Differential equivalence of APN functions: results and open problems

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3d International Workshop on
Boolean Functions and Their Applications

Loen, Norway
18 June 2018

The differential equivalence of APN functions

The associated Boolean function

Let F be a vectorial Boolean function from $\mathbb{F}_2^n$ to itself.

Definition 1 ([1])

The associated Boolean function $\gamma_F(a, b)$ *in* $2n$ *variables of* F *is defined as follows: it takes value* 1 *iff* $a \neq \mathbf{0}$ *and* $F(x) + F(x + a) = b$ *has solutions.*

Why this function is of interest?

- F is almost perfect nonlinear (APN) iff $\mathrm{wt}(\gamma_F) = 2^{2n-1} - 2^{n-1}$;
- F is almost bent (AB) iff γ is a bent function.

[1] Carlet C., Charpin P., Zinoviev V.: Codes, bent functions and permutations suitable for DES-like cryptosystems. Des. Codes Cryptogr. 15, 125–156 (1998).

The differential equivalence: definition

We introduce the following notation.

Definition 2 ([2])

Two functions F, G from $\mathbb{F}_2^n$ to itself are called differentially equivalent if $\gamma_F = \gamma_G$. Denote the differential equivalence class of F by $\mathcal{DE}_F$.

Further we will focus only on APN functions.

Proposition 1

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ be an APN function, $n > 1$. Then $F_{c,d}(x) = F(x + c) + d$ is differentially equivalent to F for all $c, d \in \mathbb{F}_2^n$ and all the functions $F_{c,d}$ are pairwise distinct.

We call functions $F_{c,d}$ as trivially differentially equivalent functions to F.

[2] Gorodilova A.A.: On a remarkable property of APN Gold functions // Cryptology ePrint Archive, Report 2016/286 (2016).

General open problem on the differential equivalence

Problem 1 ([3])

Is it possible to find a systematic way, given an APN function F, to build another function G such that $\gamma_F = \gamma_G$?

Problem 1 (modified)

- *Is it possible to describe the differential equivalence class of a given APN function?*
- *Do there exist functions which are not trivially differentially equivalent to a given APN function?*

[3] Carlet C.: Open Questions on Nonlinearity and on APN Functions. Arithmetic of Finite Fields, Lecture Notes in Computer Science. 9061, 83–107 (2015).

EA-invariant

Definition 3

F and G are called extended affine equivalent (EA-equivalent) if $G = A' \circ F \circ A'' + A$, where A', A'' are affine permutations and A is affine.

Proposition 2

Let F, G be EA-equivalent functions. Then $|\mathcal{DE}_F| = |\mathcal{DE}_G|$.

So, we can study the differential equivalence classes of EA-representatives.

Open problem on CCZ-invariant

Definition 4 ([1])

Two functions F and G are said to be Carlet-Charpin-Zinoviev equivalent (CCZ-equivalent) if their graphs $\mathcal{G}_F = \{(x, F(x)) : x \in \mathbb{F}_2^n\}$ and $\mathcal{G}_G = \{(x, G(x)) : x \in \mathbb{F}_2^n\}$ are affine equivalent.

Problem 2

Is the cardinality of the differential equivalence class of an APN function a CCZ-equivalence invariant?

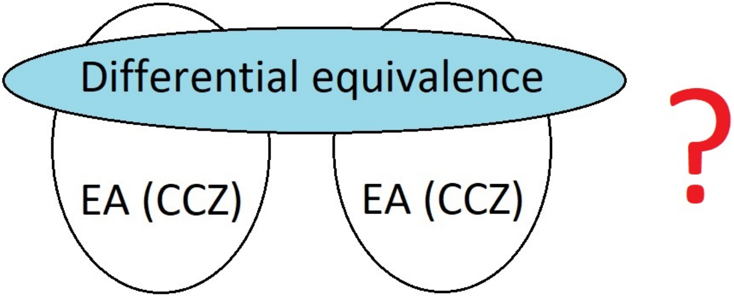
As stated in [4] the answer to this question is positive.

[4] Canteaut A., Boura C., Jean J. and Suder V.: On Sboxes sharing the same DDT. Abstracts of BFA-2018.

Open problem on connection between the differential equivalence and EA- (CCZ-) equivalence

Problem 3

Do there exist two differentially equivalent APN functions which are not EA- (CCZ-) equivalent?



By now such two APN functions have not been found.

The differential equivalence of quadratic APN functions

Quadratic APN functions

Definition 5

F is quadratic if degree of its algebraic normal form is 2.

Let

$$B_a(F) = \{F(x) + F(x+a) : \ x \in \mathbb{F}_2^n\}$$

for a vector $a \in \mathbb{F}_2^n$.

- F is APN iff $|B_a(F)| = 2^{n-1}$ for all nonzero a.
- if F is quadratic, then $B_a(F)$ is an affine hyperplane for all nonzero a.

Quadratic APN functions and crooked functions

In [5] definition of the crooked functions was introduced and it was generalized to the following:

Definition 6

F is called generalized crooked if $B_a(F)$ is an affine hyperplane for all $a \neq \mathbf{0}$.

Problem 4 ([6])

Are all crooked functions quadratic?

If "yes", then there are no nonquadratic functions differentially equivalent to a given quadratic APN function.

[5] Bending T. D., Fon-Der-Flaass D.: Crooked functions, bent functions, and distance regular graphs. Electron. J. Combin. 5 (1) (1998) R34.

[6] Kyureghyan G.: Crooked maps in $\mathbb{F}_2^n$. Finite Fields Their Appl. 13(3), 713–726 (2007)

Open problem on adding affine functions

There always exist 2^{2n} trivially differentially equivalent functions to a given APN function. Do there exist other?

- If F is quadratic, then all these 2^{2n} trivial functions are obtained by adding to F affine functions $A_{c,d}(x) = F(x) + F(x+c) + d$.

Problem 5

What affine functions do not change the associated Boolean function γ_F when adding to a quadratic APN function F?

APN Gold functions

Theorem 1

Let $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be a Gold function $F(x) = x^{2^k+1}$, where $\gcd(k, n) = 1$. Then the following statements hold:

- *if $n = 4t$ for some t and $k = n/2 \pm 1$, then there exist exactly $2^{2n+n/2}$ distinct affine functions A such that F and $F + A$ are differentially equivalent; all of them are of the form $A(x) = \alpha + \lambda^{2^k} x + \lambda x^{2^k} + \delta x^{2^j}$, where $\alpha, \lambda, \delta \in \mathbb{F}_{2^n}$, $\delta = \delta^{2^{n/2}}$, and $j = k - 1$ for $k = n/2 + 1$ and $j = n - 1$ for $k = n/2 - 1$;*
- *otherwise there exist exactly 2^{2n} distinct affine functions A such that F and $F + A$ are differentially equivalent; all of them are of the form $A(x) = \alpha + \lambda^{2^k} x + \lambda x^{2^k}$, where $\alpha, \lambda \in \mathbb{F}_{2^n}$.*

Total numbers of affine functions A on $\mathbb{F}_2^n$ such that F and $F+A$ are differentially equivalent

n	# EA classes	# affine functions A: $F+A \in \mathcal{DE}_F$
2	1	2^4
3	1	2^6
4	1 [7]	2^{10}
5	2 [7]	for all 2 classes: 2^{10}
6	13 [8,9]	for 12 classes: 2^{12}; for 1 class: 2^{13}
7	$\geq$ 487 [10]	for all known 487 classes: 2^{14}
8	$\geq$ 8179 [10]	for 1 class from known 8179: 2^{20} for other 8178 classes: 2^{16}

[7] Brinkman M., Leander G.: On the classification of APN functions up to dimension five. Proc. of the International Workshop on Coding and Cryptography 2007 dedicated to the memory of Hans Dobbertin. Versailles, France, 39–48 (2007).
[8] Browning K. A., Dillon J. F., Kibler R. E., McQuistan M. T.: APN Polynomials and Related Codes. Journal of Combinatorics, Information and System Science, Special Issue in honor of Prof. D.K Ray-Chaudhuri on the occasion of his 75th birthday, vol. 34, no. 1-4, pp. 135–159 (2009).
[9] Edel Y.: Quadratic APN functions as subspaces of alternating bilinear forms. Contact Forum Coding Theory and Cryptography III, Belgium (2009), pp. 11–24 (2011).
[10] Yu Y., Wang M., Li Y.: A matrix approach for constructing quadratic APN functions. Des. Codes Cryptogr. 73, 587–600 (2014).

Properties of the associated Boolean function of a quadratic APN function

The associated function of a quadratic APN function

Let F be a quadratic APN function on $\mathbb{F}_2^n$.

Then γ_F is of the form

$$\gamma_F(a,b) = \Phi_F(a) \cdot b + \varphi_F(a) + 1,$$

where $\Phi_F : \mathbb{F}_2^n \to \mathbb{F}_2^n$, $\varphi_F : \mathbb{F}_2^n \to \mathbb{F}_2$ are uniquely defined from

$$B_a(F) = \{y \in \mathbb{F}_2^n : \ \Phi_F(a) \cdot y = \varphi_F(a)\}$$

for all $a \neq \mathbf{0}$ and $\Phi_F(\mathbf{0}) = \mathbf{0}$, $\varphi_F(\mathbf{0}) = 1$.

Note that $B_a(F)$ is a linear subspace iff $\varphi_F(a) = 0$.

Φ_F — what is this?

Let us denote $A_v^F = \{a \in \mathbb{F}_2^n : \ \Phi_F(a) = v\}$ for $v \in \mathbb{F}_2^n$.

Proposition 3 ([1])

Let F be a quadratic APN function in n variables, n is odd. Then Φ_F is a permutation; therefore, γ_F is a bent function of Maiorana–McFarland type.

Thus, when n is odd, all A_v^F, $v \in \mathbb{F}_2^n$, are pairwise distinct and each of them consists of one element. We prove the following theorem for even n.

Theorem 2

Let F be a quadratic APN function in n variables, n is even. Then $A_v^F \cup \{\mathbf{0}\}$ is a linear subspace of even dimension for any $v \in \mathbb{F}_2^n$.

Φ_F — what is this?

Value distribution of Φ_F for even n.

n	# EA classes		$\#\{v \in \mathbb{F}_2^n : \lvert A_v^F \rvert = k\}$, $k = 3$	$k = 15$
4	1		5	–
6	13	for 12 classes:	21	–
		for 1 class:	16	1
8	≥ 8179	for 7680 classes:	85	–
		for 487 classes:	80	1
		for 12 classes:	75	2

Φ_F — what is this?

Theorem 3

Let F be a quadratic APN function in n variables, n is odd, $n \geq 3$. Then $\deg(\Phi_F) \leq n-2$.

The bound of theorem 4 is tight for all known quadratic APN functions in not more than 8 variables (including also even numbers).

Moreover, it holds that all their component functions are of degree $n-2$.

For example, for an APN Gold function we have $\Phi_F(a) = (a^{2^k+1})^{-1}$, $\Phi_F(\mathbf{0}) = \mathbf{0}$, and $\deg(\Phi_F) = n-2$.

The linear spectrum of quadratic APN functions

The linear spectrum: definition

Let $F, L : \mathbb{F}_2^n \to \mathbb{F}_2^n$, where F is a quadratic APN function and L is linear.

Then $B_a(F+L)$ equals $B_a(F)$ or $\mathbb{F}_2^n \setminus B_a(F)$ for all $a \in \mathbb{F}_2^n$.

Let us denote $k_L^F = |\{a \in \mathbb{F}_2^n \setminus \{\mathbf{0}\} : \; B_a(F) = B_a(F+L)\}|$.

Definition 7

The linear spectrum of a quadratic APN function F in n variables is the vector $\Lambda^F = (\lambda_0^F, \ldots, \lambda_{2^n-1}^F)$, where λ_k^F is the number of linear functions L such that $k_L^F = k$.

It is easy to see that $\sum_{k=0}^{2^n-1} \lambda_k^F = 2^{n^2}$.

The linear spectrum: invariant

Proposition 4

The linear spectrum of a quadratic APN function is

- *a differential equivalence invariant;*
- *a EA-equivalence invariant.*

Let F be a quadratic APN function in n variables. Then

$n = 3$	$\Lambda^F = (0, 56, 0, 280, 0, 168, 0, 8)$
$n = 4$	$\Lambda^F = (0, 0, 0, 0, 0, 15552, 0, 25920, 0, 17280, 0, 5760, 0, 960, 0, 64)$
$n = 5$	2 classes with distinct spectra
$n = 6$	13 classes with pairwise distinct spectra except one pair having equal spectrum

The linear spectrum: zero values

Theorem 4

Let F be a quadratic APN function in n variables, $n > 1$. Then the following statements hold:

- $\lambda_k^F = 0$ *for all even* k, $0 \leq k \leq 2^n - 2$;
- *if* n *is even, then* $\lambda_k^F = 0$ *for all* $0 \leq k < (2^n - 1)/3$.

Let F be a quadratic APN function in n variables. Then

$n = 3$	$\Lambda^F = (0,\ 56,\ 0,\ 280,\ 0,\ 168,\ 0,\ 8)$
$n = 4$	$\Lambda^F = (0,\ 0,\ 0,\ 0,\ 0,\ 15552,\ 0,\ 25920,\ 0,\ 17280,\ 0,\ 5760,\ 0,\ 960,\ 0,\ 64)$
$n = 5$	2 classes with distinct spectra
$n = 6$	13 classes with pairwise distinct spectra except one pair having equal spectra

Differential equivalent functions in small number of variables

Based on results about the linear spectra, properties of γ_F, we computationally obtained a classification of differentially nonequivalent quadratic APN functions up to 6 variables.

Theorem 5

Let F be a quadratic APN function in n variables, $n = 2, 3, 4, 5, 6$. Then each differentially equivalent to F quadratic APN function G is represented as follows: $G = F + A$, where A is an affine function. Moreover, the number K of such functions A equals 2^{2n} for all functions except functions from two EA-equivalence classes with the following representatives:

- *$n = 4$: APN Gold function $F(x) = x^3$, $K = 2^{10}$;*
- *$n = 6$: APN function*
  $F(x) = \alpha^7 x^3 + x^5 + \alpha^3 x^9 + \alpha^4 x^{10} + x^{17} + \alpha^6 x^{18}$, $K = 2^{13}$.

Thank you for your attention!