# On sampling in $\mathbb{R}^n$, $n > 1$

Based on joint work with Alexander Olevskii

Bergen, 2013

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For simplicity of presentation, we shall consider the sampling problem only for the Paley–Wiener and Bernstein spaces.

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There is a number of results on sampling of entire functions of one variable. However, there is a fundamental difference between one and several variables: In dimension $n = 1$ the zeros of an entire function $f$ are discrete, and there is a precise connection between the asymptotic behavior/density of the zeros and the growth of $f$. In several dimensions, the zero sets are analytic manifolds, and standard complex variable techniques do no longer apply. That is why only a few general results are known in higher dimensions, most notably the classical Landau's necessary density condition and Beurling's sufficient condition for the unit ball.

# Paley–Wiener and Bernstein spaces

Let $\mathcal{S} \subset \mathbb{R}^n$ be a bounded subset with positive Lebesgue measure. We set $x = (x_1, ..., x_n)$, $t = (t_1, ..., t_n)$, $x \cdot t = x_1 t_1 + ... + x_n t_n$, $|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + ... + x_n^2}$.

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The **Paley-Wiener space** $PW_{\mathcal{S}}$ is the set of all $f \in L^2(\mathbb{R}^N)$ whose spectrum belongs to $\mathcal{S}$, i.e.

$$PW_{\mathcal{S}} := \{f \in L^2(\mathbb{R}^n) : \hat{f}(x) = 0, x \in \mathbb{R}^n \setminus \mathcal{S}\}.$$

Here $\hat{f}$ denotes the Fourier transform of $f$,

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**Example**. The function $f(t) := 2\frac{\sin t}{t}$ is the Fourier transform of the function $\mathbf{1}_{[-1,1]}$. Hence, $f \in PW_{[-1,1]}$. Similarly, we have

$$f(t_1)f(t_2) = 4\frac{\sin t_1}{t_1}\frac{\sin t_2}{t_2} \in PW_{[-1,1]^2}.$$

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The **Bernstein space** $B_\mathcal{S}$ is the set of all bounded functions on $\mathbb{R}^n$ whose spectrum belongs to $\mathcal{S}$. The latter means that

$$\int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = 0, \quad f \in B_\mathcal{S},$$

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**Example**. The exponential function $f(t) := e^{it\cdot\xi}$ is the Fourier transform of the delta-measure $\delta_\xi$. It belongs to $B_\mathcal{S}$ if and only if $\xi \in \mathcal{S}$.

# Lower uniform density

Recall that a set $\Lambda$ is called **uniformly discrete** if

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Given a set $\Lambda \subset \mathbb{R}^n$, the density

$$D^-(\Lambda) := \lim_{r\to\infty} \frac{\min_{x\in\mathbb{R}^n} \#(\Lambda \cap (x + r\mathcal{B}))}{\mathrm{Vol}\,(r\mathcal{B})}$$

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Condition $D^-(\Lambda) = 1$ means that (asymptotically as $r \to \infty$) the number of points of $\Lambda$ in each ball $x + r\mathcal{B}$ is at least as large as the number of points of $\mathbb{Z}^n$ in this ball.

# Uniform density



We say that $\Lambda$ possesses a uniform density $D(\Lambda)$, if $\Lambda$ is regularly distributed in the sense that

$$\max_x \left| \frac{\#(\Lambda \cap (x + r\mathcal{B}))}{\mathrm{Vol}\,(r\mathcal{B})} - D(\Lambda) \right| \to 0, \ \ r \to \infty.$$

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Condition $D(\Lambda) = 1$ means that (asymptotically as $r \to \infty$) the number of points of $\Lambda$ in each ball $x + r\mathcal{B}$ is "approximately the same" as the number of points of $\mathbb{Z}^n$ in this ball.

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$$C_1\|f\|_2 \leq \|f|_\Lambda\|_2 \leq C_2\|f\|_2, \text{for every} \quad f \in PW_S,$$

where

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Every sampling set $\Lambda$ is either uniformly discrete or a union of several uniformly discrete sets.

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A set $\Lambda \subset \mathbb{R}^n$ is called a sampling set for $B_S$, if there exists a positive constant $C$ such that

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Here a sampling set $\Lambda$ can be any subset of $\mathbb{R}^n$.

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**Theorem**. *Given a compact $S \subset \mathbb{R}^n, n \geq 1$, a uniformly discrete $\Lambda$ and $\epsilon > 0$.*

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*(i) If $\Lambda$ is a sampling set for $B_{\mathcal{S}+\epsilon\mathcal{B}}$, then $\Lambda$ is a sampling set for $PW_{\mathcal{S}}$.*

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This theorem will be proved at the end of this talk.

# Classical case: $S = [a, b] \subset \mathbb{R}$ (very shortly!)

**Beurling's theorem** (from: "Balayage of Fourier-Stieltjes transforms", 1977). $\Lambda \subset \mathbb{R}$ *is a sampling set for* $B_{[a,b]}$ *if and only if*

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Hence, Sampling sequences for $B_{[a,b]}$ can be described in terms of the density $D^-(\Lambda)$!

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The "critical" situation is when $D^-(\Lambda) = \frac{b-a}{2\pi}$. Example: $\Lambda = \mathbb{Z}$ is an SS for $PW_{[-\pi,\pi]}$, while $\Lambda = \mathbb{Z} \setminus \{0\}$ is NOT an SS for $PW_{[-\pi,\pi]}$.

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**Ortega-Cerdá, J., Seip, K.** (2002): Complete description of sampling sequences for $PW_{[a,b]}$.

# Landau's necessary density condition for sampling

**Landau's necessary density condition** (1967): *Condition*

$$D^-(\Lambda) \geq \frac{mes(\mathcal{S})}{(2\pi)^n}$$

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*is necessary for stable sampling in $PW_{\mathcal{S}}$, for every bounded set $\mathcal{S} \subset \mathbb{R}^n, n \geq 1$.*

Take any bounded set $\mathcal{S} \subset \mathbb{R}^n$ of measure $(2\pi)^n$. Landau's result may be understood as saying that if $\Lambda$ an SS for $PW_{\mathcal{S}}$, then $\Lambda$ is everywhere at least as dense as $\mathbb{Z}^n$.

# Sufficient conditions for sampling

The situation becomes much more delicate for disconnected spectra in $\mathbb{R}$, already when $S$ is a union of two intervals. If $S$ is not an interval, then sufficient conditions for sampling cannot be given in terms of density of $\Lambda$. The ARITHMETICAL STRUCTURE of $\Lambda$ comes into the play.

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**Example**. Let $\mathcal{S} = [-\pi - 1, -\pi + 1] \cup [\pi - 1, \pi + 1]$. It is easy to check that the function

$$f(t) := \sin \pi t \frac{\sin t}{t} \in PW_{\mathcal{S}}.$$

Since $f(j) = 0, j \in \mathbb{Z}$, the set of integers $\mathbb{Z}$ is not an SS for $PW_{\mathcal{S}}$.

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Since $f(j) = 0, j \in \mathbb{Z}$, the set of integers $\mathbb{Z}$ is not an SS for $PW_{\mathcal{S}}$. However, it follows from a general result on universal sampling (Olevskii, U., 2006) that for every $\delta > 0$ there is a "small perturbation" of integers $\Lambda := \{j + \delta_j\}, |\delta_j| < \delta$, which is an SS for $PW_{\mathcal{S}}$.

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This is not so even for sampling on lattices!

The "simplest" sampling sets are lattices i.e. sets $\Lambda = T\mathbb{Z}^n$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear operator. Each lattice possesses a uniform density $D(\Lambda) = 1/\mathrm{Vol}(T[0,1]^n)$. Denote by $\Lambda^* := (T^*)^{-1}\mathbb{Z}^n$ the dual lattice.

# Sampling on lattices

**Claim**. *Let* $\Lambda$ *be a lattice.*
*(i) Assume* $S \subset \mathbb{R}^n$ *is a bounded set. Then* $\Lambda$ *is a sampling set for* $PW_{\mathcal{S}}$ *if and only if*

$$mes((\mathcal{S} + 2\pi\lambda^*) \cap \mathcal{S}) = 0, \text{ for every } \lambda^* \in \Lambda^*.$$

*(ii) Assume* $S \subset \mathbb{R}^n$ *is a compact set. Then* $\Lambda$ *is a sampling set for* $B_{\mathcal{S}}$ *if and only if*

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**Proof**. For simplicity assume that $\Lambda = \mathbb{Z}^n$. Then $\Lambda^* = \mathbb{Z}^n$.

# Sampling on lattices



(i) 1. Assume $\mathrm{mes}((\mathcal{S} + 2\pi k) \cap \mathcal{S}) > 0, k \in \mathbb{Z}^n$. Then there exist a set of positive measure $U \subset \mathcal{S}$ and $k \in \mathbb{Z}^n$ such that $U + 2\pi k \subset \mathcal{S}$.

# Sampling on lattices



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Take any non-trivial $L^2-$function $F$ which vanishes outside $U$, and set

$$f(t):=\hat{F}(t)-e^{it\cdot 2\pi k}\hat{F}(t)=(1-e^{it\cdot 2\pi k})\hat{F}(t).$$

Clearly, $f$ is the Fourier transform of $F(x)-F(x-2\pi k)$. The support of the latter belongs to $U\cup(U+2\pi k)\subset\mathcal{S}$, so that $f\in PW_{\mathcal{S}}$. On the other hand, we see that $f$ vanishes on $\mathbb{Z}^n$, so that $\mathbb{Z}^n$ is not an SS for $PW_{\mathcal{S}}$. (In fact, $\mathbb{Z}^n$ is not even a uniqueness set for $PW_{\mathcal{S}}$!)

2. Assume $\text{mes}((\mathcal{S} + 2\pi k) \cap \mathcal{S}) = 0, k \in \mathbb{Z}^n$. For every function $F \in L^2(\mathcal{S})$ set

$$F^*(x) := \sum_{j \in \mathbb{Z}^n} F(x - 2\pi j) \cdot \mathbf{1}_{[0,2\pi]^n}(x).$$

($F^*$ is the "projection" of $F$ onto $[0, 2\pi]^n$.)

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It is clear that $\text{supp} F^* \subseteq [0, 2\pi]^n$ and that $\|F^*\|_2 = \|F\|_2$.

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$$F^*(x) := \sum_{j \in \mathbb{Z}^n} F(x - 2\pi j) \cdot \mathbf{1}_{[0,2\pi]^n}(x).$$

($F^*$ is the "projection" of $F$ onto $[0, 2\pi]^n$.)

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Since the Fourier transform of $F(x - 2\pi k)$ is $e^{it\cdot 2\pi k}\hat{F}(t)$, we see that $\hat{F}^*(j) = \hat{F}(j)$ for $j \in \mathbb{Z}^n$. The trigonometric system $\{\exp(ik \cdot x), k \in \mathbb{Z}^n\}$ forms an orthogonal basis in $L^2$ on the torus $[0, 2\pi]^n$. Hence, by Parseval's identity,

$$\|\hat{F}\|_2 = \|\hat{F}^*\|_2 = \|\hat{F}^*|_{\mathbb{Z}^n}\|_2 = \|\hat{F}|_{\mathbb{Z}^n}\|_2,$$

which proves that $\mathbb{Z}^n$ is an SS for $PW_{\mathcal{S}}$.

(ii) Assume $(\mathcal{S} + 2\pi\mathbb{Z}^n) \cap \mathcal{S} \neq \emptyset$. Then there exist $\xi \in \mathcal{S}$ and $k \in \mathbb{Z}^n$ such that $\xi + 2\pi k \in \mathcal{S}$. The function

$$f(x) := e^{-ix\cdot\xi} - e^{-ix\cdot(\xi+2\pi k)} = (1 - e^{-ix\cdot 2\pi k})e^{-ix\cdot\xi}$$

is the Fourier transform of the measure $\delta_\xi - \delta_{\xi+2\pi k}$. Since the support of this measure belongs to $\mathcal{S}$, we have $f \in B_{\mathcal{S}}$. On the other hand, it is clear that $f(j) = 0$ for every $j \in \mathbb{Z}^n$, so that $\mathbb{Z}^n$ is not a sampling set for $B_{\mathcal{S}}$. (In fact, $\mathbb{Z}^n$ is not even a uniqueness set for $B_{\mathcal{S}}$!)

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Assume $(\mathcal{S} + 2\pi\mathbb{Z}^n) \cap \mathcal{S} = \emptyset$. Then there exists $\epsilon > 0$ such that $(\mathcal{S} + \epsilon\mathcal{B}) \cap \mathcal{S} = \emptyset$. It follows from part (i) that $\mathbb{Z}^n$ is an SS for $PW_{\mathcal{S}+\epsilon\mathcal{B}}$. Hence, the theorem above (about connection between sampling for $PW-$ and $B-$spaces) shows that $\mathbb{Z}^n$ is a sampling set for $B_{\mathcal{S}}$.

# Sampling on lattices

**Corollary**. Given a convex closed set $\mathcal{S} \subset \mathbb{R}^n$, $n \geq 2$. For every $\epsilon > 0$ there is a lattice $\Lambda \subset \mathbb{R}^n$ with $D(\Lambda) > 1/\epsilon$ which is NOT a sampling set for both $B_{\mathcal{S}}$ and $PW_{\mathcal{S}}$.

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Indeed, by applying appropriate change of variables, one can see that $\Lambda = T\mathbb{Z}^n$ *is an SS for* $PW_\mathcal{S}$ *if and only if* $\mathbb{Z}^n$ *is an SS for* $PW_{T^*(\mathcal{S})}$. Then given any bounded set $\mathcal{S}$, one can find a linear transformation $T$ with arbitrarily small determinant (i.e. $T\mathbb{Z}^n$ has a small density) such that $T^*(\mathcal{S}) \cap \mathbb{Z}^n$ contains at least two different points. This implies that $T\mathbb{Z}^n$ is not an SS for both $B_\mathcal{S}$ and $PW_\mathcal{S}$.

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Observe however (A.O., A.U., 2006) that certain arbitrarily small perturbations of each lattice $\Lambda$ produce universal sampling sets (i.e. sampling sets for all spaces $B_\mathcal{S}$ and $PW_\mathcal{S}$, where $\mathcal{S}$ is compact and $\text{mes}(\mathcal{S}) < (2\pi)^n/D(\Lambda)$!)

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$$\Lambda + \rho\mathcal{B} = \mathbb{R}^n.$$

*Then*

$$\|f\|_\infty \leq \frac{1}{1 - \sin\rho} \|f|_\Lambda\|_\infty, \quad \textit{for every } f \in B_{\mathcal{B}}, \qquad (1)$$

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Condition $\Lambda + \rho\mathcal{B} = \mathbb{R}^n$ means that $\Lambda$ is an $\rho$-net, i.e. for every $x$ there exists $\lambda \in \Lambda$ with $|x - \lambda| \leq \rho$. Hence, each $\rho$-net, where $\rho < \pi/2$, is a sampling set for $B_\mathcal{B}$. This is sharp. Beurling shows that the theorem ceases to be true for $\pi/2-$nets.

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Beurling mentions (no proof is available) that a better than (1) estimate holds:

$$\|f\|_\infty \leq \frac{1}{\cos\rho}\|f|\Lambda\|_\infty.$$

# Beurling's sufficient condition for an arbitrary convex set

Let $\mathcal{K}$ be a closed convex central-symmetric body. Denote by

$$\mathcal{K}^o := \{x \in \mathbb{R}^n : x \cdot t \leq 1 \text{ for all } t \in \mathcal{K}\}$$

the polar body of $\mathcal{K}$. Observe that $\mathcal{B}^o = \mathcal{B}$.

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the polar body of $\mathcal{K}$. Observe that $\mathcal{B}^o = \mathcal{B}$.

Beurling also mentions (no proof is available) that every set $\Lambda$ satisfying $\Lambda + \rho\mathcal{K}^o = \mathbb{R}^n$ with some $\rho < \pi/2$ is an SS for $B_{\mathcal{K}}$.

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Condition $\Lambda + \rho \mathcal{K}^o = \mathbb{R}^n$ means that for every $x \in \mathbb{R}^n$ there exists $\lambda \in \Lambda$ such that $\|x - \lambda\|_{\mathcal{K}^o} \le \rho$, i.e. $\Lambda$ is a $\rho$-net in the $\| \cdot \|_{\mathcal{K}^o}$−norm.

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**Proposition**. *Suppose $\mathcal{S}$ and $\mathcal{K}$ are closed convex central-symmetric bodies, and suppose that $\mathcal{S}$ contains a point $x_0$ with $\|x_0\|_{\mathcal{K}^o} = \pi/2$. Then there exists $\Lambda \subset \mathbb{R}^n$ with $\Lambda + \mathcal{S} = \mathbb{R}^n$ which is NOT a uniqueness set for $B_{\mathcal{K}}$ (i.e. there exists $f \in B_{\mathcal{K}}$ satisfying $f|_\Lambda = 0$.)*

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**Corollary**. *Suppose $\mathcal{S}$ has the property that every set $\Lambda \subset \mathbb{R}^n$ such that $\Lambda + \mathcal{S} = \mathbb{R}^n$ is a sampling set for $B_{\mathcal{K}}$. Then $\mathcal{S} \subset \rho\mathcal{K}^o$ for some $\rho < \pi/2$.*

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Observe that the estimate

$$\|f\|_\infty \leq \frac{1}{\cos\rho}\|f|_\Lambda\|_\infty, \quad \text{for every } f \in B_\mathcal{K},$$

is also best possible.

# Proof of Proposition

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By assumption, there exist $x_0 \in \mathcal{S}$ and $t_0 \in \mathcal{K}$ such that $x_0 \cdot t_0 = \pi/2$. Observe that the spectrum of the function $\sin(x \cdot t_0)$ consists of $\pm t_0 \in \mathcal{K}$, and so $\sin(x \cdot t_0) \in B_{\mathcal{K}}$.

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Let $\Lambda := \{x \in \mathbb{R}^n : x \cdot t_0 \in \pi\mathbb{Z}\}$ be the zero set of $\sin(x \cdot t_0)$. Denote by $I = \{\tau x_0 : -1 \leq \tau \leq 1\} \subseteq \mathcal{S}$ the interval from $-x_0$ to $x_0$. Clearly, for every point $y \in \mathbb{R}^n$ there exists $n \in \mathbb{Z}$ and $-1 \leq \tau \leq 1$ such that $y \cdot t_0 = \pi n - \tau\pi/2$. Hence, $y - \tau x_0 \in \Lambda$, which implies $\Lambda + I = \mathbb{R}^n$.

# Some remarks

1. Recall, that by Landau's necessary condition,

$$D^-(\Lambda) \geq \frac{\mathrm{Vol}\,(\mathcal{K})}{(2\pi)^n}.$$

However, every $\rho-$net (in the $\|\cdot\|_{\mathcal{K}^\circ}-$norm) must be "much denser":

$$D^-(\Lambda) \geq \mathrm{Vol}(\mathcal{K})C^n n^n, \quad C \text{ is some absolute constant,}$$

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The latter inequality follows from the Blaschke-Santaló inequality: $\mathrm{Vol}(\mathcal{K}) \cdot \mathrm{Vol}(\mathcal{K}^o) \leq (\mathrm{Vol}(\mathcal{B}))^2$.

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Our proof is based on the following result by Duffin and Schaeffer:

# Duffin and Schaeffer's result

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*Let* $f \in B_{[-\tau,\tau]}$ *be a real function satisfying* $-1 \leq f(x) \leq 1$ *for all* $x \in \mathbb{R}$*. Then for every real a the function* $\cos(\tau z + a) - f(z)$ *vanishes identically or else it has only real zeros. Moreover it has a zero in every interval where* $\cos(\tau z + a)$ *varies between -1 and 1 and all the zeros are simple, except perhaps at points on the real axis where* $f(x) = \pm 1$*.*

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This result (and its extensions) have been used to obtain various inequalities for entire functions of exponential type.

# Proof of Duffin and Schaeffer's result

**Proof**. We may assume $\tau = 1, a = 0$. Set

$$f_\epsilon(z) := \frac{\sin \epsilon z}{\epsilon z}(1-\epsilon)f((1-\epsilon)z), 0 < \epsilon < 1.$$

We have $|f_\epsilon(z)| \le e^{|y|}/(\epsilon|z|)$. Let $\mathcal{S}_N$ be a closed rectangular contour consisting of segments of the lines $x = \pm 2\pi N, y = \pm N$. Then if $N \in \mathbb{N}$ is sufficiently large, we have $|f_\epsilon(z) < |\cos z|$, $z \in \mathcal{S}_N$. Hence, by Rouché's theorem the function $\cos z - f_\epsilon(z)$ has the same number of zeros in $\mathcal{S}_N$ as $\cos z$. Since $|f_\epsilon(x) \le 1 - \epsilon$, so $\cos x - f_\epsilon(x)$ is alternately plus and minus at the $2K + 1$ points $\pm \pi k, |k| \le N$, and so inside $\mathcal{S}_N$ it has at least $2K$ real zeros. Hence there are no complex zeros, and there is exactly one (simple) zero in each interval $(\pi k, \pi(k+1))$, $|k| \le N$. Taking larger values of $N$ we see that $\cos x - f_\epsilon(x)$ has exclusively real and simple zeros, which lie in the intervals $(\pi k, \pi(k+1))$. To prove the theorem, we let $\epsilon \to 0$.

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**Clunie, Rahman, Walker** (1998): *Let* $f \in B_{[-\tau,\tau]}$ *satisfy* $|f(0)| = \max_{u\in\mathbb{R}} |f(u)|$. *Then*

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2. Since $f$ has a local maximum at 0, we see that $\cos x - f(x)$ has a repeated zero at the origin. By Duffin and Schaeffer's theorem, $\cos x - f(x) \neq 0$ on $(0, \pi/2]$ and $[-\pi/2, 0)$. This can be only satisfied if either $f(x) \equiv \cos x$ or $f(x) > \cos x$ on $(0, \pi/2] \cup [-\pi/2, 0)$, which proves (2).

# Example: Separation of real zeros

Given a function $f(x), x \in \mathbb{R}$, denote by

$$M(f) := \sup\{b - a : f(x) \neq 0, x \in (a, b)\}$$

the longest zero-free interval of $f$.

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**Clunie, Rahman, Walker** (1998): *Let* $f \in B_{[-\tau,\tau]}, f \neq 0$. *Then either* $f(x) = c \sin(\tau x + a)$ *or* $M(f) > \pi/\tau$.

# Proof of Main result, $n = 1$



When $n = 1$, we have $\mathcal{K} = [-\tau, \tau]$, for some $\tau > 0$, and so $\mathcal{K}^o = [-1/\tau, 1/\tau]$. Suppose a set $\Lambda \subset \mathbb{R}$ satisfies $\Lambda + \rho\mathcal{K}^o = \Lambda + [-\rho/\tau, \rho/\tau] = \mathbb{R}$, for some $\rho < \pi/2$. Then for every $x \in \mathbb{R}$ there exists $\lambda \in \Lambda$ with $|x - \lambda| \leq \rho/\tau < \pi/2\tau$.

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1. Suppose $|f|$ attains maximum at a point $x_0$. We may assume $x_0 = 0$. There exists $\lambda_0 \in \Lambda$ with $|\lambda_0| \leq \rho/\tau$. By the Corollary above,

$$\|f\|_\infty = |f(0)| \leq \frac{|f(u)|}{\cos \tau u} \leq \frac{|f(\lambda_0)|}{\cos \rho} \leq \frac{1}{\cos \rho} \|f|_\Lambda\|_\infty.$$

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2. If $|f|$ does not attains maximum on $\mathbb{R}$ (this may happen!), we apply the argument above to the function $f_\epsilon(x) := f(x)\sin(\epsilon x)/(\epsilon x)$ (where $\epsilon$ must be small enough), and then let $\epsilon \to 0$.

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# Proof of the Main Result, $n \geq 2$



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Consider the function of one variable $g(z) := f(z\lambda_0)$. One may check that $g \in B_{[-\tau,\tau]}$ with $\tau = \|\lambda_0\|_{\mathcal{K}^o}(\leq \rho < \pi/2!)$. Also, $\|f\|_\infty = \|g\|_\infty = |g(0)|$ and $g(1) = f(\lambda_0)$. By the Corollary above, we conclude

$$\|f\|_\infty = |g(0)| \leq \frac{|g(u)|}{\cos \tau u} \leq \frac{|g(1)|}{\cos \tau} \leq \frac{|f(\lambda_0)|}{\cos \rho} \leq \frac{1}{\cos \rho}\|f|_\Lambda\|_\infty.$$

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2. If $|f|$ does not attain maximum on $\mathbb{R}^n$, we apply the argument above to $f_\epsilon(x) := f(x)\varphi(\epsilon x)$, where $\varphi$ is any function from $PW_{\mathcal{B}}$ satisfying $\varphi(0) = 1$, and then let $\epsilon \to 0$.

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# Connection between sampling and interpolation in $PW-$ and $B-$spaces

**Theorem** *Suppose $\mathcal{S} \subset \mathbb{R}^n$ is a compact, $\Lambda$ is a uniformly discrete set in $\mathbb{R}^n$ and $\epsilon > 0$.*

*(i) If $\Lambda$ is an SS for $PW_{\mathcal{S}+\epsilon\mathcal{B}}$ then it is a SS for $B_{\mathcal{S}}$.*
*(ii) If $\Lambda$ is an SS for $B_{\mathcal{S}+\epsilon\mathcal{B}}$ then it is a SS for $PW_{\mathcal{S}}$.*
*(iii) If $\Lambda$ is an IS for $PW_{\mathcal{S}}$ then it is an IS for $B_{\mathcal{S}+\epsilon\mathcal{B}}$.*
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**Proof** (we only prove (i) and (ii)). We shall use a well-known inequality: for every bounded set $\mathcal{S} \subset \mathbb{R}^n$ and every uniformly discrete set $\Lambda$ there exists $C$ such that

$$\|f|_\Lambda\|_2 \leq C\|f\|_2, \quad \text{for every } f \in PW_{\mathcal{S}}.$$

# Connection between sampling and interpolation in $PW-$ and $B-$spaces

(i) Suppose $\Lambda$ is an SS for $PW_{\mathcal{S}+\epsilon\mathcal{B}}$. Assume that $\Lambda$ is not an SS for $B_{\mathcal{S}}$. This means that there are functions $g_j \in B_{\mathcal{S}}$ satisfying $|g_j(x_j)| = 1$, for some $x_j \in \mathbb{R}^n$ and $\|g_j|_\Lambda\|_\infty < 1/j$. Fix any function $\Phi \in PW_{\epsilon\mathcal{B}}$ satisfying $\|\Phi\|_\infty = \Phi(0) = 1$. Then the function $f_j(x) := \Phi(x - x_j)g_j(x)$ belongs to $PW_{\mathcal{S}+\epsilon\mathcal{B}}$ and satisfies $|f_j(x_j)| = 1$. From this, by Bernstein's inequality for entire functions of exponential type, we get $\|f_j\|_2 \geq K$, where $K > 0$ depends only on the diameter of $\mathcal{S}$. On the other hand, by the inequality above, we have

$$\|f_j|_\Lambda\|_2 \leq \frac{1}{j}\|\Phi(x - x_j)|_\Lambda\|_2 \leq C\frac{1}{j}\|\Phi\|_2 \to 0,\ j \to \infty.$$

This is a contradiction, and so $\Lambda$ is an SS for $B_{\mathcal{S}}$.

(ii) Suppose $\Lambda$ is an SS for $B_{\mathcal{S}+\epsilon\mathcal{B}}$ then it is a SS for $PW_{\mathcal{S}}$. We may assume that $\mathcal{S} \subset [-\pi, \pi]^n$.
Take a function $h \in PW_{\epsilon\mathcal{B}}$ with $h(0) = 1$, and such that

$$C_1 := \sup_{x\in[0,1]^n} \sum_{k\in\mathbb{Z}^n} |h(k-x)|^2 < \infty.$$

Clearly, for every $f \in PW_{\mathcal{S}}$ we have $f(\cdot)h(k-\cdot) \in B_{\mathcal{S}+\epsilon\mathcal{B}}$ and $|f(k)| \leq \max_{x\in\mathbb{R}^n} |f(x)h(k-x)|$. Hence,

$$\|f\|_2^2 \leq \sum_{k\in\mathbb{Z}^n} |f(k)|^2 \leq \sum_{k\in\mathbb{Z}^n} \max_{x\in\mathbb{R}^n} |f(x)h(k-x)|^2$$

$$\leq C \sum_{k\in\mathbb{Z}^n} \max_{\lambda\in\Lambda} |f(\lambda)h(k-\lambda)|^2 \leq C \sum_{k\in\mathbb{Z}^n} \sum_{\lambda\in\Lambda} |f(\lambda)h(k-\lambda)|^2$$

$$\leq CC_1 \sum_{\lambda\in\Lambda} |f(\lambda)|^2.$$