Sub-Riemannian structures on 3D Lie groups and their applications

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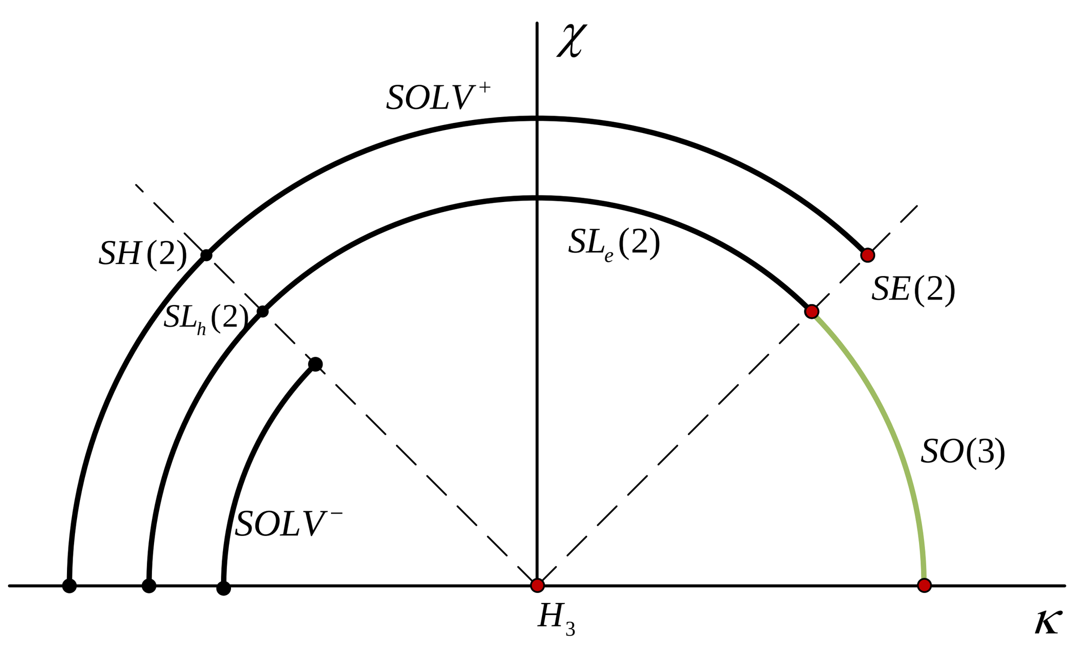
Classification of SR-structures

Left-invariant contact structures on 3D Lie groups are the most basic examples of sub-Riemannian structures. In the recent article of A.A. Agrachev and D. Barilari a full classification of such structures was given in terms of two local invariants κ and χ. One important problem is to study the geodesics and minimal curves on these sub-Riemannian manifolds. In the picture below you can see the diagram describing all possible cases up to local isometries, homogeneous scalings and reflections of the vector fields which span the distribution.



Different sub-Riemannian structures are presented by different points on the circles. The red points correspond to those sub-Riemannian structures, where minimal length curves are completely studied. The middle circle corresponds to the so-called unimodular groups, which have a great variety of applications.

Hamiltonian system

Given a 3D unimodular group G, suppose that the distribution Δ is locally spanned by two left-invariant vector fields f_1, f_2. Then geodesic equations have the following hamiltonian form:

$$\begin{cases} \dot g = p_1 f_1(g) + p_2 f_2(g) \\ \dot p_1 = p_3 p_2 \\ \dot p_2 = -p_0 3 p_1 \\ \dot p_3 = 2\chi p_1 p_2 \end{cases}$$

where $g \in G$, $p = (p_1, p_2, p_3) \in T^*_{Id}G$.
Since the hamiltonian of this system is

$$H = \frac{p_1^2 + p_2^2}{2}$$

we can consider only the level sets $H^{-1}(c)$. The case $H^{-1}(1/2)$ corresponds to geodesics parametrized by length. In this case substitution $p_1 = \cos\alpha$, $p_2 = -\sin\alpha$ reduces the vertical subsystem of the hamiltonian system to the equations of the mathematical pendulum.

The Heisenberg group

The SR-structure on the Heisenberg is probably the very first example of a SR-manifold. It was studied initially by Brockett, Vershik, Agrachev and many others. The SR-problem on the Heisenberg group has a clear geometric meaning: it's just an isoperimetric problem on the Euclidean plane. There is also a physical interpretation to this problem, namely, the sub-Riemannian geodesics correspond to the motion of a charged particle in a constant magnetic field.

The most important feature of the Heisenberg case is that it provides a local structure to all contact 3D structures. Due to this property the Heisenberg group becomes an important tool in control theory, because it allows to construct an approximation of a 3D control system with two controls, that captures some features of the initial system, such as controllability.

SO(3) and SU(2)

The group SO(3) consists of all real matrices $R \in \mathbb{R}^{3x3}$ such that $\det R = 1$ and $R^T R = Id$. Similarly the group SU(2) consists of all complex matrices $U \in \mathbb{C}^{2x2}$ such that $\det U = 1$ and $U^* U = Id$. Both groups locally have the same sub-Riemannian structures.

There is an one-parametric family of sub-Riemannian structures defined on these groups. The only fully studied case is the point $\chi = 0$, which was considered by U. Boscain and F. Rossi.

SO(3) and the rigid body

It is well known that the Riemannian geodesics on SO(3) describe the motion of a free 3D rigid body. If we fix one stationary frame at the rotation point and a moving frame attached to the body, then the orientation of the body can be described by a rotation matrix $R \in SO(3)$, which satisfies

$$\dot R = R\Omega = R \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $\Omega \in so(3)$ is the angular velocity matrix, and, according to the least action principle, it must minimize the integral of the kinetic energy over time (the action functional):

$$J(u) = \frac{1}{2}\int_0^T I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 dt \to \min$$

The constants I_j, $j = 1, 2, 3$ depend only on the shape and the mass distribution of the body. They are called the principal moments of inertia and in general define non-equivalent Riemannian structures. For a real rigid body this constants must satisfy the triangle inequalities:

$$I_1 + I_2 \geq I_3$$
$$I_2 + I_3 \geq I_1$$
$$I_3 + I_1 \geq I_2$$

If we apply the Pontryagin Maximum Principle, we'll obtain the following hamiltonian system

$$\begin{aligned} \dot R &= R \begin{pmatrix} 0 & -\frac{p_3}{I_3} & \frac{p_2}{I_2} \\ \frac{p_3}{I_3} & 0 & -\frac{p_1}{I_1} \\ -\frac{p_2}{I_2} & \frac{p_1}{I_1} & 0 \end{pmatrix} \\ \dot p_1 &= \frac{I_2 - I_3}{I_2 I_3} p_3 p_2 \\ \dot p_2 &= -\frac{I_1 - I_3}{I_1 I_3} p_0 3 p_1 \\ \dot p_3 &= \frac{I_1 - I_2}{I_1 I_2} p_1 p_2 \end{aligned} \quad (1)$$

where $p = (p_1, p_2, p_3) \in T^*_{Id}SU(2)$.

To obtain the equations for the sub-Riemannian geodesics one can simply assume that one $I_j \to \infty$ in the equations (1). This a common way to define sub-Riemannian structures via penalty metrics. From the triangle inequalities it follows, that there is no physical body which corresponds to the sub-Riemannian problem. But still, we can use the developed framework from geometric mechanics for studying sub-Riemannian geodesics.

References

Unfortunately there is no room for all the references. But feel free to ask the poster author for links, if you are interested in a specific topic presented here.

SU(2) and 2-level systems

From postulates of quantum mechanics it follows that the state ψ of a two-level system (such as an electron) is described by a unit length vector in a two-dimensional complex vector space. According to the same postulates the state of the system is observable only up to a multiplication on a unit complex number.

The two-level system can be transferred to a new state by an unitary transformation U. But since everything is defined only up to a multiplication, one can assume that $U \in SU(2)$. The dynamics of the state is in many ways similar to the dynamics of a point in the rigid body case. If there is an external magnetic field $\vec{B} = (B_1, B_2, B_3)$, the unitary transform evolves in some suitable coordinates according to the following equations:

$$\dot U = U\frac{iH}{2} = \frac{U}{2}(B_1 i\sigma_1 + B_2 i\sigma_2 + B_3 i\sigma_3)$$

where σ_i are the so-called Pauli matrices, which are hermitian. The basis of su(2) consists of all anti-hermitian matrices and thus $iH \in su(2)$.

If we assume, that the dynamics of the quantum system is such that the following functional is minimized:

$$J(u) = \frac{1}{2}\int_0^T I_1 B_1^2 + I_2 B^2 + I_3 B_3^2 dt \to \min$$

then we get a Riemannian problem, which is locally equivalent to the Riemannian problem on SO(3). In the same manner, via penalty metrics, we can obtain the equations for sub-Riemannian geodesics from the Riemannian case.

The considered application of the SR-problem on SU(2) has a peculiarity. Since everything is defined up to a multiplication on an unit complex number, we must consider admissible curves, which connect to distinct circles on SU(2). This problem was studied in detail by U. Boscain, T. Chambrion, J.-P. Gauthier.

Applications of SE(2)

The sub-Riemannian structure on SE(2) can be defined in the following way:

$$\dot x = u_1 \cos\theta,$$
$$\dot y = u_1 \sin\theta,$$
$$\dot\theta = u_2$$

This structures gives a simple model for a car-like robot. If we assume $u_1 = 0$ then the corresponding motion will be simply a rotation. If $u_2 = 0$, then the robot will move along a straight line. The sub-Riemannian distance is defined by the usual quadratic functional

$$J(u) = \frac{1}{2}\int_0^T u_1^2 + u_2^2 dt \to \min \quad (2)$$

This structure was completely studied in a series of papers by Yu.L. Sachkov.

We can consider a slightly different functional

$$J(u) = \frac{1}{2}\int_0^T a^2 u_1^2 + u_2^2 dt \to \min$$

which is widely used in the image reconstruction programs, which use the Petitot-Citti-Sarti model of the visual cortex. The idea is that in general the projections of the minimizing curves for the functional (2) may have cusps. Introducing parameter a allows cuspless curve reconstruction.