NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4110 Matematikk 3**

Academic contact during examination: Antoine Julien[a], Markus Szymik[b]

Phone: [a]73597782 , [b]41116793

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Examination time (from–to): 09:00–13:00

Permitted examination support material: C: Simple Calculator (Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College, or Hewlett Packard HP30S), Rottmann: Matematiske formelsamling

Other information:
Give reasons for all answers, ensuring that it is clear how the answer has been reached. Each exercise has the same weight.

Language: English

Number of pages: 3

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Checked by:

Date Signature

Problem 1 In this exercise, we consider the complex numbers

$$z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{and} \quad z_2 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

a) Write z_1/z_2 in the form $z_1/z_2 = a + ib$ (do not use the cos or sin functions).

b) Compute the modulus and an argument of z_1 and z_2. Write z_1 and z_2 in polar form.

c) Write z_1/z_2 in the form $z_1/z_2 = \rho e^{i\theta}$.

d) Deduce from the above the values of $\cos(\pi/12)$ and $\sin(\pi/12)$.

Problem 2 In this exercise, we consider the differential equation

$$y'' - 4y' + 4y = g(x).$$

a) Compute the general solution of the homogeneous equation.

b) Compute a particular solution when $g(x) = e^{-2x}$ and when $g(x) = e^{2x}$.

c) Compute the general solution of the equation when

$$g(x) = \frac{1}{4}(e^{-2x} + e^{2x}).$$

Problem 3 Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & a \end{bmatrix}.$$

a) For which values of a is this matrix invertible?

b) Compute A^{-1}, when this inverse exists.

Problem 4 In this exercise, we consider the matrix A given by

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

a) Find the eigenvalues of A?

b) Find a non-zero eigenvector for each eigenvalue of A.

c) Find a basis of $\mathbb{R}^3$ made of eigenvectors of A.

d) Find an *orthonormal* basis of $\mathbb{R}^3$ made of eigenvectors of A.

e) Find an orthogonal matrix P and a diagonal matrix D such that $D = P^{\mathrm{T}}AP$.

Problem 5

a) Given the data pairs

$$\begin{aligned} a_1 &= 1, \quad b_1 = 2, \\ a_2 &= 2, \quad b_2 = 3, \\ a_3 &= 3, \quad b_3 = 5, \end{aligned}$$

express the system

$$\begin{aligned} a_1x_1 + x_2 &= b_1 \\ a_2x_1 + x_2 &= b_2 \\ a_3x_1 + x_2 &= b_3 \end{aligned}$$

of linear equations in matrix form $A\mathbf{x} = \mathbf{b}$: What are A, $\mathbf{x}$ and $\mathbf{b}$?

b) For A and $\mathbf{b}$ as in (b), show that $A\mathbf{x} = \mathbf{b}$ does not have a solution.

c) Use the least squares method to find an approximate solution $\mathbf{x}$ for the equation $A\mathbf{x} = \mathbf{b}$.

d) For $\mathbf{x}$ as in (d), sketch the three data points and the line $b = x_1a + x_2$ into a coordinate system.

e) For $\mathbf{x}$ as in (d), compute $4x_1 + x_2$?

Problem 6

a) Solve the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + 2x_2 + 4x_3 &= 3 \\ x_1 + 3x_2 + 9x_3 &= 5. \end{aligned}$$

b) Let

$$p_{\mathbf{x}}(t) = x_1 + x_2 t + x_3 t^2$$

denote the polynomial with real coefficients $x_1, x_2, x_3 \in \mathbb{R}$. The transformation

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} p_{\mathbf{x}}(1) \\ p_{\mathbf{x}}(2) \\ p_{\mathbf{x}}(3) \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + 2x_2 + 4x_3 \\ x_1 + 3x_2 + 9x_3 \end{bmatrix}$$

is linear. Find the matrix A that describes this linear transformation.

c) For A as in (b), show that A is invertible.

d) For A as in (b), find $\mathbf{x}$ such that

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

e) For $\mathbf{x}$ as in (d), compute $p_{\mathbf{x}}(4) = x_1 + 4x_2 + 16x_3$.

Problem 7 Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero, independent vectors in $\mathbb{R}^3$. Let $\mathbf{w}$ be a nonzero vector in $\mathbb{R}^3$. Show that there exists a non-zero linear combination of $\mathbf{u}$ and $\mathbf{v}$ which is orthogonal to $\mathbf{w}$.