SYMMETRIES OF PARABOLIC GEOMETRIES

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Generalization of symmetries

We generalize the classical symmetries on symmetric spaces to a larger class of geometries on connected smooth manifolds M.

Definition. A *symmetry of a geometric structure* on M centered at $x \in M$ is a globally defined diffeomorphism s_x of M such that

(1) $s_x(x) = x$,
(2) $T_x s_x = -\mathrm{id}$ on $T_x M$, or at least on a maximal possible subspace of $T_x M$, where it is reasonable,
(3) s_x is an automorphism of the geometric structure.

If there is a symmetry at each $x \in M$, we call the geometry *symmetric*.

Example. Consider a manifold M together with an affine connection ∇. Denote by $[\nabla]$ the class of connections projectively equivalent to ∇. Thus $\hat{\nabla} \in [\nabla]$ if and only if

$$\hat{\nabla}_\xi(\eta) = \nabla_\xi(\eta) + \Upsilon_\xi(\eta) + \Upsilon_\eta(\xi)$$

for some $\Upsilon \in \Omega^1(M)$. Then a *projective symmetry* s_x is an automorphism of M such that

(1) $s_x(x) = x$,
(2) $T_x s_x = -\mathrm{id}$ on $T_x M$,
(3) $s_x^* \nabla_\xi(\eta) = \nabla_\xi(\eta) + \Upsilon_\xi(\eta) + \Upsilon_\eta(\xi)$ for some $\Upsilon \in \Omega^1(M)$,

and $(M, [\nabla])$ is called *projectively symmetric*, if there is a symmetry at each point.

Observations:

- Symmetries on $(M, [\nabla])$ are not determined uniquely.
- If (M, ∇) is affine symmetric space, then $(M, [\nabla])$ is projectively symmetric space. The converse does not hold.

Definition. Let us choose a symmetry s_x at each $x \in M$ on a symmetric geometric structure on M. A *system of symmetries* is a map

$$S : M \times M \to M, \quad S(x, y) \mapsto s_x(y).$$

If S is smooth in both variables, we call S a *smooth system*.

Questions.

- How the existence of a geometric structure restricts the possible symmetries and systems of symmetries?
- How the existence of symmetries and systems of symmetries restricts the possible geometric structure?
- Are there some additional restrictions under which have symmetric geometric structures 'usual' properties?

Basic concepts

We focus on geometric structures, which can be described as parabolic geometries. We suppose that we have effective parabolic geometries with connected base manifolds.

It is natural to consider the following conditions on a *symmetry* s_x of a (geometric structure given by a) parabolic geometry:

(1) $s_x(x) = x$,
(2) $T_x s_x = -\operatorname{id}$ on $T_x^{-1}M$,
(3) s_x is a base automorphism of some automorphism of the parabolic geometry.

Results for general parabolic geometries

For all flat models, there is a complete description of their symmetries, see [20].

Theorem. *On a (connected component of a) flat model* $(G \to G/P, \omega_G)$*:*

- *All symmetries at the origin* eP *are left multiplications by elements of the form* $g_0 \exp Z$*, where* $g_0 \in G_0$ *such that* $\operatorname{Ad}_{g_0} = -\operatorname{id}$ *on* $\mathfrak{g}_{-1}$*, and* $Z \in \mathfrak{g}^1$ *is arbitrary.*

- If there is such g_0, then there is an infinite amount of symmetries at each point. If there is no such g_0, then no parabolic geometry of the same type is symmetric.
- In general, there can exist non–involutive symmetries.

For each parabolic geometry carrying a smooth system of symmetries, there is the following fact, see [8, 9].

Theorem. *Each parabolic geometry $(\mathcal{G} \to M, \omega)$ with a smooth system of symmetries S is a homogeneous parabolic geometry. More precisely, the group $K \subset \mathrm{Aut}(\mathcal{G}, \omega)$ generated by S acts transitively on M, and we get $M \simeq K/H$, where H is a stabilizer of a point $x \in M$.*

There exist symmetric parabolic geometries, which are not homogeneous. However, each system of symmetries on such a geometry is non–smooth. An example of such geometry is a homogeneous model of the projective geometry where one removes two points, see [22, 18].

Construction of symmetric parabolic geometries

There are two main ingredients to construct a symmetric parabolic geometry on a homogeneous space K/H, see [9].

(1) An *extension* (α, i) of (K, H) to (G, P), where:
... $i : H \to P$ is a Lie group homomorphism,
... $\alpha : \mathfrak{k} \to \mathfrak{g}$ is a linear map extending $i' : \mathfrak{h} \to \mathfrak{p}$ such that:
(a) $\underline{\alpha} : \mathfrak{k}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{p}$ is vector space isomorphism,
(b) $\alpha \circ \mathrm{Ad}(l) = \mathrm{Ad}(i(l)) \circ \alpha$ for $l \in H$.

(2) A (smooth) map $\sigma : K \to H \cap i^{-1}(\Sigma_{eP})$ such that

$$\sigma(k \cdot l) = l^{-1}\sigma(k)l$$

for all $l \in H$, where $\Sigma_{eP} = \{g_0 \exp(Z) : g_0 \in G_0$ such that $\mathrm{Ad}_{g_0} = -\mathrm{id}$ on $\mathfrak{g}_{-1}$, $Z \in \mathfrak{g}^1$ is arbitrary$\}$ is the set of symmetries of homogeneous model at the origin, and by $i^{-1}(\Sigma_{eP})$ we denote the preimage of this set with respect to i.

Theorem.

- *With the data as above, there is a parabolic geometry* $(K \times_i P \to K/H, \omega_\alpha)$ *of type* (G, P)*, where* $\omega_\alpha = \alpha \circ \omega_K$ *on* $K \times_i i(H)$*. This geometry carries a (smooth) system of symmetries of the form* $s_{kH} := k\sigma(k)k^{-1}$*.*
- *One can construct each parabolic geometry* $(\mathcal{G} \to M, \omega)$ *with a smooth system of symmetries in this way.*

In particular, if $H \cap i^{-1}(\Sigma_{eP})$ has one element, then K/H is a *reflexion space*, i.e. a fiber bundle over a symmetric space, see [15, 6].

Some more restrictions for |1|–graded geometries

The symmetry s_x at x reverts the whole T_xM, and simultaneously preserves the corresponding G_0–structure. For example, if $G_0 = CO(p, q)$, the symmetry s_x preserves the corresponding conformal class of pseudometrics. However, this does not mean, that s_x preserves some metric from the conformal class (and corresponding Levi–Civita connection).

The classifications of simple Lie algebras, their real forms, and parabolic subalgebras allow us to classify |1|–graded parabolic geometries (with simple Lie algebras). There is the following fact, see [21].

Theorem. *The following types of normal symmetric |1|–graded parabolic geometries have to be locally flat:*

- *projective structures of dimension* 2*,*
- *almost Grassmannian geometries of type* $(p, q), p, q > 2$*,*
- *geometries modeled on quaternionic Grassmannians (but not the almost quaternionic ones),*
- *conformal structures in all signatures of dimension* 3*,*
- *geometries for the algebra* $\mathfrak{sp}(p, p)$*, where* $p > 2$*,*
- *all geometries coming from the algebras of types* C_ℓ*,*
- *spinorial geometries in the* D_ℓ *types with* $\ell > 4$*,*
- *all exotic geometries.*

The remaining geometries from the classification are:

- projective structures of dimension > 2,
- conformal structures of dimension > 3,
- para-quaternionic structures,
- almost quaternionic structures.

Geometries of these types can carry symmetries at points with non–zero harmonic curvature.

Theorem. *For each $|1|$–graded parabolic geometry $(\mathcal{G} \to M, \omega)$, we have the following restrictions, see [21, 22]:*

(1) *Each symmetry s_x at each $x \in M$ is involutive.*
(2) *Symmetric $|1|$–graded geometries are torsion–free.*
(3) *For each symmetry s_x, there exists a Weyl connection ∇, which is invariant with respect to s_x on a neighborhood of x.*
(4) *The Weyl curvature is parallel with respect to ∇ at x.*
(5) *For a system of symmetries S, there exists a Weyl connection ∇, which is invariant with respect to all symmetries from the system S, if and only if*
   - *S is smooth,*
   - *$s_x \circ s_y \circ s_x = s_{s_x(y)}$ for each $x, y \in M$.*

   In such case, the connection ∇ is given uniquely, and (M, ∇) forms an affine symmetric space.

Construction of symmetric $|1|$–graded geometries on symmetric spaces

There is a natural Cartan geometry on each affine locally symmetric space (M, ∇), see [5, 6].

A locally symmetric space is a locally flat Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (K, H, h), where:

- K is a Lie group with a Lie subgroup H, and K/H is connected,
- $h \in H$ such that $h^2 = \mathrm{id}_K$, and H is open in the centralizer of h in K,
- maximal normal subgroup of K contained in H is trivial.

Symmetries are locally given by $S_{kH} = khk^{-1}$ and the extensions (i, α) of (K, H) to $|1|$–graded geometries such that $i(h) = g_0$ provide all symmetric $|1|$–graded geometries on locally symmetric spaces of type (K, H, h). The classification result from [7] is:

Theorem. *There is a symmetric $|1|$–graded geometry of type (G, P) on a locally symmetric space of type (K, H, h) if and only if there is an invariant G_0–structure on the locally symmetric space.*
On a locally symmetric space of type (K, H, h), there is a bijection between:

- *equivalence classes of $|1|$–graded geometries of type (G, P) (up to outer automorphisms of the Lie group $Ad(H)$ induced by automorphisms of K).*
- *pairs consisting of a conjugacy class of inclusions $i : H \to G_0$ and an element of the centralizer of $i(H)$ in $Gl(\mathfrak{g}_{-1})$ contained in $Gl(\mathfrak{g}_{-1})/G_0$.*

Some more restrictions and construction of symmetric parabolic contact geometries

The symmetry s_x at x reverts the contact distribution $T_x^{-1}M$, and simultaneously preserves the corresponding geometric structure on $T^{-1}M$. For example, the symmetry s_x preserves the complex structure on $T^{-1}M$ in the CR–case, or the para–complex structure in the Lagrangian case.

The classification of simple Lie algebras, their real forms, and their contact gradings allow us to classify parabolic contact geometries.

Theorem. *The following types of normal symmetric parabolic contact geometries have to be locally flat, see* [23]*:*

- *Lie contact structures, $\mathfrak{g} = \mathfrak{so}(p + 2, q + 2), p + q \geq 3$,*
- *parabolic contact geometries corresponding to exotic algebras $(E_6, E_7, E_8, F_4, G_2)$.*

The remaining geometries from the classification are:

- contact projective structures,
- Lagrangian contact structures,
- partially integrable almost CR–structures of hypersurface type,

Geometries of these types can carry symmetries at points with non–zero harmonic curvature.

Theorem. *For each parabolic contact geometry $(\mathcal{G} \to M, \omega)$, there are the following restriction, see* [23]*:*

1. *Symmetric parabolic contact geometries are torsion–free.*
2. *For each symmetry s_x, the differential $T_x s_x$ is involutive (even when s_x is not involutive).*
3. *There is the decomposition $T_x M \simeq T_x^{-1} M \oplus T_x^{+} M$ into ± 1–eigenspaces, where $T_x^{-1} M$ is the contact distribution, and $T_x^{+} M$ is its one–dimensional complement.*
4. *If $\kappa_H(x) \neq 0$, then each symmetry s_x at x is involutive.*
5. *For each involutive symmetry s_x, there exists a Weyl connection ∇, which is invariant with respect to s_x on a neighborhood of x.*
6. *If $s_x \circ s_y \circ s_x = s_{s_x}(y)$ holds for all $x, y \in M$ and symmetries are involutive, then there is a subclass of Weyl connections such that $s_x^* \nabla_\xi(\eta) = \nabla_\xi(\eta) - \{\{\Upsilon, \xi_+\}, \eta\}$ for suitable $\Upsilon \in \Gamma(\mathcal{A}^2 M)$, where ξ_+ is a projection of ξ into $T^+ M$ in direction $T^{-1} M$.*

- For each smooth system of symmetries S, there is a decomposition $TM \simeq T^{-1}M \oplus T^+M$, where T^+M is a one–dimensional (integrable) distribution.
- Denote by F_x an integral submanifold of T^+M through x, and by N the space of all integral submanifolds of T^+M.
- As above, denote by K the transitive group generated by S acting on M. Thus $M \simeq K/H$, where H is a stabilizer of $x \in M$.

With this notation, there are the following facts, see [8].

Theorem. *For each parabolic contact geometry* $(\mathcal{G} \to K/H, \omega)$ *with a smooth system of involutive symmetries such that* $s_x \circ s_y \circ s_x = s_{s_x(y)}$ *holds for all* $x, y \in K/H$*:*

(1) *The space* N *forms a smooth manifold.*
(2) *The group* K *acts correctly and transitively on* N*. In particular, each symmetry* s_x *on* M *determines a classical symmetry on* N*, and thus,* N *is a symmetric space.*
(3) *We can write* $N = K/L$*, where* L *is a stabilizer of an integral submanifold* F_x*.*
(4) *For a Lagrangian contact structure on* K/H*: If the para–complex structure at* eH *is* L*–invariant, and if there is some* H*–invariant vector in* $T^+_{eH}(K/H)$*, then* N *is a para–pseudo–Hermitian symmetric space.*
(5) *For a CR–structure on* K/H*: If the complex structure at* eH *is* L*–invariant, and if there is some* H*–invariant vector in* $T^+_{eH}(K/H)$*, then* N *is a pseudo–Hermitian symmetric space.*

The construction of symmetric parabolic contact geometries from previous Theorem is described in [5]. If the group K is semisimple, the following Theorem describes the possible extensions:

Theorem. *Let* K/H *be a connected homogeneous space such that:*

- *there is* $h \in K$ *such, that* $h^2 = \mathrm{id}$*,* H *is contained in the centralizer* L *of* h *in* K*, and* $\dim(L/H) = 1$*;*
- K *is semisimple and the maximal normal subgroup of* K *contained in* H *is trivial.*

Let (i, α) *be (up to equivalence) an extension of* (K, H) *to a regular parabolic contact geometry of type* (G, P)*. Then:*

(1) $i(h) = g_0$, $\alpha(\mathfrak{l}) \subset \mathfrak{g}_{-2} + \mathfrak{g}_0 + \mathfrak{g}_2$, $\alpha(\mathfrak{k}/\mathfrak{l}) \subset \mathfrak{g}_{-1} + \mathfrak{g}_1$,
(2) $i(H) \subset G_0$ *and* $\mathfrak{l}/\mathfrak{h}$ *is in center of* $\mathfrak{l}$*,*
(3) *the semisimple symmetric space* K/L *has only pseudo–hermitian or para–pseudo–hermitian simple factors.*

Examples

The following examples from [5, 8] will illustrate that the geometric structures on the symmetric space N and on the contact distribution are not always compatible. Let us consider:

- a simply connected Lie group K with given Lie algebra $\mathfrak{k}$,
- a subgroup $L \subset K$ with algebra $\mathfrak{l}$,
- a subgroup $H \subset L$, which is contained in the centralizer L of a chosen $h \in K$ such that $h^2 = \mathrm{id}$, with algebra $\mathfrak{h}$.

Then $M = K/H$ is connected, $L/H = F_{eH}$, and $N = K/L$ is corresponding symmetric space with $s_{eH} = h$ is such case. We denote examples by a triple $(\mathfrak{k}, \mathfrak{l}, \mathfrak{h})$.

Example $(\mathfrak{sl}(n+1,\mathbb{R}), \mathfrak{gl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}))$

Let us start with a Lagrangian contact structure given by an extension $\alpha : \mathfrak{sl}(n+1,\mathbb{R}) \to \mathfrak{sl}(n+2,\mathbb{R})$ of the form

$$\alpha(\left(\begin{smallmatrix} a & Y^T \\ X & A \end{smallmatrix}\right)) = \begin{pmatrix} \frac{1}{2}a & \frac{1}{2}Y^T & \frac{1}{4}a \\ X & A & \frac{1}{2}X \\ a & Y^T & \frac{1}{2}a \end{pmatrix}.$$

Since the conditions in the Theorem are satisfied, we recover the structure of para–pseudo–Hermitian symmetric space.

Example $(\mathfrak{so}(p+2,q), \mathfrak{so}(p,q) \oplus \mathfrak{so}(2), \mathfrak{so}(p,q))$

Let us start with a class of Lagrangian contact structure given by class of extensions $\alpha : \mathfrak{so}(p+2,q) \to \mathfrak{sl}(p+q+1,\mathbb{R})$ parametrized by $t \geq 0$ of the form

$$\alpha(\begin{pmatrix} 0 & a & -X^T\mathbb{I} \\ -a & 0 & -Y^T\mathbb{I} \\ X & Y & A \end{pmatrix}) =$$

$$\begin{pmatrix} \frac{n}{2(n+1)}ta & -(\frac{(n+2)}{2(n+1)}t^2+1)X^T\mathbb{I} - \frac{n}{2(n+1)}tY^T\mathbb{I} & -\frac{((3n+2)(n+2)}{4(n+1)^2}t^2a - a \\ X & A - \frac{1}{n+1}tEa & -\frac{(n+2)}{2(n+1)}tX - Y \\ a & tX^T\mathbb{I} + Y^T\mathbb{I} & \frac{n}{2(n+1)}ta \end{pmatrix}$$

Conditions of the Theorem are not satisfied, and we do not recover the structure of para–pseudo–Hermitian symmetric space. There is no induced compatible para–pseudo–Hermitian structure.

In fact, corresponding symmetric space is a pseudo–Hermitian symmetric space in this case, and the pseudo–Hermitian structure has nothing to do with the Lagrangian contact structure.

Example $(\mathfrak{so}(p+1,q+1), \mathfrak{so}(p,q) \oplus \mathfrak{so}(1,1), \mathfrak{so}(p,q))$
Let us start with a two types of extensions to Lagrangian contact structures. First type is given by class of extensions $\alpha : \mathfrak{so}(p+1, q+1) \to \mathfrak{sl}(p+q+1, \mathbb{R})$ parametrized by $t \geq 0$ of the form

$$\alpha_1(\begin{pmatrix} 0 & a & -X^T\mathbb{I} \\ a & 0 & Y^T\mathbb{I} \\ X & Y & A \end{pmatrix}) =$$

$$\begin{pmatrix} \frac{n}{2(n+1)}ta & -(-\frac{(n+2)}{2(n+1)}t^2+1)X^T\mathbb{I}-\frac{n}{2(n+1)}tY^T\mathbb{I} & -\frac{((3n+2)(n+2)}{4(n+1)^2}t^2a-a \\ X & A-\frac{1}{n+1}tEa & -\frac{(n+2)}{2(n+1)}tX+Y \\ a & -tX^T\mathbb{I}+Y^T\mathbb{I} & \frac{n}{2(n+1)}ta \end{pmatrix}.$$

The second type is given by an extension of the form

$$\alpha_2(\begin{pmatrix} 0 & a & -X^T\mathbb{I} \\ a & 0 & Y^T\mathbb{I} \\ X & Y & A \end{pmatrix}) = \begin{pmatrix} \frac{n}{2(n+1)}a & -\frac{n}{4(n+1)}X^T\mathbb{I}+\frac{n}{4(n+1)}Y^T\mathbb{I} & \frac{n^2}{8(n+1)^2}a \\ X+Y & A-\frac{1}{n+1}Ea & \frac{n}{4(n+1)}(X+Y) \\ 2a & -X^T\mathbb{I}+Y^T\mathbb{I} & \frac{n}{2(n+1)}a \end{pmatrix}.$$

For the first type of extension, the conditions of the Theorem are not satisfied, and we do not recover the structure of para–pseudo–Hermitian symmetric space.

For the second type of extension, the conditions in the Theorem are satisfied, we recover the structure of para–pseudo–Hermitian symmetric space.

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