similar.

Language: English

Number of pages: 7

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1 Compute the following limits.

a) $\lim_{x \to 2} (x^2 + 2x - 1)$

b) $\lim_{h \to 0} \left(\frac{\cos(x+h) - \cos(x)}{h} \right)$

Solution: **a)** Since $x^2 + 2x - 1$ is continuous, we have

$$\lim_{x \to 2} (x^2 + 2x - 1) = (2)^2 + 2(2) - 1 = 7.$$

b) Note that the limit in the question is of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

where $f(x) = \cos(x)$. But this is just the definition of the derivative of $f(x)$. Thus

$$\lim_{h \to 0} \left(\frac{\cos(x+h) - \cos(x)}{h} \right) = \frac{d}{dx}(\cos(x)) = -\sin(x).$$

Problem 2 Determine if the following matrices have inverses. Explain why or why not. Do not compute the inverses.

a) $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{bmatrix}$

Solution: **a)** The determinant of A is

$$\det A = 0 \cdot 3 - 1 \cdot 2 = -2.$$

Therefore the matrix has an inverse.

b) Since B is not a square matrix, it does not have an inverse.

c) The determinant of C is

$$\det C = 1 \cdot \det \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Each of the smaller determinants is zero, and so the determinant of C is zero. Thus, C does not have an inverse.

Problem 3 Differentiate the function

$$g(x) = x^2 e^x + \cos(\ln |x|).$$

Solution: We apply the product rule to the first term, and the chain rule to the second term to obtain

$$g'(x) = 2xe^x + x^2 e^x - \sin(\ln |x|) \cdot \frac{1}{x}.$$

Problem 4 Solve the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

Solution: The solution to the equation $A\boldsymbol{x} = \boldsymbol{b}$ is given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$, where A^{-1} is the inverse of A. Using the procedure discussed in lecture, the inverse is given by

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Multiplying by b then gives

$$A^{-1}b = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

Problem 5 Compute the matrices corresponding to the following linear transformations.

a) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 2x_1 \end{pmatrix}$

b) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 4x_2 \\ -2x_1 + 3x_2 \end{pmatrix}$

Solution **a)** We have

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and so the matrix corresponding to T is

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

b) As before, we compute

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

and

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix},$$

so the matrix for T is

$$\begin{pmatrix} 2 & -1 \\ 1 & 4 \\ -2 & 3 \end{pmatrix}.$$

Problem 6 Compute the following integrals.

a) $\int_0^1 x + x^2 + x^{10}\ dx$

b) $\int x \sin x\ dx$

c) $\int_0^{\frac{\pi}{2}} \cos(x) e^{\sin(x)}\ dx$

Solution: **a)** By the power rule, we have

$$\begin{aligned}\int_0^1 x + x^2 + x^{10}\ dx &= \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{11}x^{11}\Big|_0^1 \\ &= \frac{1}{2}(1)^2 + \frac{1}{3}(1)^3 + \frac{1}{11}(11)^{11} - \left(\frac{1}{2}(0)^2 + \frac{1}{3}(0)^3 + \frac{1}{11}(0)^{11}\right) \\ &= \frac{61}{66}.\end{aligned}$$

b) We use integration by parts.

$$\begin{aligned} u &= x & dv &= \sin x\ dx \\ du &= dx & v &= -\cos x \end{aligned}$$

Applying the IBP formula, we get

$$\begin{aligned}\int x \sin x\ dx &= -x\cos x - \int -\cos x\ dx \\ &= \sin x - x\cos x + C\end{aligned}$$

c) We apply u-Substitution. Let $u = \sin x$. Then $du = \cos x\ dx$. This allows us to rewrite the integral as

$$\int \cos x\ e^{\sin x}\ dx = \int e^u\ du = e^u + C = e^{\sin x} + C.$$

Thus, the definite integral is

$$\int_0^{\frac{\pi}{2}} \cos(x)e^{\sin(x)}\,dx = e^{\sin x}\Big|_0^{\pi/2} = e^{\sin\left(\frac{\pi}{2}\right)} - e^0 = e^1 - 1 = e - 1.$$

Problem 7 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Compute the following matrices, if they are defined. If they are not defined, explain why not.

a) $2A$

b) $A + B$

c) AB

d) BA

Solution: **a)** We have

$$\begin{aligned} 2A &= 2\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2\cdot 1 & 2\cdot 2 & 2\cdot(-1) \\ 2\cdot 0 & 2\cdot 3 & 2\cdot(-2) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & -4 \end{bmatrix} \end{aligned}$$

b) Addition and subtraction of matrices is only defined for matrices of the same size. So this sum is not defined.

c) This product is defined, and is given by

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 3 \\ 6 & 1 \end{bmatrix} \end{aligned}$$

d) This product is also defined, and is given by

$$BA = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 8 & -5 \\ 2 & 7 & -4 \\ 0 & 3 & -2 \end{bmatrix}$$

Problem 8 A spherical balloon is being inflated in such a way that its radius is increasing at a rate of 3 cm/min. If the radius of the balloon is initially 5 cm, how fast is the volume increasing after the balloon has been inflated for 2 minutes? (*Hint: You will need that the volume of a sphere is* $V = \frac{4}{3}\pi r^3$.)

Solution: We can solve this using implicit differentiation. The idea is that the question is asking for the rate of change of V, as a function of time. However, the formula given to you in the question tells you the volume as a function of radius. In addition, you are given the rate of change of the radius. Thus, we can think of the formula in the hint as

$$V(r(t)) = \frac{4}{3}\pi(r(t))^3.$$

Differentiating implicitly, we get

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$
$$= 4\pi(r(t))^2\frac{dr}{dt}.$$

If the balloon starts with a radius of 5 cm and increases by 3 cm every minute, then when $t = 2$, $r = 11$. Thus

$$\frac{dV}{dt} = 4\pi(r(t))^2\frac{dr}{dt}$$
$$= 4\pi(11)^2(3)$$
$$= 1452\pi \ cm^3/min.$$

Problem 9 Solve the following initial value problems.

a) $\dfrac{dy}{dx} = x^2 + 2x - 5, \ y(0) = -4$

b) $\dfrac{dy}{dx} = \dfrac{y^2}{x}, \ y(1) = 5$

Solution: **a)** This is a Type I problem, and so we solve it by integrating:

$$y = \int x^2 + 2x - 5 \; dx = \frac{1}{3}x^3 + x^2 - 5x + C.$$

Applying the initial condition, we get

$$-4 = \frac{1}{3}(0)^3 + (0)^2 - 5(0) + C = C.$$

Thus

$$y = \int x^2 + 2x - 5 \; dx = \frac{1}{3}x^3 + x^2 - 5x - 4.$$

b) This is a Type II problem, and we solve it by separating the variables:

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x}$$

Next, we integrate both sides:

$$\int \frac{1}{y^2}\frac{dy}{dx} \; dx = \int \frac{1}{x} \; dx$$

Using the definition of the differential, we can write the left side as

$$\int \frac{1}{y^2}\frac{dy}{dx} \; dx = \int \frac{1}{y^2} \; dy$$

Thus

$$\int \frac{1}{y^2} \; dy = \int \frac{1}{x} \; dx.$$

The integrals become

$$-\frac{1}{y} = \ln|x| + C.$$

Applying the initial condition, we get

$$-\frac{1}{5} = \ln|1| + C = C.$$

So $C = -1/5$. Thus

$$y = \frac{1}{\frac{1}{5} - \ln|x|}.$$