# Are the scales of quasi-arithmetic means necessarily comparable among each other?

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## Abstract

Quasi-arithmetic (QA) mean is defined for any continuous strictly monotone function $f\colon U \to \mathbb{R}$. We assume $U$ to be an open and bounded interval. When $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of points in $U$ and $\underline{w} = (w_1, \ldots, w_n)$ is a sequence of weights ($w_i > 0$, $w_1 + \cdots + w_n = 1$), then the mean $\mathfrak{M}_f(\underline{a}, \underline{w}) := f^{-1}\left(\sum_{i=1}^n w_i f(a_i)\right)$ is directly generalizing the way in witch the power means have been defined.

This family of means was shown, by Kolmogorov in 1930, to be very vast and ubiquitous. In fact, he proved that if a mean satisfied a short list of natural axioms, then it had to be a QA mean for certain function $f$.

For a family $\{k_t \colon t \in I\}$ of functions ($I$ – an open interval) sending $U$ to $\mathbb{R}$, $(\mathfrak{M}_{k_t})_{t \in I}$ is called a *scale* on $U$ if the mapping $I \ni t \mapsto k_t^{-1}\left(\sum_{i=1}^n w_i k_t(a_i)\right)$ is a continuous bijection between $I$ and $(\min \underline{a}, \max \underline{a})$, for any fixed non-constant sequence $\underline{a}$ and any weights $\underline{w}$.

We are going to construct two scales $\{\mathfrak{M}_{k_t} \colon t \in I\}$ and $\{\mathfrak{M}_{l_u} \colon u \in J\}$ which are not comparable. That is to say,

- there is no $t \in I$ and $u \in J$ such that $\mathfrak{M}_{k_t}$ is greater then $\mathfrak{M}_{l_u}$,
- there is no $t \in I$ and $u \in J$ such that $\mathfrak{M}_{l_u}$ is greater then $\mathfrak{M}_{k_t}$.

This result is fairly unexpectable because (by the definition) for any non-constant vector $\underline{a}$ with weights $\underline{w}$ both mappings $I \ni t \mapsto \mathfrak{M}_{k_t}(\underline{a}, \underline{w})$ and $J \ni u \mapsto \mathfrak{M}_{l_u}(\underline{a}, \underline{w})$ are strictly increasing, 1-1 and onto $(\min \underline{a}, \max \underline{a})$.


## Definition

*Quasi-arithmetic mean* (or: *generalized mean*) is defined for any continuous strictly monotone function $f \colon U \to \mathbb{R}$, $U$ – an open interval. When $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of points in $U$ and $\underline{w} = (w_1, \ldots, w_n)$ is a sequence of *weights* ($w_i > 0$, $w_1 + \cdots + w_n = 1$), then the mean $\mathfrak{M}_f(\underline{a}, \underline{w})$ is defined by the equality
$\mathfrak{M}_f(\underline{a}, \underline{w}) := f^{-1}\left(\sum_{i=1}^n w_i f(a_i)\right)$.

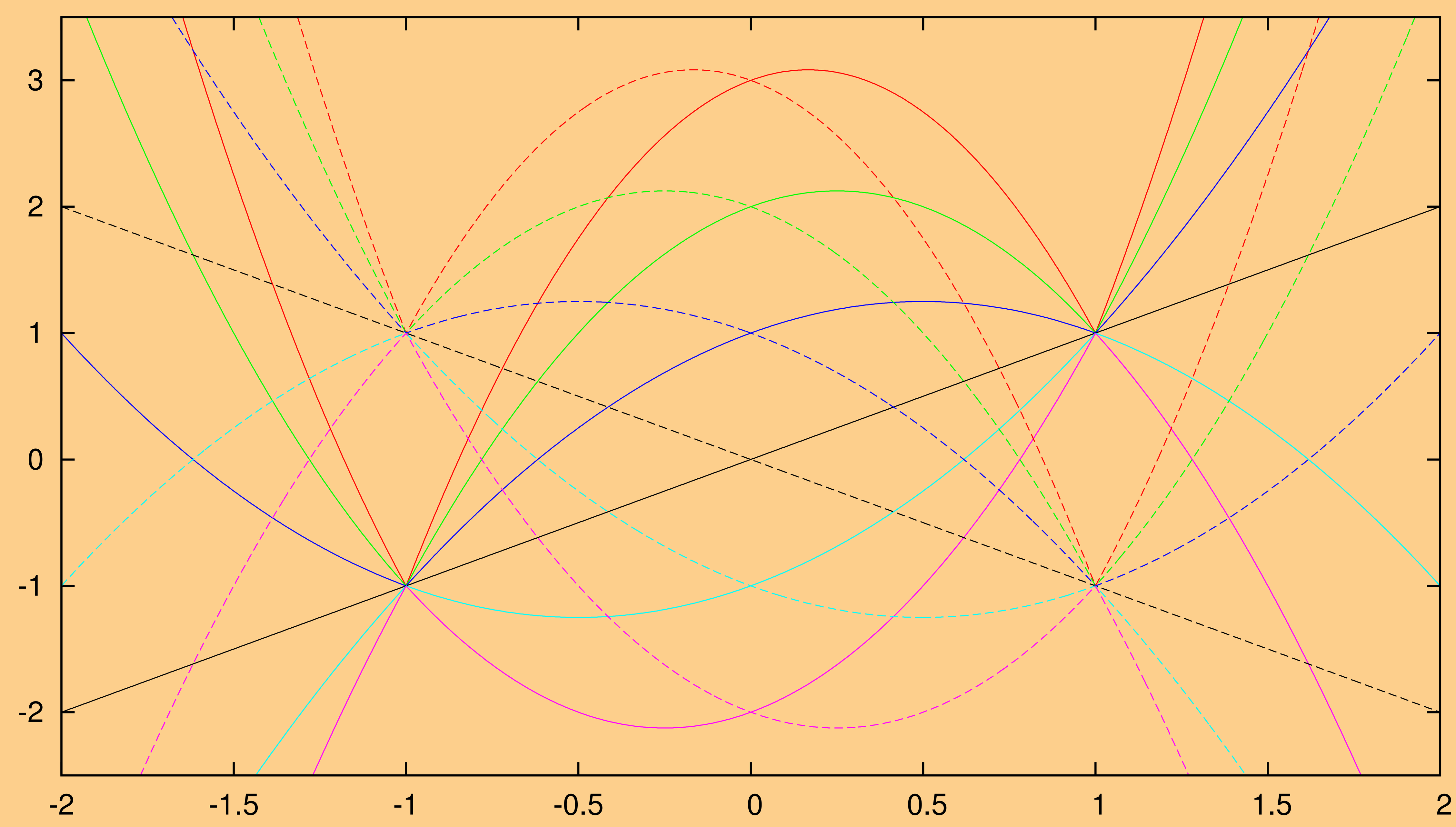
## Theorem (Mikusiǹski 1948)

Let $U$ be an interval, $f, g \in \mathcal{C}^{2\neq}(U)$ (functions in $\mathcal{C}^2(U)$ with nowhere vanishing first derivative). Then $\mathfrak{M}_f(\underline{a}, \underline{w}) > \mathfrak{M}_g(\underline{a}, \underline{w})$ for all nonconstant vectors $\underline{a}$ with weights $\underline{w}$ (in such a case the means are called comparable) if and only if $A(f) > A(g)$ on a dense subset of $U$, where $A(f) := \frac{f''}{f'}$.

## Fact (P. 2013)

Let $U$ be a closed, bounded interval and $f \in \mathcal{C}^{2\neq}(U)$. Moreover, let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions from $\mathcal{C}^{2\neq}(U)$ satisfying $A(f_n) \to A(f)$ in $L^1(U)$. Then $\mathfrak{M}_{f_n} \rightrightarrows \mathfrak{M}_f$ uniformly with respect to $\underline{a}$ and $\underline{w}$.

## Non-comparable families




Let $U = (-2, 2)$ and let us consider two families $(\mathfrak{M}_{k_\alpha})_{\alpha \in \mathbb{R}}$ and $(\mathfrak{M}_{l_\beta})_{\beta \in \mathbb{R}}$ such that
$A(k_\alpha)(x) = -x + \alpha(x+1)(x-1)$ (dotted),
$A(l_\beta)(x) = x + \beta(x+1)(x-1)$ (continuous).
Then $A(k_\alpha)(-1) = 1$, $A(k_\alpha)(1) = -1$;
$A(l_\beta)(-1) = -1$, $A(l_\beta)(1) = 1$.

One may observe that every continuous line intersects with every dotted line. Applying Mikusiǹski's theorem one obtains that $\mathfrak{M}_{k_\alpha}$ is comparable with $\mathfrak{M}_{l_\beta}$ for no $\alpha$ and $\beta$. Moreover, by the strengthening of Theorem both $(\mathfrak{M}_{k_\alpha})_{\alpha \in \mathbb{R}}$ and $(\mathfrak{M}_{l_\beta})_{\beta \in \mathbb{R}}$ are scales.

## Definition

Let $\{k_t \colon t \in I\}$ be a family of continuous, strictly monotone functions ($I$ – an open interval) sending $U$ to $\mathbb{R}$, $(\mathfrak{M}_{k_t})_{t \in I}$ is called a *scale* on $U$ if the mapping
$I \ni t \mapsto k_t^{-1}\left(\sum_{i=1}^n w_i k_t(a_i)\right)$ is a continuous bijection between $I$ and $(\min \underline{a}, \max \underline{a})$, for any fixed non-constant sequence $\underline{a}$ and any weights $\underline{w}$.

## Theorem (P. 2013)

Let $U$ be an interval, $I = (a, b)$ an open interval, $(k_\alpha)_{\alpha \in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all $\alpha$.

If $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ is an increasing scale then there exists a dense subset $X \subset U$ such that the mapping $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing, 1-1 and onto for all $x \in X$.

## Theorem (P. 2013)

Let $U$ be an interval, $I = (a, b)$ – an open interval, $(k_\alpha)_{\alpha \in I}$ – a family of functions on $U$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all $\alpha$.

If $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing and 1-1 on a dense subset of $U$, and is onto for all $x \in U$, then $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ is an increasing scale on $U$.

## Recent strengthening

Let $U$ be an interval, $I = (a, b)$ – an open interval, $(k_\alpha)_{\alpha \in I}$ – a family of functions on $U$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all $\alpha$.

If $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing and 1-1 on a dense subset of $U$, and is onto for **almost** all $x \in U$, then $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ is an increasing scale on $U$.

## Sketch of the proof

One has only to prove that for any vector $\underline{a}$ with weights $\underline{w}$, $\lim_{\alpha \to b} \mathfrak{M}_{k_\alpha}(\underline{a}, \underline{w}) = \max(\underline{a})$ (case with min is analogous). Let us fix $M > 0$ and $k_{\alpha,M}$ be such a function that $A(k_{\alpha,M}) = \min(M, k_\alpha)$. Then $\lim_{\alpha \to b} A(k_{\alpha,M})(x) = M$ for almost all $x$. Hence $\mathfrak{M}_{k_{\alpha,M}} \to \mathfrak{M}_{e^{Mx}}$ uniformly.

In particular there exists $\alpha_M$ such that for all $\underline{a}$, $\underline{w}$ and $\alpha > \alpha_M$ the inequality $\mathfrak{M}_{k_\alpha}(\underline{a}, \underline{w}) \ge \mathfrak{M}_{k_{\alpha,M}}(\underline{a}, \underline{w}) > \mathfrak{M}_{e^{Mx}}(\underline{a}, \underline{w}) - \frac{1}{M}$ holds. Taking $M \to \infty$ one gets $\mathfrak{M}_{k_\alpha}(\underline{a}, \underline{w}) \to \max(\underline{a})$.

## Open problem

How to unify presented theorems so as to get a set of conditions that would simultaneously be necessary and sufficient?

## Source code

The source code is available at `http://www.mimuw.edu.pl/~ppasteczka/poster_Bergen.pdf`

## References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