Journey into differential and graph theoretical properties of (generalized) Boolean function

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The objects of the investigation: (Generalized) Boolean functions I

- Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$
- *Generalized Boolean function* $f : \mathbb{F}_2^n \to \mathbb{Z}_q$ $(q \geq 2)$; its set $\mathcal{GB}_n^q$; when $q = 2$, $\mathcal{B}_n$;
- *(Generalized) Walsh-Hadamard transform*:
$$\mathcal{H}_f^{(q)}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{F}_2^n} \zeta_q^{f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}},\ \zeta_q = e^{\frac{2\pi i}{q}};\ \text{(use } \mathcal{W}_f\text{, if } q = 2\text{)}$$
- Fourier transform: $\mathcal{F}_f(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{F}_2^n} f(\mathbf{x})(-1)^{\mathbf{u}\cdot\mathbf{x}}$
- Let $2^{k-1} < q \leq 2^k$. Then $\mathcal{GB}_n^q \ni f \longleftrightarrow \{a_i\}_{0 \leq i \leq k-1} \subset \mathcal{B}_n$:
$$f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1}a_{k-1}(\mathbf{x}), \forall \mathbf{x} \in \mathbb{F}_2^n.$$

Characterizing generalized bent $f : \mathbb{F}_2^n \to \mathbb{Z}_{2^k}$

- $f : \mathcal{GB}_n^q$ is *generalized bent* (*gbent*) if $|\mathcal{H}_f(\mathbf{u})| = 2^{n/2}, \forall \mathbf{u}$.

Theorem (Various Authors 2015–'17)

Let $f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-2}a_{k-2}(\mathbf{x}) + 2^{k-1}a_{k-1}(\mathbf{x})$ be a function in $\mathcal{GB}_n^{2^k}$, $k > 1$, $a_i \in \mathcal{B}_n$, $0 \le i \le k-1$, and $\tilde{f} \in a_{k-1} \oplus \langle a_0, a_1, \ldots a_{k-2} \rangle$. Then f is gbent iff $\tilde{f}$ is bent (n even), respectively, semibent (n odd), with an (explicit) extra condition on the Walsh-Hadamard coeff.

Differential properties of generalized Boolean functions I

- $\mathbf{u} \in \mathbb{F}_2^n$ is a *linear structure* of $f \in \mathcal{GB}_n^q$ if the derivative $D_{\mathbf{u}} f(\mathbf{x}) := f(\mathbf{x} \oplus \mathbf{u}) - f(\mathbf{x}) = c \in \mathbb{Z}_q$ constant, for all $\mathbf{x} \in \mathbb{F}_2^n$.
- Let $S_f = \{\mathbf{x} \in \mathbb{F}_2^n \,|\, \mathcal{H}_f(\mathbf{x}) \neq 0\} \neq \emptyset$ (gen.WH support)

Theorem (Martinsen–Meidl–Pott–S., 2018)

Let $f \in \mathcal{GB}_n^{2^k}$, with $f(\mathbf{x}) = \sum_{i=0}^{k-1} 2^i a_i(\mathbf{x})$, $a_i \in \mathcal{B}_n$. The following are equivalent:

(*i*) $\mathbf{a}$ *is a linear structure for* f.

(*ii*) $\mathbf{a}$ *is a linear structure for* a_i, *s.t.* $a_i(\mathbf{a}) = a_i(\mathbf{0}), 0 \leq i < k-1$.

(*iii*) $\mathbf{a}$ *satisfies* $\zeta^{f(\mathbf{a})-f(\mathbf{0})} = (-1)^{\mathbf{a}\cdot\mathbf{w}}$, *for all* $\mathbf{w} \in S_f$.

Differential properties of generalized Boolean functions II

- We say that $f \in \mathcal{GB}_n^{2^k}$ satisfies the (*generalized*) *propagation criterion of order ℓ* ($1 \le \ell \le n$), $gPC(\ell)$, iff the autocorrelation $\mathcal{C}_f(\mathbf{v}) = \sum_{\mathbf{x} \in \mathbb{V}_n} \zeta^{f(\mathbf{x}) - f(\mathbf{x} \oplus \mathbf{v})} = 0$, for all vectors $\mathbf{v} \in \mathbb{F}_2^n$ of weight $0 < wt(\mathbf{v}) \le \ell$.
- f is gbent $\iff gPC(n)$.

Theorem (Martinsen–Meidl–Pott–S., 2018)

Let $f \in \mathcal{GB}_n^{2^k}$, and $A_j^{(\mathbf{w})} = (D_{\mathbf{w}} f)^{-1}(j) = \{\mathbf{x} | f(\mathbf{x} \oplus \mathbf{w}) - f(\mathbf{x}) = j\}$. Then f is $gPC(\ell)$ if and only if, for $1 \le wt(\mathbf{w}) \le \ell$,

$$|A_0^{(\mathbf{0})}| = 2^n, |A_j^{(\mathbf{0})}| = 0, |A_j^{(\mathbf{w})}| = |A_{j+2^{k-1}}^{(\mathbf{w})}|, \ \forall\, 0 \le j \le 2^{k-1} - 1.$$

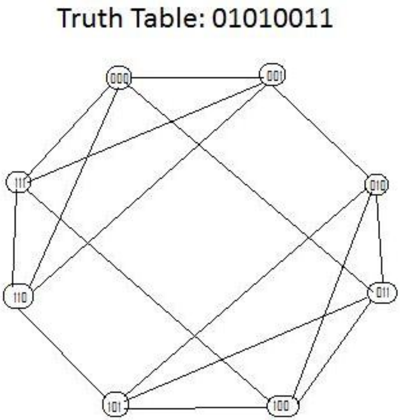
Can one "visualize" some cryptographic properties of a Boolean function?

- *Cayley graph* of $f : \mathbb{F}_2^n \to \mathbb{F}_2$, $G_f = (\mathbb{F}_2^n, E_f)$,

$$E_f = \{(\mathbf{w}, \mathbf{u}) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \,:\, f(\mathbf{w} \oplus \mathbf{u}) = 1\}.$$

- Adjacency matrix $A_f = \{a_{i,j}\}$, $a_{i,j} := f(\mathbf{i} \oplus \mathbf{j})$ (where $\mathbf{i}$ is the binary representation as an n-bit vector of the index i);
- *Spectrum* of G_f is the set of eigenvalues of A_f (G_f).
- Cayley graph G_f has eigenvalues $\lambda_i = \mathcal{W}_f(\mathbf{i})$, $\forall i$.

Cayley graph example: $f(x_1, x_2, x_3) = x_1x_2 \oplus x_1x_3 \oplus x_3$



Strongly regular graphs

- A graph is regular of degree r (or r-regular) if every vertex has degree r;
- The Cayley graph of a Boolean function is always a regular graph of degree $wt(f)$.
- We say that an r-regular graph G with v vertices is a strongly regular graph (SRG) with parameters (v, r, e, d) if $\exists$ integers $e, d \geq 0$ s.t. for all vertices $\mathbf{u}, \mathbf{v}$:
  - the number of vertices adjacent to both $\mathbf{u}, \mathbf{v}$ is e if $\mathbf{u}, \mathbf{v}$ are adjacent,
  - the number of vertices adjacent to both $\mathbf{u}, \mathbf{v}$ is d if $\mathbf{u}, \mathbf{v}$ are nonadjacent.
- We assume throughout that G_f is connected (in fact, one can show that all connected components of G_f are isomorphic).

Bernasconi-Codenotti correspondence

- Shrikhande & Bhagwandas '65: A connected r-regular graph is strongly regular iff $\exists$ exactly three distinct eigenvalues $\lambda_0 = r, \lambda_1, \lambda_2$
  (also, $e = r + \lambda_1\lambda_2 + \lambda_1 + \lambda_2$, $d = r + \lambda_1\lambda_2$).
- The parameters satisfy $r(r - e - 1) = d(v - r - 1)$.
- The adjacency matrix A satisfies (J is the all 1 matrix)

$$A^2 = (d - e)A + (r - e)I + eJ.$$

- Bernasconi-Codenotti correspondence: Bent functions exactly correspond to strongly regular graphs with $e = d$.

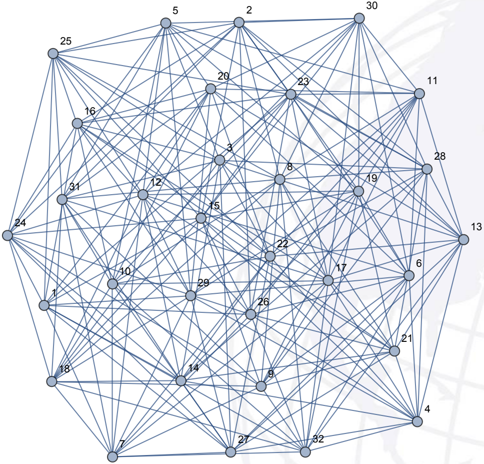
 P.J. Cameron: "*Strongly regular graphs lie on the cusp between highly structured and unstructured. For example, there is a unique srg with parameters* $(36, 10, 4, 2)$*, but there are* 32548 *non-isomorphic srg with parameters* $(36, 15, 6, 6)$*. In light of this, it will be difficult to develop a theory of random strongly regular graphs!*"

Plateaued functions and their Cayley graphs

- $f \in \mathcal{GB}_n^{2^k}$ is called *s-plateaued* if $|\mathcal{H}_f(\mathbf{u})| \in \{0, 2^{(n+s)/2}\}$ for all $\mathbf{u} \in \mathbb{F}_2^n$.
- For $k = 1$: $s = 0$ (n even), f is bent; if $s = 1$ (n odd), or $s = 2$ (n even), we call f semibent.
- Advantages: they can be balanced and highly nonlinear with no linear structures.
- In general, the spectrum of the Cayley graph of an s-plateaued $f : \mathbb{F}_2^n \to \mathbb{F}_2$ will be 4-valued (so, not srg!): if the WH transform of f takes values in $\{0, \pm 2^{\frac{n+s}{2}}\}$, then the Fourier transform of f takes values in $\{wt(f), 0, \pm 2^{\frac{n+s}{2}-1}\}$;

Cayley graphs of plateaued Boolean functions: example

Cayley graph of the semibent
$f(\mathbf{x}) = x_1x_2 \oplus x_3x_4 \oplus x_1x_4x_5 \oplus x_2x_3x_5 \oplus x_3x_4x_5$



Cayley graphs of plateaued Boolean functions with $wt(f) = 2^{(n+s-2)/2}$

There is one case when we do obtain an srg:

Theorem (Riera–Solé–S. 2018)

If $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is s-plateaued and $wt(f) = 2^{(n+s-2)/2}$, then G_f (if connected) is the complete bipartite graph between $\mathrm{supp(f)}$ and $\overline{\mathrm{supp(f)}}$ (if disconnected, it is a union of complete bipartite graphs). Moreover, G_f is strongly regular with $(e, d) = \left(0, 2^{(n+s-2)/2}\right)$.

Strongly walk-regular graphs

- van Dam and Omidi: G is strongly ℓ-walk-regular of parameters $(\sigma_\ell, \mu_\ell, \nu_\ell)$ if there are $\sigma_\ell, \mu_\ell, \nu_\ell$ walks of length ℓ between every two adjacent, every two non-adjacent, and every two identical vertices, respectively.
- Every strongly regular graph of parameters (v, r, e, d) is a strongly 2-walk-regular graph with parameters (e, d, r).

Cayley graphs of plateaued Boolean functions with $wt(f) \neq 2^{(n+s-2)/2}$

Theorem (Riera–Solé–S. 2018)

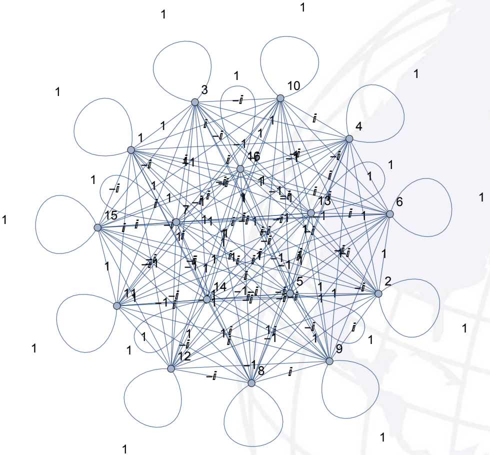
Let $f : \mathbb{F}_2^n \to \mathbb{F}_2$ be a Boolean function, and assume that G_f is connected and $r := wt(f) \neq 2^{(n+s-2)/2}$. Then, f is s-plateaued (with 4-valued spectra) if and only if G_f is strongly 3-walk-regular of parameters $(\sigma, \mu = \nu) = (2^{-n}r^3 + 2^{n+s-2} - 2^{s-2}r, 2^{-n}r^3 - 2^{s-2}r)$.

1. In fact, we showed that it is ℓ-walk regular for all odd ℓ, and found the parameters explicitly.

Go2OpenQues

Generalized Boolean and their Cayley graphs I

- For $f \in \mathcal{GB}_n^q$, *(gen.) Cayley graph* G_f: $\mathbb{V}_n$ vertices; $(\mathbf{u}, \mathbf{v})$ edge of (multiplicative) weight $\zeta^{f(\mathbf{u} \oplus \mathbf{v})}$ (additively $f(\mathbf{u} \oplus \mathbf{v})$).



Strong regularity for weighted graphs

- Let $X, Y \subseteq \mathbb{Z}_{2^k}$. A weighted regular $G = (V, E, w)$, $V \subseteq \mathbb{V}_n$, $w : E \to \mathbb{Z}_{2^k}$ is a (gen.) *$(X; Y)$-strongly regular* of parameters $(e_{X,Y}, d_{X,Y})$ iff # vertices $\mathbf{c}$ adjacent to both $\mathbf{a}, \mathbf{b}$, with $w(\mathbf{a}, \mathbf{c}), w(\mathbf{b}, \mathbf{c}) \in Y$, is exactly $e_{X,Y}$, if $w(\mathbf{a}, \mathbf{b}) \in X$, resp., $d_{X,Y}$, if $w(\mathbf{a}, \mathbf{b}) \in \bar{X}$.
- One can weaken the condition and define a $(X_1, X_2; Y)$-srg notion, where $X_1 \cap X_2 = \emptyset$, not necessarily a bisection; or even allowing a multi-section, and all of these variations can be fresh areas of research for graph theory experts.
- Note that this is a natural extension of the classical definition: for $q = 2$, and $X = \{1\}$, the classical strongly regular graph is then equivalent to an $(X; X)$-strongly regular graph.

Bernasconi-Codenotti strong regularity for gbents

Theorem (Riera–S.–Gangopadhyay 2018)

Let $f \in \mathcal{GB}_n^4$, n even. Then f is gbent iff G_f is $(X; \bar{X})$-strongly regular with $e_X = d_X$, for both $X = \{0, 1\}$, and $X = \{0, 3\}$.

Theorem (Riera–S.–Gangopadhyay 2018)

If $f = a_0 + 2a_1 + \cdots + 2^{k-1}a_{k-1}$, $k \geq 2$, $a_i \in \mathcal{B}_n$, is gbent (n even) then the associated weighted Cayley graph is $(X_{\mathbf{c}}^0; X_{\mathbf{c}}^1)$-strongly regular with explicit $X_{\mathbf{c}}^0, X_{\mathbf{c}}^1$.

Food for thought

- How do the Cayley graphs for generalized semibent/plateaued look like?
- Can one investigate other cryptographic properties of Boolean functions in terms of their Cayley graphs?
- Investigate the "APN property" for functions $: \mathbb{F}_2^n \to \mathbb{Z}_{2^n}$;
- Construct functions with small differential spectra;
- Look at other functions, like rotation symmetric in the generalized context and their differential properties;
- Define the nonlinearity in that environment;
- Define some of these properties (depending upon the Walsh-Hadamard transform with respect to other characters, and/or combine multiple characters.

Theorem (Pante Stanica: http://faculty/nps.edu/pstanica)

Thank you for your attention!

Proof.

None required, but questions are welcome! □