Perturbations of Binary de Bruijn Sequences

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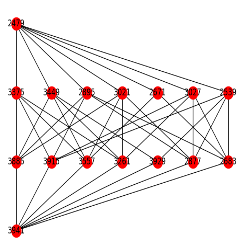
Two binary de Bruijn sequences $\mathbf{v}$ and $\mathbf{u}$ of order n can be obtained from each other by applying the cross join method, possibly repeatedly [2]. Here we explicitly determine the list of non-ovelapping cross join pairs between any two sequences. The *truth table distance* is the smallest number of assignments in the *truth table* of the feedback function of $\mathbf{v}$ that must be changed from 0 to 1 or vice versa to become the truth table of $\mathbf{u}$. If $\mathbf{v}$ and $\mathbf{u}$ have distance $2k$, then there are k cross join pairs between them.

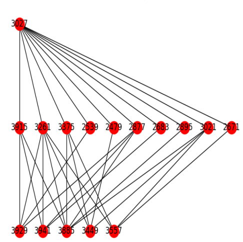
Let $\mathbf{v}$ be a de Bruijn sequence constructed by adding a 0 to the longest string of zeros in an m-sequence $\mathbf{v}'$ of length $2^n - 1$. The Fryer's formula in [1, Sect. 1] gives the number of de Bruijn sequences of distance $2k$ for all $1 \leq k \leq 2^{n-1} - 1$. The situation when $\mathbf{v}'$ is **not** an m-sequence is less clear. We study the perturbation of such sequences and provide a complete classification for some small orders.

As an example, we present the situation for $n = 4$. The vertices are the sequences in their decimal representations. The graph on the left shows that from vertex 2479, *i.e.*, $\mathbf{v} = (0000100110101111)$ there are 7 sequences each of cross join distances $k = 1$ and $k = 2$ and a unique sequence with $k = 3$. This matches the Fryer's formula since the characteristic polynomial $x^4 + x + 1$ is primitive. Sequence $\mathbf{u} = (0000101111010011)$ at the top of the middle figure has a different cross join pattern. Its perturbations are not given by the Fryer's formula, albeit quite close to it. Some de Bruijn sequences which do not correspond to any m-sequence still exhibit the Fryer's formula. The complete information on the cross join patterns is in the figure on the right.

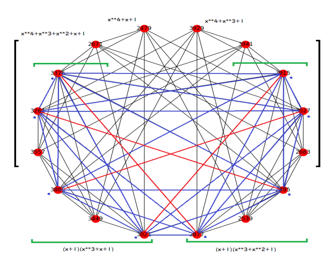
Symmetric $(1, 7, 7, 1)$



Asymmetric $(1, 10, 5, 0)$



Classification $n = 4$



References

[1] D. Coppersmith, R. C. Rhoades, and J. M. Vanderkam, *Counting de Bruijn sequences as perturbations of linear recursions*, [Online] Available at `https://arxiv.org/pdf/1705.07835.pdf`

[2] J. Mykkeltveit and J. Szmidt, *On cross joining de Bruijn sequences*, Contemporary Mathematics, **632**, pp. 333-344 (2015).