Geodesics and the topology of short-path spaces

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Introduction

Carnot groups are local models for the sub-Riemannian manifolds; they approximate both the metric and the algebraic structures of the bracket-generating distribution given with the manifold. In a similar way the Carnot group may contain informations about the topology of "short" paths joining two points separate by an infinitesimal distance. On a Carnot group it corresponds to consider paths starting at the origin and arriving to a point p_ϵ parametrized continuously by ϵ such that for $\epsilon = 0$ we get the identity of the group; moreover we bound the energy of the allowed paths by a small constant in order to get the local information. We call this space $\Omega_{\epsilon p}$; we study the asymptotic behaviour of its topology as $\epsilon \to 0$. More precisely we consider the total Betti number (the sum of all Betti numbers) as a function of ϵ and we look for its growth rate with respect to $1/\epsilon$. Being geodesics critical points of the energy, we can relate the homological information we find with the number of geodesics, although we find a bound on the homology asymptotically sharper than the one given by Morse inequalities.

Preliminaries: sub-Riemannian manifolds

We are given a step 2 sub-Riemannian manifold; a smooth manifold M with a distribution Δ and a smooth scalar product g on Δ. The distribution is required to be bracket generating of step 2: it means that vector fields in Δ and their brackets generate all the vector fields on the manifold. The main consequence (for general bracket generating) is the *Chow-Rashevsky theorem*: every two points p and q in M are joined by curves with velocities in Δ; these curves are said *admissible*. It is possible to endow M with a metric space structure: the distance between p and q is the infimum of the lengths of curves joining them. For admissible curves it is also possible to define energy

$$J(\gamma) = \frac{1}{2}\int \|\dot{\gamma}\|^2 \, dt \, .$$

Preliminaries: nilpotent approximation

The sub-Riemannian tangent space is defined to be the metric tangent space (according to the notion of Gromov, see [1]) and it is constructed by Bellaiche in the paper [2]. Fixed $x \in M$, consider the ball of radius ϵ centered in x; then dilate by $1/\epsilon$ the metric structure obtaining a ball of radius 1. The limit in Gromov-Hausdorff convergence as $\epsilon \to 0$ is the ball of radius 1 in the metric tangent space. This construction approximates not only the metric structure, considering the different "weight" of dilations depending on the number of brackets needed to get a given direction, but also the algebraic structure of the vector fields. The result is a graded nilpotent Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ with a scalar product on $\mathfrak{g}_1$. By exponentiating $\mathfrak{g}$ we get a simply connected Lie group G with a bracket generating distribution $\mathfrak{g}_1$ endowed with a left-invariant metric: these data define a sub-Riemannian manifold as well, which is a *Carnot group*.

The space of curves with bounded energy

We now fix our setting: a generic step 2 Carnot group G. We consider curves starting in the identity element e with a given point $p \in G$. Then we apply dilations as in the process to obtain the nilpotent approximation and we look what happens to the topology of the space of curves from e to p with energy bounded by a constant, denoted by $\Omega_{\epsilon p}$; now p is not fixed but it "moves" according to the identification of the ball of radius 1 with the ball of radius ϵ. A Carnot group is the sub-Riemannian tangent space to itself, thus this process leaves it fixed; but the point moves along a curve with parameter ϵ such that for $\epsilon \to 0$ the point converges to e. When $\epsilon = 0$ we get the space of closed curves and we find that it is contractible; no homological information is left. So we are concerned with the asymptotic behaviour of the homology of $\Omega_{\epsilon p}$ by taking $\epsilon \to 0$; more precisely we study the asymptotic behaviour of the *total Betti number*, the sum of all the Betti numbers.

The Riemannian case

In the Riemannian case if we take two points q and p_ϵ sufficiently close, and the constant bounding the energy sufficiently small, it's well known that they are joined by a unique geodesic: by Morse inequalities the space of curves with bounded energy $\Omega_{\epsilon p}$ is contractible and in particular its topology is constant for $\epsilon \to 0$.

The Heisenberg group

The Heisenberg group is the simplest example of a Carnot Group: its Lie algebra is generated by the vector fields X, Y, Z with the commutator relation $[X, Y] = Z$ and with X, Y being an orthonormal basis for the sub-Riemannian metric. Here there are two different behaviours: for the generic point p the dilations process moves the point close to the identity in a direction of the plane generated by X, Y; the asymptotic behaviour of the topology of $\Omega_{\epsilon p}$ turns out to be trivial like in the Riemannian case.
Conversely, if we take a point in the z-axis (obtained by exponentiating Z), the number of geodesics increases as $\epsilon \to 0$: more precisely its growth rate is $\Theta(1/\epsilon)$. The space $\Omega_{\epsilon p}$ has the same homotopy type of a sphere S^k, thus the total Betti number is constant; but the dimension k of the sphere S^k increases indefinitely as $\epsilon \to 0$. Thus we find a generator of the homology that increases in dimension, and in the limit we get S^∞ which is contractible and no information is left.

Asymptotics for Betti numbers

We now consider the case of corank $l > 1$. As in the Heisenberg group the number of bounded geodesics increases indefinitely as $\epsilon \to 0$. We prove that the growth rate of the number of geodesics is related to the corank l of the distribution:

$$\Theta(\epsilon^{-1}) \leq \text{Number of geodesics going to } p_\epsilon \leq \Theta(\epsilon^{1-l}).$$

Now we look at the topology: for every ϵ we have many nontrivial homology groups $H_i(\Omega_{\epsilon p})$; for ϵ sufficiently small every homology groups becomes trivial, but new non trivial ones arise in higher dimensions. Therefore we want to approximate how many homology groups we have for every $\epsilon > 0$ by looking at the total Betti number.

Growth rate of Betti numbers

The total Betti number increases at most like ϵ^{1-l}.

$$b(\Omega_{\epsilon p}) \leq \Theta(\epsilon^{1-l}).$$

It is worth to notice that this bound is asymptotically sharper by one order than the one given by Morse inequalities, since the number of geodesics increases as ϵ^{-l}.

The case of corank 2

In the case of corank 2 we prove that the bound on the growth rate of the total Betti number is in fact an equality:

$$b(\Omega_{\epsilon p}) = \Theta(\epsilon^{-1}).$$

Moreover there is an explicit formula for the leading coefficient depending on the point and on the structure constants of the Lie algebra. Thus we see that the homology "increases" indefinitely as $\epsilon \to 0$; moreover by Morse inequalities we conclude that the number of geodesics increases indefinitely as well.

References

[1] Mikhael Gromov.
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[2] André Bellaïche.
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Information

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