Composition Operators in Sobolev Spaces on Carnot Group

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Objectives

Mainly we study mappings that induce composition operators on Sobolev spaces. The research involves such subject as

- Sobolev spaces on Carnot groups
- Composition operators on Sobolev spaces
- Metric properties of various domains on Carnot groups
- Set functions

Carnot Group

A **Carnot group** $\mathbb{G}$ is a connected, simply connected, stratified nilpotent Lie group. This means that the Lie algebra $\mathfrak{g}$ of the group $\mathbb{G}$ admits a nilpotent stratification:

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_m,$$

and $[V_1, V_j] = V_{j+1}$ for $j = 1, \dots, m-1$, whereas $[V_1, V_m] = \{0\}$. Space V_1 is called horizontal subspace. (Heisenberg group is the simplest example of a non-commutative Carnot group.)

Sobolev Spaces

Let the vector fields $X_1, X_2, \dots, X_n$ form a Lie basis of algebra $\mathfrak{g}$ of group $\mathbb{G}$. A Locally summable function $f : U \to \mathbb{R}$ (U – domain in $\mathbb{G}$) belongs **Sobolev space** $L^1_p(U)$ if it has weak derivatives $X_1 f, X_2 f, \dots, X_n f$ in Sobolev sense and finite seminorm

$$\|f \mid L^1_p(U)\| = \|\nabla_{\mathcal{L}} f \mid L_p(U)\|$$

where

$$\nabla_{\mathcal{L}} f = (X_1 f, X_2 f, \dots, X_n f)$$

is a *generalized subgradient* of the function.

Composition Operator

Let $\varphi : D \to D'$ is a measurable mapping and $L^1_q(D)$, $L^1_p(D')$ are Sobolev spaces on these domains, then we would "define" composition operator as $\varphi^* f = f \circ \varphi$ (see fig. 1). However the composition $f \circ \varphi$ of measurable mapping with function of Sobolev class could not be defined correctly in such way.

So we need further consideration. Actually the idea of the definition is following:

1. To define bounded composition operator on some nice space (like $C^\infty(D')$ or $C(D')$) which is dense in original space $L^1_p(D')$;
2. To extend by continuity in the whole space $L^1_p(D')$;

Results

Now we are working on a special case when $q = p$ and operator φ^* is an isomorphism. The following theorem provide necessary and sufficient conditions for a measurable mapping to induce an isomorphism.

Theorem 1. *Let $p \geq 1$, $p \neq \nu$, and D, D' are domains on a Carnot group $\mathbb{G}$. Measurable mapping $\varphi : D \to D'$ induces an isomorphism of Sobolev spaces*

$$\varphi^* : L^1_p(D') \to L^1_p(D),$$

if and only if φ coincides almost everywhere with a quasi-isometric homeomorphism $\Phi : D \to \Phi(D)$, for which Sobolev spaces $L^1_p(\Phi(D))$ and $L^1_p(D')$ are equivalent.

Bounded composition operator

Let D and D' are domains in a Carnot group $\mathbb{G}$ and a measurable mapping $\varphi : D \to D'$ is defined almost everywhere on D.

If function $f \in L^1_p(D')$ is continuous then composition $f \circ \varphi$ is defined almost everywhere on D.

Assume that $f \circ \varphi \in L^1_q(D)$ and $\|f \circ \varphi \mid L^1_q(D)\| \leqslant K \|f \mid L^1_p(D')\|$

for all $f \in L^1_p(D') \cap C(D')$. We can define *composition operator*:

$$L^1_p(D') \cap C(D') \ni f \mapsto \varphi^* f = f \circ \varphi \in L^1_q(D)$$

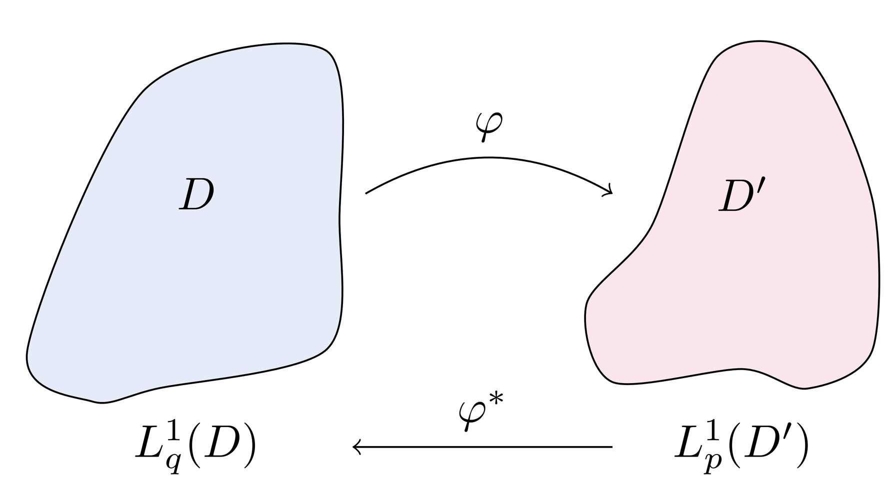

Figure 1: Composition operator

Quasi-isometric Homeomorphism

Definition. *Homeomorphism $\Phi : D \to D'$, $D, D' \subset \mathbb{G}$, of Sobolev class $W^1_{1,\mathrm{loc}}(D)$ is called **quasi-isometric**, if $|D\Phi(x)| \leqslant L$ and $0 < \alpha \leqslant |\det D\Phi(x)|$ for almost all $x \in D$, where constants L and α not depend on x.* Equivalent definition involves following relations

$$\varlimsup_{y \to x} \frac{|\Phi(y) - \Phi(x)|}{|y - x|} \leqslant M \qquad \varlimsup_{y \to z} \frac{|\Phi^{-1}(y) - \Phi^{-1}(z)|}{|y - z|} \leqslant M$$

Toy Example

Sometimes we deal with solutions of some problem which are elements of Sobolev space on a not so good domain (like D on fig. 2). On the other hand with help of described above operators we can transform this problem to the problem on Sobolev space on more appropriate domain (for example on a ball D').

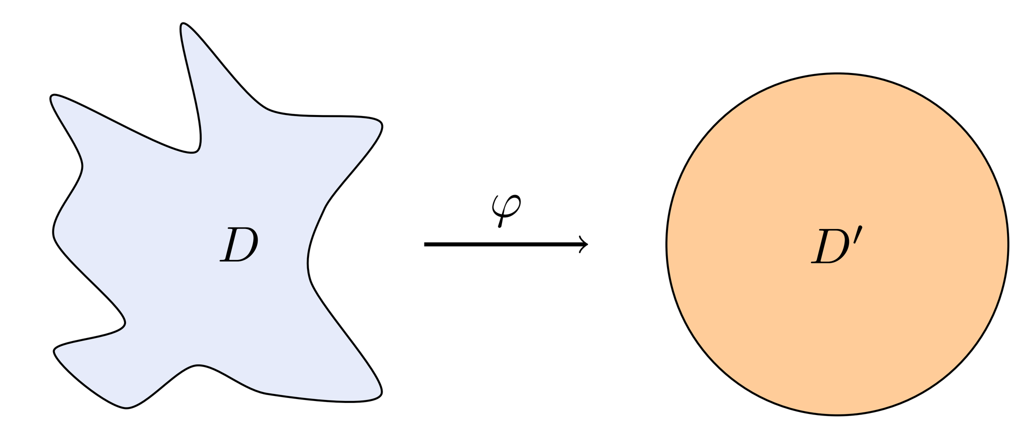

Figure 2: "Good" domain instead of "Bad"

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