On Cross-Join Method for de Bruijn Sequences and Zech Logarithms

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The Feedback Shift Registers - *FSRs*

- Let $\mathbb{F}_2$ be the binary field and $\mathbb{F}_2^n$ the n-dimensional vector space over $\mathbb{F}_2$. Let us consider a mapping

$$\mathfrak{F} : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$$

$$\mathfrak{F}(x_0, \ldots, x_{n-1}) = (x_1, x_2, \ldots, x_{n-1}, f(x_0, \ldots, x_{n-1})) \quad (1)$$

where f is a Boolean function of n variables of the form

$$f(x_0, \ldots, x_{n-1}) = x_0 + F(x_1, \ldots, x_{n-1}), \quad (2)$$

and F is a Boolean function of $n - 1$ variables.

- The formula (1) defines a nonsingular *FSR* of order n.
- A nonsingular register decomposes the space $\mathbb{F}_2^n$ into a finite number of disjoint cycles.

Generating Binary Sequences

- If there is only one cycle (of length 2^n), then we have a de Bruijn sequence.
- The number of cyclically non-equivalent de Bruijn sequences of order n is (published 1946)

$$B_n = 2^{2^{n-1}-n}$$

- In fact, these sequences were discovered by Fench mathematician C. Flye Sainte-Marie in 1984 and he proved the above formula.
- Consider the binary sequence $\mathbf{s} = (s_0, s_1, \ldots)$ with given n-initial elements $(s_0, \ldots, s_{n-1})$. The next elements, for $i \geqslant 0$, are calculated from the formula

$$s_{i+n} = f(s_i, s_{i+1}, \ldots, s_{i+n-1}) = s_i + F(s_{i+1}, \ldots, s_{i+n-1}).$$

Nicolaas Govert de Bruijn, Dutch mathematician
9 July 1918 - 17 February 2012

Oberwolfach, 1960

Nonlinear Feedback Shift Registers

- *The Algebraic Normal Form (ANF)* of a Boolean function f of n variables is given by

$$f(x_0, x_1, \ldots, x_{n-1}) = \sum a_{i_1,\ldots,i_t} x_{i_1} \cdots x_{i_t} \text{ with } a_{i_1,\ldots,i_t} \in \mathbb{F}_2,$$

  where the sum is over all t-subsets

$$\{i_1, \ldots, i_t\} \subset \{0, 1, \ldots, n-1\}.$$

- In particular we have the linear recurrence

$$f(x_0, x_1, \ldots, x_{n-1}) = x_0 + c_1 x_1 + \ldots + c_{n-1} x_{n-1}.$$

  and the corresponding Linear Feedback Shift Register (*LFSR*).
- When the Boolean function F is a non-linear one, we have a Nonlinear Feedback Shift Register (*NLFSR*).

Solomon Golomb (30 May 1932 - 1 May 2016) and Guang Gong, SETA 2012

Cross-Join Pairs of States

- Let $(s_t) = (s_0, s_1, \cdots, s_{2^n-2}, s_{2^n-1})$ be a de Bruijn sequence.
- Let $S_i = (s_i, s_{i+1}, \cdots, s_{i+(n-1)})$ denote a state. Consider the de Bruijn sequence as a sequence of its states

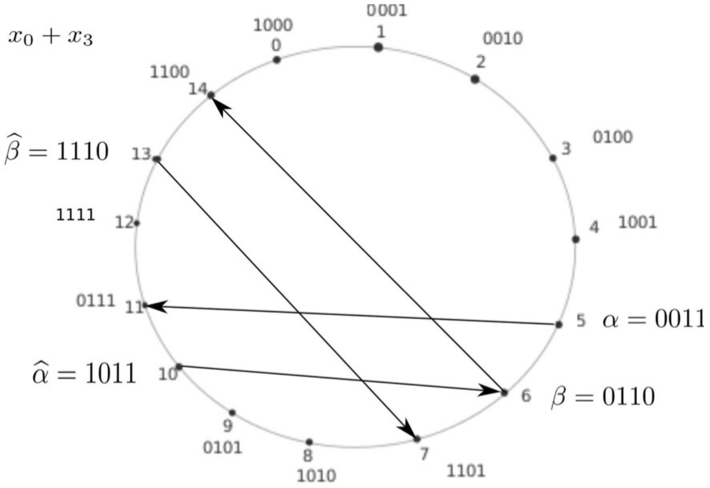
$$(S_t) = (S_0, S_1, \cdots, S_{2^n-2}, S_{2^n-1})$$

.

Definition

Two pairs of states $(a, \widehat{a})$ and $(b, \widehat{b})$ constitute cross-join pairs of states if $a = (a_0, A), \quad \widehat{a} = (\overline{a_0}, A)$ and $b = (b_0, B), \quad \widehat{b} = (\overline{b_0}, B)$, where $\overline{u} = u + 1$ is the negation of the bit u and the states appear in the order $a,\ b,\ \widehat{a},\ \widehat{b}$ in the sequence of states of a given de Bruijn sequence.
We write $A = (a_1, \cdots, a_{n-1})$ and $B = (b_1, \cdots, b_{n-1})$.

Cross-Join Pairs of States - an Example for $n = 4$



$\alpha = 0011 \qquad \beta = 0110$

$\widehat{\alpha} = 1011 \qquad \widehat{\beta} = 1110$

$\overline{x_1}x_2x_3 \qquad x_1x_2\overline{x_3}$

$x_0 + x_3 + \overline{x_1}x_2x3 + x_1x_2\overline{x_3} = x_0 + x_3 + x_1x_2 + x_2x_3$

de Bruijn Sequences and the Cross-Join Pair Operation

Let $\{s_n\}$ be a de Bruijn sequence of order n (or modified de Bruijn sequence with period $2^n - 1$) generated by the feedback Boolean function f of the form (2). Let $(a, \widehat{a})$ and $(b, \widehat{b})$ are cross-join pairs of states for that sequence. Then the feedback Boolean function

$$f(x_0.x_1, \ldots, x_{n-1}) + \prod_{i=1}^{n-1}(x_i + a_i + 1) + \prod_{i=1}^{n-1}(x_i + b_i + 1) \qquad (3)$$

generates new de Bruijn sequence. We call (3) the cross-join pair operation.

Theorem 1. (*J. Mykkeltveit and J. Szmidt*, 2015)
Let (u_t), (v_t) be two de Bruijn sequences of order n. Then (v_t) can be obtained from (u_t) by repeated applications of the cross-join operation.

The List of NLFSRs for $n = 4$

- 1: $x_0 + x_1$
- 2: $x_0 + x_3$
- 3: $x_0 + x_1 + \overline{x_1}x_2x_3 + \overline{x_1}x_2\overline{x_3} = x_0 + x_1 + x_2 + x_1x_2$
- 4: $x_0 + x_3 + \overline{x_1}x_2x_3 + \overline{x_1}x_2\overline{x_3} = x_0 + x_2 + x_3 + x_1x_2$
- 5: $x_0 + x_1 + (\overline{x_1}x_2x_3 + \overline{x_1}x_2\overline{x_3}) + (x_1x_2\overline{x_3} + x_1\overline{x_2}x_3) =$ $x_0 + x_1 + x_2 + x_1x_3$
- 6: $x_0 + x_3 + (\overline{x_1}x_2x_3 + \overline{x_1}x_2\overline{x_3}) + (x_1x_2\overline{x_3} + x_1\overline{x_2}x_3) =$ $x_0 + x_2 + x_3 + x_1x_3$
- 7: $x_0 + x_3 + \overline{x_1}x_2\overline{x_3} + \overline{x_1x_2}x_3 = x_0 + x_2 + x_1x_2 + x_1x_3$
- 8: $x_0 + x_1 + \overline{x_1}x_2\overline{x_3} + \overline{x_1x_2}x_3 = x_0 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3$
- notation: $\overline{x_i} = x_i + 1$

The list of NLFSRs for n = 4

- 9: $x_0 + x_1 + x_1 x_2 \overline{x_3} + \overline{x_1} x_2 \overline{x_3} = x_0 + x_1 + x_2 + x_2 x_3$
- 10: $x_0 + x_3 + x_1 x_2 \overline{x_3} + \overline{x_1} x_2 \overline{x_3} = x_0 + x_2 + x_3 + x_2 x_3$
- 11; $x_0 + x_1 + \overline{x_1} x_2 x_3 + x_1 x_2 \overline{x_2} = x_0 + x_1 + x_1 x_2 + x_2 x_3$
- 12: $x_0 + x_1 + x_1 \overline{x_2 x_3} + \overline{x_1 x_2} x_3 = x_0 + x_3 + x_1 x_2 + x_2 x_3$
- 13: $x_0 + x_1 + x_1 \overline{x_2 x_3} + \overline{x_1} x_2 \overline{x_3} = x_0 + x_2 + x_1 x_3 + x_2 x_3$
- 14: $x_0 + x_3 + x_1 \overline{x_2 x_3} + x_1 \overline{x_2} x_3 = x_0 + x_1 + x_2 + x_3 + x_1 x_3 + x_2 x_3$
- 15:
  $x_0 + x_1 + x_1 \overline{x_2} x_3 + \overline{x_1} x_2 \overline{x_3} = x_0 + x_1 + x_2 + x_1 x_2 + x_1 x_3 + x_2 x_3$
- 16:
  $x_0 + x_3 + x_1 \overline{x_2} x_3 + \overline{x_1} x_2 \overline{x_3} = x_0 + x_2 + x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3$

Finite Fields, Primitive Polynomials and *m*-Sequences

- Let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + 1$ be a primitive polynomial of degree n with binary coefficients.
- Then the linear recurrence

$$g(x_0, x_1, \ldots, x_{n-1}) = x_0 + c_1 x_1 + \cdots + c_{n-1} x_{n-1}$$

  generates the *m*-sequence which is a binary sequence of the period $2^n - 1$.
- Let a be a root of the polynomial $p(x)$, i.e. $p(a) = 0$ in the Galois field $GF(2^n)$ constructed by the polynomial $p(x)$.
- The sequence of elements $\{1, a, a^2, \ldots, a^{2^n-2}\}$ in $GF(2^n)$ has period $2^n - 1$ and directly leads to a binary *m*-sequence.

Evariste Galois (25 October 1811 - 31 May 1832)

Zech Logarithms in $GF(2^n)$

- Let $j \in \{1, \ldots, 2^n - 2\}$
- Then the integer $Z(j)$ such that

$$1 + a^j = a^{Z(j)}$$

  is the Zech logarithm of j.
- Then we have a one-to-one function

$$Z : \{1, \ldots, 2^n - 2\} \longrightarrow \{1, \ldots, 2^n - 2\}$$

- The Zech logarithms are tabularized. There are effective algorithms to calculate them.
- The `Magma` computer algebra system can calculate the Zech logarithms for $n \leqslant 430$, *i.e.*, in $GF(2^{430})$.

The Feedback Functions of the Constructed NFSRs

- Take the primitive polynomial $x^5 + x^2 + 1$.
- The values of the feedback function at the points of 'the jumps', say $Z(2) = 5$ and $Z(4) = 10$ are

$$A = (0,0,0,0,1) \text{ and } B = (0,0,1,0,0).$$

- The feedback function of the NLFSR is $f =$

$$x_0 + x_2 + (x_1+1)(x_2+1)(x_3+1)x_4 + (x_1+1)x_2(x_3+1)(x_4+1)$$
$$= x_0 + x_4 + x_1x_2x_3 + x_1x_2 + x_1x_3x_4 + x_1x_4 + x_2x_3 + x_3x_4.$$

- The quadratic feedback function for the register of order 5 obtained by applying the cross-join operation twice is

$$x_0 + x_4 + x_2x_3 + x_3x_4.$$

- The quadratic feedback function for the register of order 6 obtained similarly is

$$x_0 + x_1 + x_2 + x_5 + x_1x_2 + x_1x_5.$$

The Cross-Join Pair for LFSR of Order $n = 31$

- Let a be a root of the primitive polynomial $p(x) = x^{31} + x^3 + 1$.
- We use the mapping $Z(2n) = 2Z(n)$ for the Zech logarithm. The cross-join pairs $c := (3, 6, 31, 62)$ abbreviates the pair of states $(a^3, a^6, 1 + a^3 = a^{31}, 1 + a^6 = a^{62})$ since $Z(3) = 31$.
- The states of LFSR at 'the jumps':
  $A = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
  $0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0)$, $A_{28} = 1$,
  $B = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$
  $0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0)$, $B_{25} = 1$.
- The corresponding feedback function of the constructed NLFSR

$$f = x_0 + x_3 + \prod_{i=1}^{30} (x_i + A_i + 1) + \prod_{i=1}^{30} (x_i + B_i + 1).$$

It is a Boolean function of degree 29.

The Cross-Join Pairs for Order $n = 127$

- Use the primitive polynomial $p(x) = x^{127} + x + 1$.
- Since $Z(1) = 127$, making $Z(2) = 254$, we have the sequence of mutually disjoint cross-join pairs:

$$c_i = (2^{8i}, 2^{1+8i}, 127 \cdot 2^{8i}, 127 \cdot 2^{1+8i}) \text{ for } i = 0, 1, \ldots, 15.$$

- From this family we can construct $2^{16} - 1$ NFSRs of order $n = 127$ which generate sequences of the period $2^{127} - 1$.
- An Example: the cross-join pairs $c_3 = (2^{24}, 2^{25}, 127 \cdot 2^{24}, 127 \cdot 2^{127})$.
- The corresponding Boolean feedback function has algebraic degree 125.

The Quadratic NLFSRs of Order $n \in \{27, 28, 29\}$

- For $n = 27$

$$x_0 + x_1 + x_2 + x_4 + x_8 + x_{10} + x_{11} + x_{14} + x_{17} + x_{19} + x_{21} + x_6 x_{10}.$$

- For $n = 28$

$$x_0 + x_4 + x_5 + x_6 + x_8 + x_{11} + x_{14} + x_{18} + x_{19} + x_{21} + x_{22} + x_{26} + x_{27} + x_8 x_{27}.$$

- For $n = 29$

$$x_0 + x_3 + x_5 + x_6 + x_{11} + x_{12} + x_{16} + x_{19} + x_{22} + x_{23} + x_{27} + x_{20} x_{28}$$

and

$$x_0 + x_4 + x_6 + x_7 + x_9 + x_{10} + x_{11} + x_{12} +$$
$$x_{16} + x_{17} + x_{21} + x_{25} + x_{26} + x_{17} x_{21}$$

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THANK YOU