Approximation by Logarithmic Derivatives of Complex Polynomials and Their Modifications

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Definition

Simple partial fraction (SPF)

Logarithmic derivatives of complex polynomials

$$\rho_n(z) = \sum_{k=1}^{n} \frac{1}{z - z_k}, \qquad z, z_k \in \mathbb{C}.$$

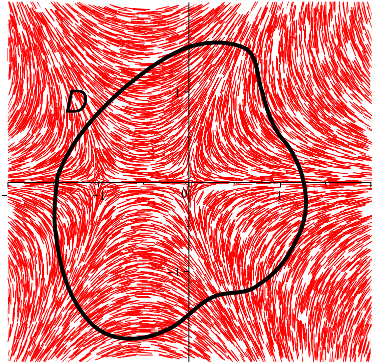
Indeed, if $P_n(z) = \prod_{k=1}^{n}(z - z_k)$ then

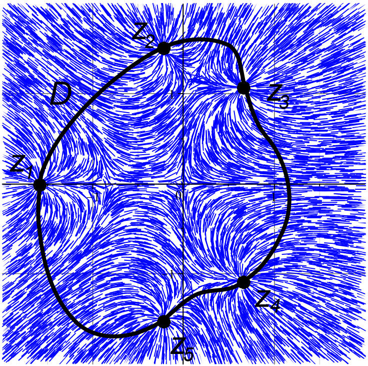
$$\frac{d}{dz}\mathrm{Ln} P_n(z) = \frac{P_n'(z)}{P_n(z)} = \frac{d}{dz}\sum_{k=1}^{n}\mathrm{Ln}(z - z_k) \equiv \rho_n(z).$$

- Macintyre and Fuchs (1940), Gonchar (1955), Dolzhenko (1963).
- In approximation theory: Korevaar (1964), Chui (1970, 1985).

Th. (Korevaar, 1964)

If f is a function analytic in a bounded Jordan domain $D \subset \mathbb{C}$, then there exist SPFs ρ_n with poles $z_k \in \partial D$, $k = \overline{1, n}$, which approximate f uniformly on each compact subset of D.





? Find points $z_k \in \partial D$ where we should put electrons in order to approximate the given electrostatic field on D.

Gorin's problem (1962)

Let ρ_n be SPF with the poles $z_k \in \mathbb{C}\backslash\mathbb{R}$. How close to $\mathbb{R}$ can be z_k if $|\rho_n(x)| \leqslant 1$, $x \in \mathbb{R}$? In other words, we need to find lower estimate of the distance

$$d(n) = \inf \left\{ \min_{k=\overline{1,n}} |\mathrm{Im} z_k| : |\rho_n(x)| \leqslant 1 \right\}.$$

- Nikolaev (1965), Gelfond (1966), Katsnelson (1967), etc.

Th. (Danchenko, 1994)

$$d(n) \asymp \frac{\ln \ln n}{\ln n}, \qquad n \to \infty.$$

? Dolzhenko: Uniform approximation by SPFs with free poles?

Analog of Mergelyan's theorem (Danchenkos, 1999)

Let $D \subset \mathbb{C}$ be a compact set such that $\mathbb{C} \backslash D$ is connected. Then, every function f continuous on D and analytic in the interior of D can be approximated uniformly on D with SPFs ρ_n.

Polynomials and SPFs

Similarity (Danchenko, 2001, 2008; Kosukhin, 2001)

- The same rate of convergence for a wide class of functions and domains.
- Jackson's, Bernstein's, Walsh's, etc. theorems hold.
- For any analytic function, there exists a unique ρ_n of n-multiple interpolation in a neighborhood of the origin.

Difference (Danchenko and Kondakova, 2010; Kosukhin and Borodin, 2005; Protasov, 2009; Kayumov, 2012)

- SPF of the best uniform approximation may be non-unique.
- Interpolation of tables with ∞-elements.
- Approximation by SPFs on unbounded sets (lines, rays, half-plane).

? How to find SPF of n-multiple interpolation in a neighborhood of the origin for a certain analytic function?

Th. (Danchenko and Ch., 2011)

Let $f(z) = f_0 + f_1 z + f_2 z^2 + \cdots$ be a function analytic in a neighborhood of the origin. Let

$$T_n(\lambda) = \lambda^n - \tau_1 \lambda^{n-1} + \tau_2 \lambda^{n-2} + \ldots + (-1)^n \tau_n = 0,$$

with the following coefficients

$$\tau_1 = -f_0, \quad \tau_m = \frac{(-1)^m}{m} \left(f_{m-1} + \sum_{j=1}^{m-1} (-1)^j f_{m-j-1} \tau_j \right), \quad m = \overline{2, n}.$$

Then SPF of n-multiple interpolation in a neighborhood of the origin for f has the form

$$\rho_n(z) = \sum_{k=1}^{\mu} \frac{1}{z - z_k} = \frac{Q'(z)}{Q(z)}, \quad Q_\mu(z) = \frac{(-1)^\mu}{\tau_\mu} z^n T_n\left(\frac{1}{z}\right), \quad 1 \leqslant \mu \leqslant n,$$

where $z_k = \lambda_k^{-1}$ and λ_k, $k = \overline{1, \mu}$, are non-zero roots of T_n.

? Estimates for the poles of SPF if moduli of Taylor coefficients of f are majorized by a geometrical progression.

Th. (Ch., 2012)

If $|f_{m-1}| \leqslant a^m$, $m = \overline{1, n}$, for some $a > 0$ then

$$|z_k| > \frac{1 - \varepsilon_n}{a}, \quad k = \overline{1, n}$$

where $\varepsilon_n \sim \frac{2 \ln n}{n}$, $n \to \infty$.

! There exists f such that $|f_{m-1}| \leqslant a^m$, $m = \overline{1, n}$, and

$$|z_1| < \frac{1 - 1/(20n)}{a}$$

for sufficiently large $n \geqslant n_0$.

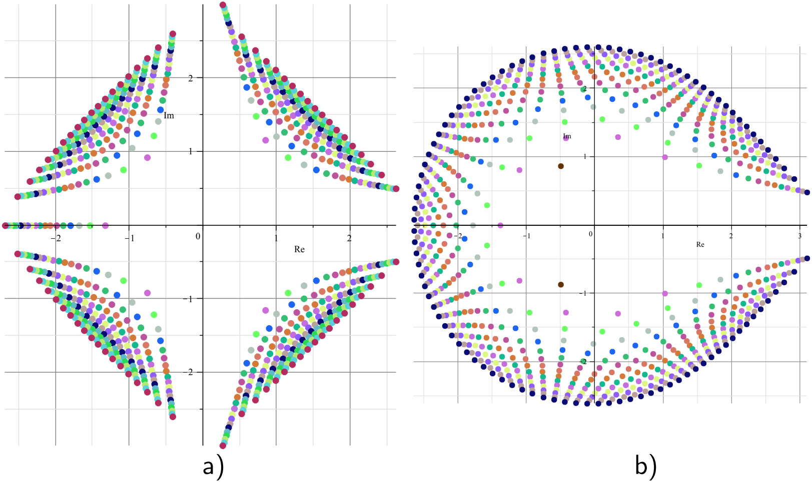


Figure : Movement of SPFs poles while n is growing: a) $f(z) = z^3 + 1$;
b) $f(z) = \exp(z)$.

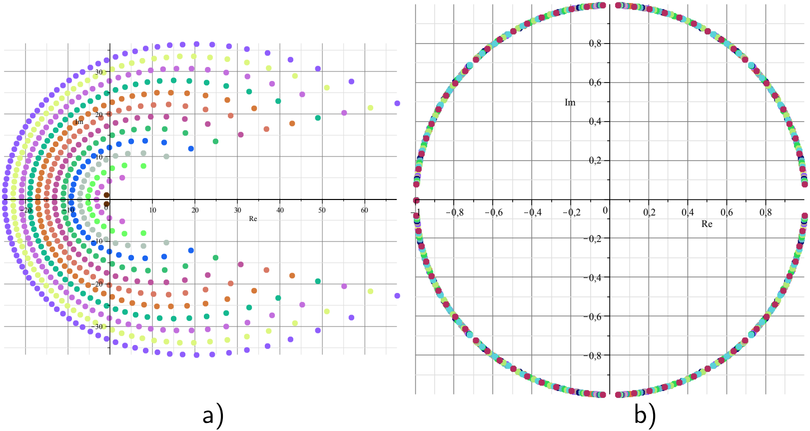


Figure : Shift of poles of SPFs while n is growing: a) $f(z) = 1$, b) $f(z) = \frac{1}{1-z}$.

Open problems

- **SPF series.** Obtain criteria of convergence of series

$$\rho_\infty(z) = \sum_{k=1}^{\infty} \frac{1}{z - z_k}, \qquad z_k \in \mathbb{C}\backslash\mathbb{R},$$

  in $L_p(\mathbb{R})$, $p > 1$, in terms of the poles z_k.
- **Chebyshev-type problem.** Find SPF ρ_n of the best uniform (with a weight (i.e., $\sqrt{1-x^2}$) or without) approximation of zero-function on the interval $I = [-1; 1]$ if distance from z_k to I is less than a given value.
- **Compositions of SPF.** Approximation by functions of the form

$$\rho_{1,n_1}(z) - \rho_{2,n_2}(z), \qquad \frac{\rho_{1,n_1}(z) - \rho_{2,n_2}(z)}{\rho_{3,n_3}(z) - \rho_{4,n_4}(z)},$$

  where ρ_{s,n_s}, $s = \overline{1,4}$, are SPFs of degree n_s.