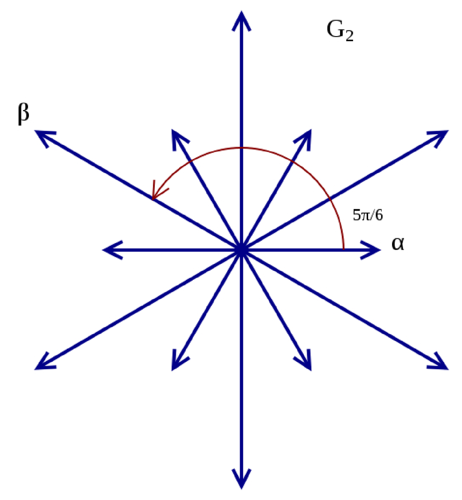

Symmetries of the Rolling Problem

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(Preprint available at arXiv:1301.2579)

Symmetries

Let $D \hookrightarrow TM$ be a distribution with sections $\Gamma(D)$. Denote by $\mathfrak{X}(M)$ the vector fields on M. The space of symmetries of D is

$$\mathrm{Sym}(D) = \Big\{ X \in \mathfrak{X}(M) \,\big|\, [X, \Gamma(D)] \subseteq \Gamma(D) \Big\}.$$

- <u>Fact:</u> $\mathrm{Sym}(D)$ is a Lie algebra.
- Why: Let $X, Y \in \mathfrak{X}(M)$. We need to show that

  $$X, Y \in \mathrm{Sym}(D) \Longrightarrow [X, Y] \in \mathrm{Sym}(D).$$

  Let $Z \in \Gamma(D)$, then by the Jacobi identity on $\mathfrak{X}(M)$ we have that

  $$[[X, Y], Z] = [X, [Y, Z]] + [Y, [Z, X]] \in \Gamma(D).$$

Examples in $\mathbb{R}^3$

Let $D \hookrightarrow T\mathbb{R}^3$. Whether a vector field

$$X = a\partial_x + b\partial_y + c\partial_z \in \mathfrak{X}(\mathbb{R}^3)$$

is in $\mathrm{Sym}(D)$ can be encoded in a system of PDEs in terms of $a, b, c \in C^\infty(\mathbb{R}^3)$.

1. If $D_1 = \mathrm{span}\{\partial_x, \partial_y\}$, then

   $$X \in \mathrm{Sym}(D_1) \Leftrightarrow \begin{cases} \partial_x c = 0, \\ \partial_y c = 0. \end{cases}$$

2. If $D_2 = \mathrm{span}\{\partial_x, Y = \partial_y + x\partial_z\}$, then

   $$X \in \mathrm{Sym}(D_2) \Leftrightarrow \begin{cases} \partial_x(xb - c) = b, \\ Y(xb - c) = -a. \end{cases}$$

3. If $D_3 = \mathrm{span}\{\partial_x, Y = \partial_y + x^2\partial_z\}$, then

   $$X \in \mathrm{Sym}(D_3) \Leftrightarrow \begin{cases} \partial_x(x^2 b - c) = 2xb, \\ Y(x^2 b - c) = -2xa. \end{cases}$$

Problems

(i) Can $\mathrm{Sym}(D)$ be finite dimensional?

(ii) If $D \cong \tilde{D}$, then $\mathrm{Sym}(D) \cong \mathrm{Sym}(\tilde{D})$. Use it to prove non-equivalence of distributions.

(iii) Can $\mathrm{Sym}(D)$ be determined for specific examples?

References

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[3] Chitour, Y., Kokkonen, P.: *Rolling Manifolds on Space Forms*. Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), no. 6, 927–954.

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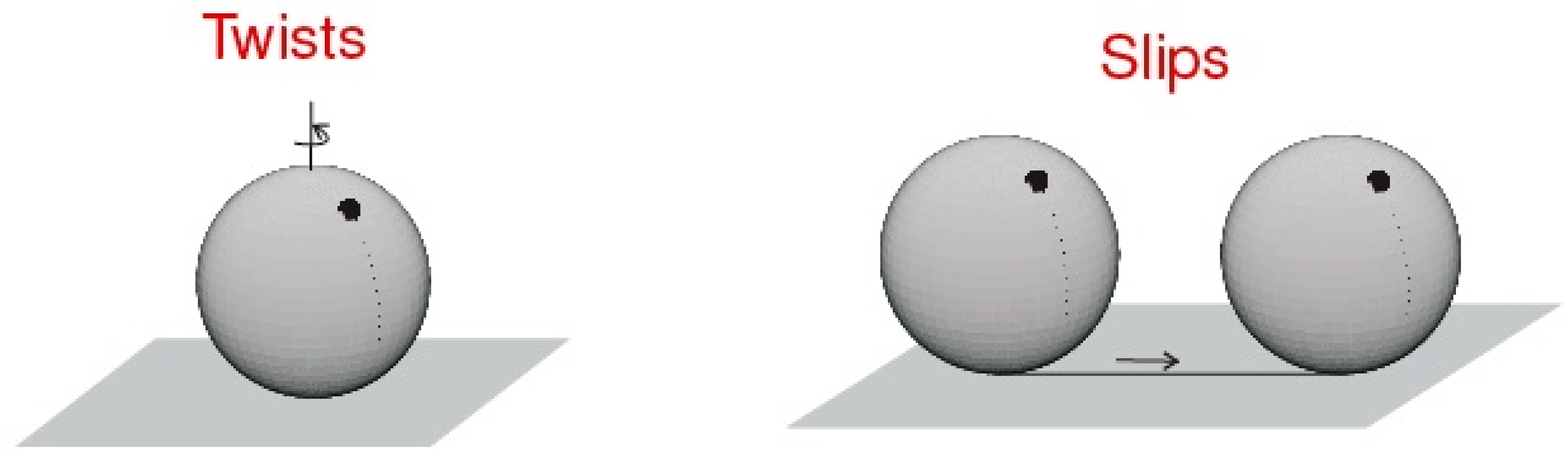
Rolling model

Let us introduce the model of two Riemannian manifolds (M, g) and $(\hat{M}, \hat{g})$ of the same dimension rolling without twisting or slipping. To do that, consider the state space

$$Q = Q(M, \hat{M}) = \Big\{ A : T_x M \overset{\text{o.p. isom.}}{\longrightarrow} T_{\hat{x}}\hat{M} \,\big|\, (x, \hat{x}) \in M \times \hat{M} \Big\}.$$



Given a point $q = (x, \hat{x}; A) \in Q$, and a vector $X \in T_x M$ the rolling lift $\mathscr{L}_{\mathrm{R}}(X)_q \in T_q Q$ at q is defined by

$$\mathscr{L}_{\mathrm{R}}(X)_q = \frac{d}{dt}\Big|_{t=0} \left(P_0^t(\hat{\gamma}) \circ A \circ P_t^0(\gamma) \right),$$

where $\gamma \colon (-\varepsilon, \varepsilon) \to M$ and $\hat{\gamma} \colon (-\varepsilon, \varepsilon) \to \hat{M}$ are curves satisfying $\gamma(0) = x$, $\dot{\gamma}(0) = X$, $\hat{\gamma}(0) = \hat{x}$, $\dot{\hat{\gamma}}(0) = AX$ and P denotes parallel transport.

The rolling distribution $\mathcal{D}_{\mathrm{R}} \hookrightarrow TQ$ is the n-dimensional smooth distribution defined, for $(x, \hat{x}; A) \in Q$, by

$$\mathcal{D}_{\mathrm{R}} = \mathscr{L}_{\mathrm{R}}(TM).$$

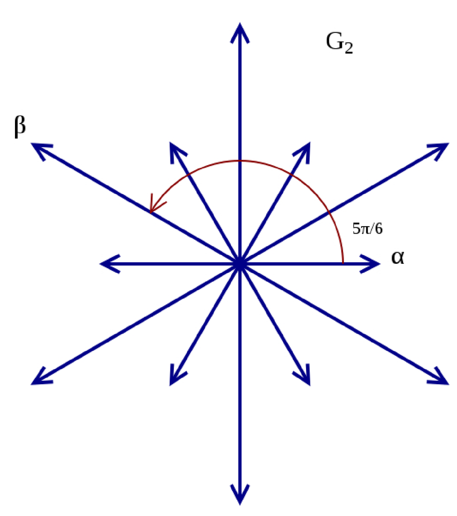
A curve $q \colon [0, \tau] \to Q$ is a rolling curve if $\dot{q}(t) \in \mathcal{D}_{\mathrm{R}q(t)}$. See [3, 4].

Cartan distribution

Two distributions $D, \tilde{D} \hookrightarrow TM$ are (locally) equivalent, if there exists a (local) diffeo $F : M \to M$ such that $F_* D = \tilde{D}$.

A generic distribution D of rank 2 on a manifold M of dimension 5, or equivalently, a regular distribution with growth vector $(2, 3, 5)$ is called a Cartan distribution (see [2]). A Cartan distribution $\overline{D}$ is called flat when it is locally equivalent to the (unique) graded nilpotent Lie algebra $\mathfrak{h}$ of step 3 with growth vector $(2, 3, 5)$.

Theorem (Cartan). *If D is a Cartan distribution then $\dim \mathrm{Sym}(D) \leq 14$. If D is flat, then equality holds and moreover $\mathrm{Sym}(\overline{D}) = \mathfrak{g}_2$.*



Our result

Consider the symmetries of the rolling distribution that are annihilated by the projection $\pi_{Q,M} \colon Q \to M$, that is

$$\mathrm{Sym}_0(\mathcal{D}_{\mathrm{R}}) = \{ S \in \mathrm{Sym}(\mathcal{D}_{\mathrm{R}}) \,\big|\, (\pi_{Q,M})_* S = 0 \}.$$

Theorem (CGK). *If there is an open dense $O \subset Q$ such that the Riemannian curvature map $R_x : \bigwedge^2 T_x M \to \bigwedge^2 T_x M$ is invertible on $\pi_{Q,M}(O)$ and $\widetilde{\mathrm{Rol}}_q$ is invertible for all $q \in O$ then, up to isomorphism,*

$$\mathrm{Sym}_0(\mathcal{D}_{\mathrm{R}}) = \mathrm{Isom}(\hat{M}, \hat{g}).$$

In particular, all the elements of $\mathrm{Sym}_0(\mathcal{D}_{\mathrm{R}})$ are induced by Killing fields of $(\hat{M}, \hat{g})$.

Moreover, if there is a principal bundle structure on $\pi_{Q,M} : Q \to M$ that renders $\mathcal{D}_{\mathrm{R}}$ to a principal bundle connection, then $(\hat{M}, \hat{g})$ is a space of constant curvature.

Rolling curvature

For any $q = (x, \hat{x}; A) \in Q$, the (normalized) rolling curvature is the linear map $\widetilde{\mathrm{Rol}}_q \colon \bigwedge^2 T_x M \to \bigwedge^2 T_x M$, given by

$$\widetilde{\mathrm{Rol}}_q(X \wedge Y) := R(X, Y) - A^{-1}\hat{R}(AX, AY)A.$$

This map permits to give a first sufficient condition for the rolling problem to be controllable.

Theorem (Chitour & Kokkonen, Grong). *If $\widetilde{\mathrm{Rol}}_q$ is an isomorphism for every $q \in Q$, then the rolling problem is completely controllable.*

Cartan and rolling

The rolling problem for two 2-dimensional spheres S_r and S_R, of radii $r < R$, has strong ties with Cartan's theorem. If $r/R = 1/3$, then $\mathcal{D}_{\mathrm{R}}$ is flat and if $r/R \neq 1/3$, then $\mathrm{Sym}(\mathcal{D}_{\mathrm{R}}) = \mathfrak{so}(3) \times \mathfrak{so}(3)$. A reason for this (see [1]) is that the 1/3 case can be rephrased as studying the flat Cartan distribution corresponding to the space of curves in S^3 of constant torsion 1/2 (see [2]).

It has been recently observed (see An, Nurowski: arXiv:1210.3536) that $\mathrm{Sym}(\mathcal{D}_{\mathrm{R}})$ can be $\mathfrak{g}_2$ in the case of surfaces of non-constant Gaussian curvature.