Boolean Functions and Trapdoors on Block Cipher

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Trapdoors Project (2005–)

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Block Cipher

Let $\mathcal{C} = \{\phi_k | k \in \mathcal{K}\}$ be a block cipher acting on $V = V_1 \oplus \cdots \oplus V_s$, with $V_i = \mathbb{F}_2^m$ for $i = 1, \ldots, s$.

Definition

An element $\gamma \in Sym(V)$ is called a bricklayer transformation (or parallel S-box) of V if for any $v = (v_1 \oplus \cdots \oplus v_s) \in V$

$$v\gamma = v_1\gamma_1 \oplus \cdots \oplus v_s\gamma_s,$$

for some $\gamma_i \in Sym(V_i)$.

Definition

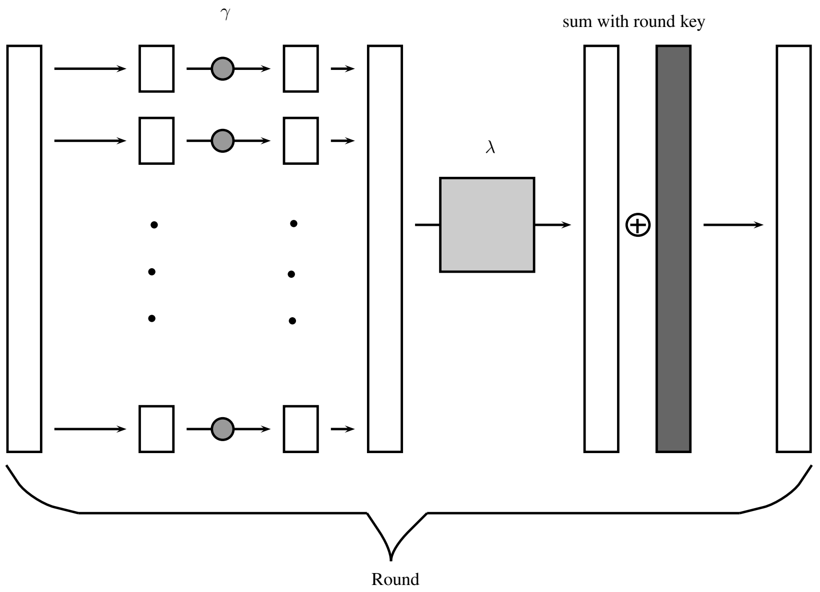
A linear map $\lambda \in GL(V)$ is called a proper mixing layer if no sum of the V_i, except $\{0\}$ and V, is λ-invariant.

Definition

A block cipher $\mathcal{C} = \{\phi_k | k \in \mathcal{K}\}$ is called translation based (TB) if

(1) each ϕ_k is the composition of ℓ round functions $\phi_{k,h}$, for $k \in \mathcal{K}$, and $h = 1, \ldots, \ell$, where in turn each round function can be written as a composition $\gamma_h \lambda_h \sigma_{\bar{k},h}$ of three permutations of V, with
   - γ_h is a bricklayer transformation depending on the round
   - λ_h is a linear permutation depending on the round
   - $\sigma_{\bar{k},h}$ is the translation by $\bar{k}$ depending on the key k and the round

(2) for at least one round the mixing layer is proper and the map $\mathcal{K} \to V$, $k \mapsto \bar{k}$ is surjective.


γ
sum with round key
λ
Round

Linear Trapdoors

Cryptographers construct $\mathcal{C}$ s.t. $\phi_k \notin AGL(V,+)$ for any key k, but there could be a hidden sum $\circ$ s.t.:

$(V,\circ)$ is a vector space and $\phi_k \in AGL(V,\circ)$

On a single round

Let us focus on a sigle round. Then the question is:

Is there any operation $\circ$ s.t. $(V, \circ)$ is a vector space and

$$\gamma_h \lambda_h \sigma_{k,h} \in AGL(V, \circ)?$$

Proposition

If $\gamma_h \lambda_h \sigma_{k,h} \in AGL(V, \circ)$ for all $k \in V$, then $\gamma_h \lambda_h \in AGL(V, \circ)$ and $T_+ = \{\text{translations with respect to } +\} \subseteq AGL(V, \circ)$.

Problems

(a) Find $\circ$ s.t. $T_+ \subseteq AGL(V, \circ)$.

(b) When γ, λ or $\gamma\lambda \in AGL(V, \circ)$?

Problem (a)

In [1], Caranti *et al.* characterized the abelian regular subgroups of $AGL(V,+)$ in terms of algebras. Let $T_\circ$ be the translation group with respect to $\circ$, we have:

Theorem

There is a one-to-one correspondence between

$$T_\circ \subseteq AGL(V,+)$$

$$\updownarrow$$

elementary abelian regular subgroups of $AGL(V,+)$

$$\updownarrow$$

commutative, associative $\mathbb{F}_2$*-algebra structures* $(V,+,\cdot)$*, with* $x\cdot x = 0$ *for all* $x \in V$ *and such that the resulting ring is radical* $(x+y+x\cdot y = x\circ y)$*.*

Problem (a)

Theorem

Let $V = \mathbb{F}_2^n$. *For* $n \leq 6$

$$T_\circ \subseteq AGL(V,+) \Leftrightarrow T_+ \subseteq AGL(V,\circ)$$

For $n \geq 7$

$$\exists T_\circ \subseteq AGL(V,+) \text{ s.t. } T_+ \nsubseteq AGL(V,\circ)$$

Problem (a)

How many $T_\circ \subseteq AGL(V,+)$

- $n = 2$ only T_+
- $n = 3$ there are 8 groups, of which 1 is T_+ and the others 7 are conjugated.
- $n = 4$ there are 106 groups, of which 1 is T_+ and the others 105 are conjugated.
- $n = 5$ there are 1954 groups, of which 1 is T_+ and the others form 2 classes of cardinality 1085 and 868.
- there are complex formulae for $n \geq 6$.

Problem (b)

Let $f : V \to V$, we denote by $\hat{f}_a : x \mapsto f(x) + f(x+a)$

Definition

f is called weakly δ-uniform if for every $a \in V \setminus \{0\}$

$$|\mathrm{Im}(\hat{f}_a)| > \frac{2^{m-1}}{\delta}$$

Definition

f is called strongly r-anti-invariant if for any two subspace U and W s.t. $f(U) = W$, we have $\dim(U) = \dim(W) < m - r$ or $U = W = V$.

Problem (b)

Definition

f is called anti-crooked (AC) if for every $a \in V \setminus \{0\}$ $\mathrm{Im}(\hat{f}_a)$ is not an affine subspace of V.

Theorem

Let f be a power function (i.e. x^d). If f is weakly-APN and not APN then f is AC.

Corollary

x^{-1} in even dimension is AC.

Problem (b)

Lemma

Let f be a power function. If there exists $a \neq 0$ s.t. $\mathrm{Im}(\hat{f}_a)$ is not an affine subspace of V, then $\mathrm{Im}(\hat{f}_{a'})$ is not an affine subspace of V for all $a' \in V \setminus \{0\}$.

Recalling that f is called crooked if f is APN and $\mathrm{Im}(\hat{f}_a)$ is an hyperplane for all $a \in V \setminus \{0\}$.

Corollary

Let f be an APN power function not crooked, then f is AC.

Problem (b)

[3] gives sufficient conditions on γ and λ in order to have $\gamma\lambda$ non linear

Theorem

Let $\mathcal{C}$ be a TB and there exists at least 1 round with S-box $\gamma = (\gamma_1, \ldots, \gamma_s)$ and mixing layer λ s.t.:

- γ_i *weakly-APN, strongly 1-AI and AC for all* $i = 1, \ldots, s$
- λ *proper*

$\Rightarrow$ $\gamma\lambda \notin AGL(V, \circ)$ *for any* $\circ$

Some differential properties of $AGL(V, \circ)$

Let

$$\delta(f) = \max_{a \neq 0, b \in V} |\hat{f}_a^{-1}(b)|.$$

Theorem

$T_+ \subseteq AGL(V, \circ) \Rightarrow$ for any $f \in AGL(V, \circ)$

$$\delta(f) \geq 2^{\frac{n}{2}}.$$

Theorem

For $n = 3, 4, 5$, if $T_+ \subseteq AGL(V, \circ)$ then $\delta(f) \geq 2^{n-1}$ for any $f \in AGL(V, \circ)$.

Remark

$n = 6$: $\delta(f)$?
For $n = 7$: *there exists* f *s.t.* $\delta(f) = 2^{n-2}$.

Remark

Let $V = (\mathbb{F}_2^m)^s$ *and* γ_i *be APN, then* $\delta(\gamma) = 2^{m(s-1)+1}$.
E.g. $s = 2$ *and* $m = 5$ *we have that if all the* γ_i *are APN then* $\delta(\gamma) = 2^6 > 2^5$.

Conclusions

(1) If γ is not in our class (i.e. weakly-APN, strongly AI, and AC) maybe there exists a hidden sum $\circ$ s.t. for any round and key-schedule $\mathcal{C}$ is linear.

(2) Even if γ is in our class maybe there exists a hidden sum $\circ$ s.t. ϕ_k is linear depending on the key-schedule.

(3) Only APN's will not be enough!

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